SOME FIXED POINT RESULTS IN SEMI–METRIC SPACE

THESIS

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BY

UMESH RAJOPADHYAYA (SUBEDI)

M.SC.(T.U.), M.PHIL.(K.U.)

UNDER THE SUPERVISION OF

PROF. DR. KANHAIYA JHA

DEPARTMENT OF NATURAL SCIENCES (MATHEMATICS)

SCHOOL OF SCIENCE

KATHMANDU UNIVERSITY, NEPAL

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Dedicated

To

My Beloved Parents

Govinda B. Rajopadhyaya

And

Rama Rajopadhyaya

Scholar's Declaration

I **Umesh Rajopadhyaya (Subedi)**, hereby declare that, the research work entitled Some Fixed Point Results in Semi-metric space submitted for the fulfillment of Doctor of Philosophy **(Ph.D.)** degree in mathematics to the Department of Natural Sciences, School of Science, Kathmandu University, Nepal, is genuine work done originally by me and has not been published or submitted elsewhere for the requirement of a degree programme. Any literature, data or works done by others and cited within this thesis has been given due acknowledgement and listed in the reference section.

....

Umesh Rajopadhyaya (Subedi)
K.U. Regd. No. 009099-08
Department of Natural Sciences(Mathematics)
School of Science
Kathmandu University
Dhulikhel, Kavre

CERTIFICATION

This is to certify that the thesis entitled **Some fixed point results in semi-metric space** which is being submitted by **Mr. Umesh Rajopadhyaya (Subedi)** in the fulfilment for the award of Doctor of Philosophy (**Ph. D.)** degree in mathematics of Kathmandu University, Nepal is a record of his own work carried out by him under my guidence and supervision.

The matter embodied in this thesis has not been submitted for the award of any degree.

....

Prof. Dr. Kanhaiya Jha
School of Science
Kathmandu University
Dhulikhel, Nepal.

Certificate of Approval

The thesis entitled **Some fixed point results in semi-metric space** that is being submitted to Kathmandu University, Dhulikhel, Kavre, Nepal for the award of the degree of **Doctor of Philosophy (Ph.D.)** in mathematics is original, and is a record of bonafide research work carried out by **Mr. Umesh Rajopadhyaya (Subedi)** in the department of Natural Sciences, School of Science, Kathmandu University, during the last three years.

APPROVED BY:

____

Prof. Dr. Kanhaiya Jha
Supervisor
Department of Natural Sciences
School of Science
Kathmandu University, Nepal

____

Prof. Dr. Mohan Bikram Gewali
Dean, School of Science
Chairman
Research committee
Kathmandu University, Nepal

____

Prof. Dr. Santosh Man Maskey
External Examiner
Professor of Mathematics, Emeritus
Tribhuvan University, Nepal

Date : 4th December 2015

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Umesh Rajopadhyaya (Subedi)

K.U. Regd. No. 009099-08

Preface

Mathematics is the backbone of modern science as it deals with the study of quantity structure and shapes. It is remarkably efficient source of new concepts and tools to provide solution to existing problem. From different perspective, mathematics can be defined as a science which involves logical reasoning based on accepted rules, laws and facts. In mathematics, analysis plays a vital role for its development. The study of several functions come under the functional analysis. Functional analysis has been divided into two parts namely linear functional analysis and non-linear functional analysis. Since 1960, fixed point theory is considered to be the part of non-linear functional analysis. Functional analysis is an abstract branch of mathematics that originated from classical analysis. It serves as an essential tool for various branches of mathematical analysis and its applications.

Polish mathematician Stephan Banach published his contraction principle in 1922. Since then, this principle has been extended and generalized in several ways. Its development started about eighty years ago and nowadays functional analystic methods and results are important in various fields of mathematics and its applications. The theory of fixed point is very extensive field which has wide applications. Fixed point theory has played a central role in the problems of non-linear functional analysis and provided a power tool in demonstrating the existence and uniqueness of solutions to various mathematical models representing phenomena arising in different fields such as in Engineering, Economics , Game Theory and Nash Equilibrium, Steady State Temperature Distribution, Epidemics, Flow of Fluids, Chemical Reactions, Neutron Transport Theory, Haar Measures, Abstract Elliptic Problems. Invariant Subspace Problems, Approximation Problems.

French mathematician Maurice Frechet introduced the concept of metric space in 1906. After 22 years from this, Austrian mathematician Karl Menger introduced semi-metric space as an important generalization of metric space.

Chapter wise cameo description of the present study is as follows.

CHAPTER ONE deals with the general introduction of functional analysis and fixed point theory. A brief survey of the development of the fixed point theory in metric space and semi-metric space has been presented and some of the well-known theorems have been stated yearwise. Also, it deals with some applications of fixed point theorem.

CHAPTER TWO is intended to obtain the fixed point theorems of semi-metric space using E.A property with different contractive conditions . It includes basic definitions and the chronological development of fixed point theorems on semi-metric space using E.A property.

CHAPTER THREE is intended to obtain some common fixed point theorems using occasionally weakly compatible mappings and occasionally converse commuting mapping in semi-metric space . It includes basic definitions and those theorems specially having the relevance for the establishment of our theorems.

CHAPTER FOUR is intended to obtain fixed point theorem in fuzzy semi-metric space in two pair of mappings with basic definitions. As a conclusion, a future research scope has been kept for further research activities.

The list of literature consulted has been placed at the end of the thesis as Bibliography. Other original contributions have been contained in chapters 2,3 and 4. A part of the research work contained in this thesis has been already published in international peer reviewed journals [49],[51], [53], [94], [95], [96]and [97].

LIST OF PUBLICATIONS

Papers in Peer Reviewed International Journals :

1. Common Fixed Point Theorem for Generalized Contractive Type Mappings in Semi-Metric Space , *International Journal of Mathematical Sciences and Engineering Applications,* **1** *(8)(2014), 139-146.*

2. *Fixed Point Theorem for Ocassionally Weakly Compatible Mappings in Semi-Metric Space,* Annals of Pure and Applied Mathematics, **2**(5) (2014), 153-157.

3. Common Fixed Point Theorem for Ocassionally Converse Commuting Mappings in Semi-metric space, *International Journal of Mathematical Analysis,* **13** *(8),2014, 627-634.*

4. *Common Fixed Point Theorem in Semi-Metric Space with Compatible Mapping of Type (E),* Bulletin of Mathematical Sciences and Applications, **10***(2014), 141 - 147.*

5. *A Common Fixed Point Theorem for Weakly Compatible Mappings in Fuzzy Semi-Metric Space,* Journal of Mathematics and System Science,**(4)***(2014), 720 - 724.*

6. *A Common Fixed Point Theorem in Semi-Metric Space Using E.A Property,* Electronic Journal of Mathematical Analysis and Applications, **1***(3)(2015), 19 - 23.*

7. *Common Fixed Point Theorem For Six Mappings In Semi-metric Space with E.A. property,* Application and Applied Mathematics:An International Journal (2015)(To Appear)

Papers in Proceedings of Conferences:

8. A Common Fixed Point Theorem in Semi-metric Space Using a New Contractive Condition, Proceedings of National Conference of Mathematics (NCM 2014): A publication of Nepal Mathematical Society (NMS), 2014, 54-56.

9. Semi-metri Space and Common Fixed Point Theorems, Proceedings of National Conference of Mathematics (NCM 2012): A Publication of Nepal Mathematical Society (NMS), 2012, 59-67.

10. Common Fixed Point Theorem for Six Mappings in Semi-metric Space with E.A Property, Proceedings of Eighteenth International Mathematics Conference (IMC 2013): A publication of Bangladesh Mathematical Society (BMS), 2013, 19-21.

Contents

Chapter 1

Introduction

This chapter includes an introduction of fixed point theory in semi-metric space with some fundamental concepts and historical development.

1.1 Introduction

Functional analysis is an important branch of mathematics that has been originated from classical analysis. Its development started about eighty years ago. It has been recognized as a basic area of mathematics as the initial platform for generalizing many of the concepts of classical mathematics. The concepts and tools developed in this area are applied to the various branches of mathematics and other disciplines.

In 1886, a famous French mathematician H. Poincare was the first one to work in the field of fixed point theory. In 1912, L.

E. J. Brouwer[10] established the fixed point theorem for the solution of the equation $f(x) = x$. He also proved fixed point theorem for a square, a sphere and their n-dimensional counter parts which was further extended by Kakutani[87]. The result to Brouwer was extended by P. J. Schauder[87] in 1930, which in turn was extended by A. N. Tychnoff[87] in 1935. Each of these theorems asserts that every continuous mapping of a compact convex set into itself has a fixed point. Brouwer's and Schauder's fixed point theorems are fundamental theorems in the area of fixed point theory and its application.

The first infinite dimensional fixed point theorem was investigated by C. D. Birkhoff and O. D. Kellogg[87] in 1922. The fixed point theory is one of the most powerful tool in modern mathematics. The theorems concerning the existence and the properties of fixed points are known as fixed point theorems.
In 1906, French mathematician M. Frechet [19] introduced a notion of metric space. This concept is strongly based on distance function and very much useful to find the distance between any two objects. Since then several extended generalized forms of metric space been introduced. The semi-metric space was introduced by K. Menger[70] in 1928 as an important notion. Also, the fuzzy metric space has been introduced by O. Kramosil and J. Michalek [63] in 1975.

With the discovery of computer and development of new software's for quick computing, a new dimension has been given to fixed point theory. The new field of study has been generated like applied mathematics, numerical analysis and algorithms. Fixed point theory has become the subject of scientific research both in deterministic and stochastic circumstances. Many nonlinear equations can be solved using fixed point theorems. Fixed point theory is an interdisciplinary topic which can be applied in various disciplines of mathematics and mathematical sciences like non linear integral, differential equations, game theory, optimization theory, mathematical economics, approximation theory, variational inequalities and boundary value problems.

Fixed Point Theory is divided into three major areas:

1. Topological Fixed Point Theory ;
2. Metric Fixed Point Theory; and
3. Discrete Fixed Point Theory.

   Historically, the boundary lines between the three areas was defined by the discovery of three major theorems:

1. Brouwer's Fixed Point Theorem in 1906;
2. Banach's Fixed Point Theorem in 1922 and
3. Tarski's Fixed Point Theorem in 1955.

Metric fixed point theory is a branch of fixed point theory which has its primary applications in functional analysis. Apart from establishing the existence of a fixed point, it often becomes necessary to prove the uniqueness of the fixed point. Besides, from computational point of view, an algorithm for calculating the value of the fixed point to a given degree of accuracy is desirable. Often this algorithm involves the iteration of the given function. In essence, the question about the existence, uniqueness and approximation of fixed point provide three significant aspect of the general fixed point principle. Among several fixed point theorems, Brouwer's fixed point theorem is well known due to its remarkable application in different fields of mathematics. The theorem is supposed to have originated from L. Brouwer's observation of a cup of coffee. If one stirs to dissolve a lump of sugar, it appears there always a point without motion. He drew the conclusion that at any moment there is a point on the surface that is not moving. The fixed point is not necessarily the point that seems to be motionless since the centre of the turbulence moves a little bit. The development of fixed point theory which is the cardinal branch of non-linear analysis has given great efforts in the advancement of non-linear analysis. The earliest results had already been obtained in 1920's.

In 1922, Polish mathematician Stephan Banach[8] established Banach's contraction principle (BCP) in his Ph.D. dissertation.It is also known to be Banach fixed point theorem or principle of contraction mapping. It has become milestone to all the students of mathematical analysis to establish new theorems by generalizing this theorm. The BCP has been considered to be very important as it is a source of existence and uniqueness theorem in different branches of sciences. This theorem provides an illustration of the unifying aspects in functional analysis. The important feature of the BCP is that it gives the existence, uniqueness and the sequence of the successive approximation converges to a solution of the problem.

Definition 1.1.1. *[19] Let X be a non empty set and d be a real function from $X \times X$ into $\mathbb{R}^+$ such that for all $x, y, z \in X$, we have*

1. $d(x, y) \geq 0$ *(Positivity);*
2. $d(x, y) = 0 \iff x = y$ *(Indiscarnible);*
3. $d(x, y) = d(y, x)$ *(Symmetricity); and*
4. $d(x, z) \leq d(x, y) + d(y, z)$ *(Triangle inequality)*

then d is called a **metric** or distance function and the pair (X, d) is called a **metric space**. The space is denoted simply by X if the metric is understood.

Definition 1.1.2. *[103] A sequence $\{x_n\}$ in a metric space (X, d) is called a* **Cauchy Sequence** *if it satisfies the following condition, for every $\varepsilon > 0$ there is an integer N such that*
$d(x_n, x_m) < \varepsilon$ whenever $n \geq N$ and $m \geq N$.

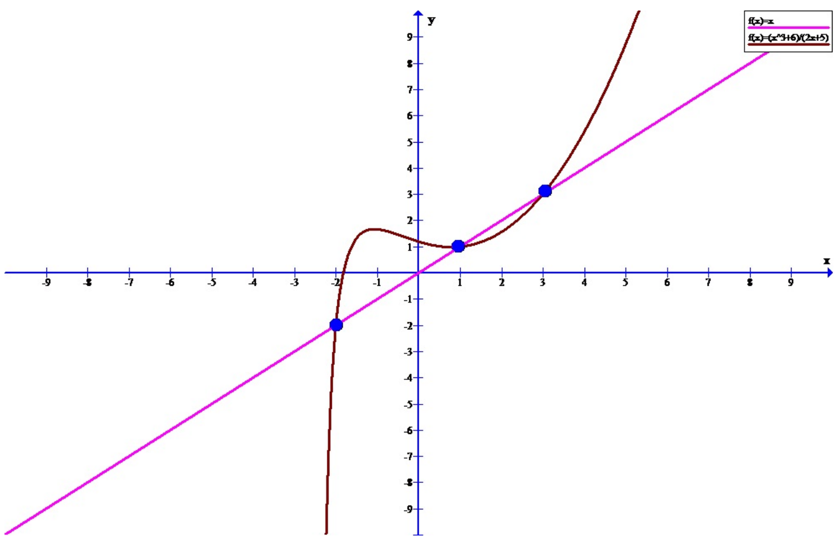
Definition 1.1.3. *[103] A sequence $\{x_n\}$ of points in X is said to* **converge** *if there is a point p in X with the following property, for every $\varepsilon > 0$ there is an integer N (depending on ε) such that $|x_n - p| < \varepsilon$ whenever $n \geq N$.*

Definition 1.1.4. *[64] A metric space (X,d) is called* **complete** *if every cauchy sequence in X converges in X. A subset M of X is called complete if the metric space (M,d) is complete.*

Definition 1.1.5. *[93] Consider a map $T : X \to X$ then any point $x \in X$ is said to be a* **fixed point** *of T if $Tx = x$.*

Example 1.1.6. *[93]*

Consider the cubic equation $x^3 - 2x^2 - 5x + 6 = 0$,then the points $x = -2, 1, 3$ are the roots of this equation. This equation can be written as one of the form $x = T(x) = \frac{x^3+6}{2x+5}$. Then, it is a function equation. Since $T(-2) = -2$, $T(1) = 1$ and $T(3) = 3$, so the points $x = -2, 1$ and 3 are three fixed points of T.



The following is the famous Banach Contraction Principal introduced by S. Banach in 1922.

Theorem 1.1.7. *[41] Any contraction mapping T defined on a non-empty complete metric space (X, d) into itself has a unique fixed point x^* on X. Moreover, if x_o is any arbitrary point in X and the sequence $\{x_n\}$ is defined by $x_{n+1} = Tx_{n+1}$ for $n = 0, 1, 2, 3....$ Then $lim_{n\longrightarrow\infty} x_n = x^*$, and we have the estimate $d(x_n, x^*) \leq \frac{k^n}{1-k} d(x_o, x_1)$.*

1.2 Some Applications of Fixed Point Theory

There are numerous applications of fixed point theory in mathematics and other fields. Brouwer's and Schauder's fixed point theorems are fundamental theorems in the area of fixed point theorem and its applications. Von Neumann in 1937 firstly used a generalization of Brouwer's fixed point theorem to prove existence of a saddle point for balanced growth equilibrium in the expansion of economy. With reference to economics, Brouwer's fixed point theorem is very helpful to calculate a certain economical equilibrium. The first such algorithm was proposed by H. Scarf [87] in 1983. A fundamental principle in computer science is iteration.

Banach contraction principle is one of the most useful application of fixed point theory in its existence and uniqueness theories. Among all the classical fixed point theorems, the contraction principle has many applications which are scattered throughout almost all the branches of mathematics. Fixed point theorem has many applications in the field like signal and image reconstruction, tomography, telecommunications, interpolation, extrapolation, signal enhancement, signal synthesis, filter synthesis. Besides these, there are many researchers involving to solve problems in fixed point theory and to expand its application in various area.

The paper of Jha [40] deals with the survey work on some applications of Banach Contraction Principle. Some brief outline of useful applications are included in the text books of Kreyszig [41]in 1978, A. Sidiqui in 1986, B. Chaudhary and S. Nanda in 1989, S. and B. Ahmad in 1996 and E. T. Copson in 1996.

1.3 Some Fixed Point Theorems in Semi-Metric Space

1.3.1 Basic Definitions

In 1928, Austrian Mathematician Karl Menger introduced semi-metric space as an important generalization of metric space.

Definition 1.3.1. *[70] A* **semi-metric (also symmetric)** *space is non-empty set X together with a function $d : X \times X \longrightarrow [0, \infty)$ satisfying the following conditions:*

1. *$d(x, y) = 0 \iff x = y$, and*
2. *$d(x, y) = d(y, x)$ for $x, y \in X$.*

Example 1.3.2. *[93] Let $X = R$ be the set of all real numbers. Let a function d be defined as follows:*
$d(x, y) = |x - y|$, x and y are both rational or irrational, and
$d(x, y) = |x - y|^{-1}$ otherwise.
Then (X,d) is a semi-metric space but is not a metric space since d doesn't satisfy triangle inequality.

Example 1.3.3. *Consider $X = [0, 1]$. Let a function d be defined as $d(x, y) = (x - y)^2$. Then (X,d) is a semi-metric space but not a metric space since d doesn't satisfy triangle inequality.*

The difference of a semi-metric space and metric space comes from the triangle inequality. In order to obtain the fixed point theorems on a semi-metric, some additional axioms W3, W4, W5, W, H.E. and C.C are needed. The properties W3, W4 and W5 were introduced by W. A. Wilson[110] in 1931, H.E. by M. Aamri and D.El. Moutawakil [2] in 2003, W by D. Mihet [72] in 2005 and C.C by S. H. Cho, G. y. Lee and J. S. Bae [15] in 2008 as a partial replacement of triangle inequality are as follows:

W3 :[110] For a sequence $\{x_n\}\in X$ and for all $x, y \in X$,
$lim_{n\to\infty} d(x_n, x) = 0$ and $lim_{n\to\infty} d(x_n, y) = 0$
implies that $d(x, y) = 0$ which gives $x = y$.

W4 :[110] For sequences $\{x_n\}$, $\{y_n\} \in X$ and $x \in X$,
$lim_{n\to\infty} d(x_n, x) = 0$ and $lim_{n\to\infty} d(x_n, y) = 0$,
implies that $lim_{n\to\infty} d(y_n, x) = 0$.

W5 :[110] For sequences $\{x_n\}$,$\{y_n\}$ and $\{z_n\} \in X$
$lim_{n\to\infty} d(x_n, y_n) = 0$ and $lim_{n\to\infty} d(y_n, z_n) = 0$
implies that $lim_{n\to\infty} d(x_n, z_n) = 0$.

H.**E**.[2] For sequences $\{x_n\}$,$\{y_n\}\in X$ and $x \in X$,
$lim_{n\to\infty} d(x_n, x) = 0$ and $lim_{n\to\infty} d(y_n, x) = 0$
imply that $lim_{n\to\infty} d(x_n, y_n) = 0$.

The following additional property which is related to the continuity of semi-metric space, is

C.**C**. [15] For a sequence $\{x_n\} \in X$, for all $x, y \in X$,
$lim_{n\to\infty} d(x_n, x) = 0$, implies that $lim_{n\to\infty} d(x_n, y) = d(x, y)$.

Example 1.3.4. *Let $X = [-2, 2]$ be a semi-metric space with $d(x, y) = (x - y)^2$. Consider a sequence $\{x_n\}$,$\{y_n\} \in X$ defined by $x_n = \frac{1}{n} + 1$ and $y_n = -\frac{1}{n} + 1$ which satisfies W3, W4, H.E and C.C properties.*

W : [72] For sequences $\{x_n\}$,$\{y_n\} \in X$, $lim_{n\to\infty} d(x_n, y_n) = 0$ and $lim_{n\to\infty} d(y_n, z_n) = 0$ implies that $lim_{n\to\infty} d(x_n, z_n) = 0$.

The following proposition shows the relationship among W3, W4 and C.C properties

Proposition 1.3.5. *[15] For axioms in semi-metric space (X,d), we have*
1) $W4 \Rightarrow W3$, *and*
2) $C.C \Rightarrow W3$.

It is important to note that $W4 \nRightarrow H.E$ and $W4 \nRightarrow C.C$ and $W3 \nRightarrow H.E$ and $W3 \nRightarrow C.C$ by the proposition 1.3.5.

Example 1.3.6. *[15] Let* $X = [0, \infty]$ *and let* $d(x, y) = |x - y|, (x \neq 0, y \neq 0)$, *and* $d(x, y) = \frac{1}{x}, (x \neq 0)$. *Then,* (X, d) *is a semi-metric space which satisfies W4 but does not satisfy H.E if we take sequences* $x_n = n$, $y_n = n + 1$. *Also* (X, d) *does not satisfy C.C.*

Definition 1.3.7. *[7] Let X be a non empty set and* $A, B : X \to X$ *be arbitrary mappings. A point* $y \in X$ *is a* **coincidence point** *for A and B if and only if* $Ay = By$.

Example 1.3.8. *Consider two self-maps A and B on* $X = R$ *the set of the real numbers defined by* $A(x) = x^2 + 1$ *and* $B(x) = e^x$. *If* $x = 0$, $A(0) = 1$ *and if* $x = 0$, $B(0) = 1$. *Also,* $A(0) = B(0) = 1$, *this imply* $A(0) = B(0)$. *Therefore, the point* $0 \in X$ *is the coincidence point of A and B.*

Definition 1.3.9. *[54] Let A and B be two self-mappings of a semi-metric space (X,d). Then A and B are said to be* **compatible** *if $lim_{n\to\infty}d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that*
$lim_{n\to\infty}d(Ax_n, t) = lim_{n\to\infty}d(Bx_n, t) = 0$, for some $t \in X$.

Definition 1.3.10. *[2] Let A and B be two self-mappings of a semi-metric space (X,d). Then A and B are said to satisfy the* **property E.A.** *or* **tangential** *if there exists a sequence $\{x_n\}$ such that*
$lim_{n\to\infty}d(Ax_n, t) = lim_{n\to\infty}d(Bx_n, t) = 0$ for some $t \in X$.

Definition 1.3.11. *[67] Let A and B be two self-mappings of a semi-metric space (X,d). Then A and B are said to be* **commuting** *if $STx = TSx$ for all $x \in X$.*
Two self mappings A and B on a semi-metric space are said to be commuting at a point $z \in X$ if $STz = TSz$.

Definition 1.3.12. *[55] Let A and B be two self-mappings of a semi-metric space (X,d). Then A and B are said to be* **weakly compatible** *if they commute at their coincidence points.*

Definition 1.3.13. *[56] Let A and B be two self-mappings of a semi-metric space (X,d). Then A and B are said to be* **occasionally weakly compatible (owc)** *if there is a point $x \in X$ which is coincidence point of A and B at which A and B commute.*

Example 1.3.14. *[49] Let us consider* $X = [2, 20]$ *with the semi-metric space* (X, d) *defined by* $d(x, y) = (x - y)^2$*. Define self maps* A *and* B *by*
$A(2) = 2$ *at* $x = 2$ *and* $A(x) = 6$ *for* $x > 2$
$B(2) = 2$ *at* $x = 2$*,* $B(x) = 12$ *for* $2 < x \leq 5$ *and* $B(x) = x - 3$ *for* $x > 5$*. Then, for* $x = 9$*, we get*
$A(9) = B(9) = 6$*.So, besides* $x = 2$*,* $x = 9$ *is another coincidence point of* A *and* B*. Also,we have* $AB(2) = BA(2)$ *but*
$AB(9) = 6$ $BA(9) = 3$*,* $AB(9) \neq BA(9)$*. Therefore* A *and* B *are occasionally weakly compatible but not weakly compatible. Hence, weakly compatible mappings are occasionally weakly compatible but not conversely in semi-metric space.*

Definition 1.3.15. *[75] Let* (X, d) *be a d-bounded semi-metric space and let C(X) be the set of all non-empty d-closed subset of* (X, d)*. Consider the function* $D : 2^X \times 2^X \longrightarrow R^+$ *defined by*
$D(A, B) = max\{sup_{a \in A} d(a, B); sup_{b \in B} d(A, b)\}$ *for* $A, B \in C(X)$*. Then,* $(C(X), D)$ *is a semi-metric space.*

Definition 1.3.16. *[93] Let* X *be a non-empty set and* $A, B : X \to X$ *be arbitrary mappings. A point* $x \in X$ *is a* **common fixed point** *for* A *and* B *if* $Ax = Bx = x$*.*

Example 1.3.17. *[93] Let* $A, B : X \to X$ *be functions, such that* $A(x) = x^2$ *and* $B(x) = xe^x$*. If* $x = 0$*, then* $A(0) = 0$ *and* $B(0) = 0$*. So,* $x = 0$ *is common fixed point of* A *and* B*.*

Definition 1.3.18. *[93] Let (X,d) be a semi-metric space. Then*

1. *(X,d) is* **S-complete** *if for every d-Cauchy sequence* $\{x_n\}$*, there exists* x *in* X *such that* $lim_{n\to\infty}d(x,x_n) = 0$*.*

2. *A mapping* $A : X \longrightarrow X$ *is* **d-continuous** *if* $lim_{n\to\infty}d(x_n,x) = 0$ *implies* $lim_{n\to\infty}d(Ax_n, A(x)) = 0$*.*
   For topological aspects we have the following definition for continuity.

3. *A mapping* $A : X \longrightarrow X$ *is* **t(d)-continuous** *if* $lim_{n\to\infty}d(x_n,x) = 0$ *with respect to t(d) implies* $lim_{n\to\infty}d(Ax_n, A(x)) = 0$ *with respect to t(d).*

Definition 1.3.19. *[38] Let* Y *be an arbitrary set and* X *be a non-empty set equipped with semi-metric d. Then the pairs* (A,S) *and* (B,T) *of mappings from* Y *into* X *are said to have the* **common limit range property** *(with respect to mappings S and T) often denoted by* (CLR_{ST}) *if there exist two sequences* x_n *and* y_n *in* Y *such that*
$lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = lim_{n\to\infty}By_n = lim_{n\to\infty}Ty_n = z$*, where* $z \in S(Y) \cap T(Y)$*.*

1.3.2 Some Fixed Point Theorems in Semi-metric Space

In this section, some fixed point theorems in semi-metric space have been stated without proof as the sources for our results.

In 2003, D. El. Moutawakil[75] established the following theorem in S-complete semi-metric space for single self mapping.

Theorem 1.3.20. *[75] Let* (X, d) *be a d-bounded and S-complete semi-metric space satisfying W4 and* $A : X \longrightarrow C(X)$ *be a multivalued mappings such that*
$d(Ax, By) \leq kd(x, y)$, $k \in [0, 1]$, *for all* $x, y \in X$.
Then, there exists $u \in X$ *such that* $u \in Au$.

In 1999, T.L. Hicks and B. E. Rhoades established the following common fixed point theorem as an extension of Banach contraction principle in semi-metric space for pair of self maps.

Theorem 1.3.21. *[29] Let (X,d) be a bounded semi-metric space that satisfies W3. Suppose* (X, d) *is S-complete and*
$A : X \longrightarrow X$ *is d-continuous. Then A has a fixed point if and only if there exists* $\alpha \in (0, 1)$ *and a d-continuous function*
$B : X \longrightarrow X$ *which commutes with A and satisfies*
$B(x) \subset A(X)$ *and* $d(Bx, By) \leq \alpha d(Ax, Ay)$ *for all* $x, y \in X$.
Indeed, A and B have a unique common fixed point if the above contraction holds.

In particular if $A = B$ and $B = I$ an identity mapping in the above theorem, then Banach contraction principle in semi-metri space reduces to BCP in usual metric space. Now, we consider a function $\emptyset : R^+ \longrightarrow R^+$ satisfying the condition $0 < \emptyset(t) < t$ for each $t > 0$.

In 2002, M. Aamri and D. El Moutawakil [2] established the following theorem for pair of self-mappings in semi-metric space using weakly compatible self-mappings.

Theorem 1.3.22. *[2] Let (X, d) be a semi-metric space that satisfies W3 and HE. Let A and B be two weakly compatible self-mappings of (X, d) such that for all $x, y \in X$,*

i) $d(Ax, Ay) \leq \emptyset(max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\})$,

ii) A and B satisfy the property E.A., and

iii) $AX \subset BX$.

If the range of A or B is a complete subspace of X, then A and B have a unique common fixed point.

In 2006, M. Imdad, J. Ali and L. Khan established the following coincidence point theorem in semi-metric space for pair of mappings.

Theorem 1.3.23. *[35] Let (X, d) be a semi-metric space that enjoys W3. Let A and B be two self-mappings of X such that*

i) A and B satisfy the property E.A,

ii) for all $x \neq y \in X$

$$
\begin{aligned}
&d(Ax, By) \\
&\quad < max\{d(Ax, Ay), \frac{k}{2}[d(Bx, Ax) + d(By, Ay)], \\
&\qquad \frac{k}{2}[d(By, Ax) + d(Bx, Ay)]\}, \ 1 \leq k < 2.
\end{aligned}
$$

If A(X) be d-closed subset of X, then A and B have a point of coincidence.

Let Φ denotes the set of all real functions $\varphi : [0, \infty) \longrightarrow [0, \infty)$ with the following properties:

i) $\varphi(0) = 0$,

ii) $\varphi(r) < r$ for all $r > 0$,and

iii) $lim_{t \longrightarrow r+\varphi(t)} < r$ for any $r > 0$.

Also, δ denotes the set of all continuous,monotone non-decreasing, real functions $F : [0, \infty] \to [0, \infty]$ such that $F(x) = 0$ if and only if $x = 0$.

In 2009, I. D. Arandelovic and D. S. Petkovic [7] established the following common fixed point theorem in semi-metric space using weakly compatible mappings for pair of self-mappings.

Theorem 1.3.24. *[7] Let (X, d) be a semi-metric space which satisfies properties W3 and H.E. Let $\varphi \in \Phi$,$F \in \delta$ and let $A, B : X \longrightarrow X$ be self-mappings of X such that:*

i) $F(d(Ax, By) \leq \varphi(F(max\{d(Bx, Ay), d(Bx, Ay), d(Ay, By)\})$ for any x, y,

ii) A and B are weakly compatible,

iii) A and B satisfy the property E.A, and

iv) $AX \subseteq BX$.

If the range of one of the mappings A or B is a complete subspace of X, then A and B have unique common fixed point.

In 2013, A. Bhatt [9] established the following common fixed point theorem in semi-metric space using g-compatible and g-reciprocally continuous pair of self-mappings.

Theorem 1.3.25. *[9] Let f and g be g-reciprocally continuous self-mappings of a semi-metric space (X, d) satisfying*

$$d(fx, fy) < max\{d(gx, gy), \frac{d(fx, gy) + d(fy, gy)}{2}, \frac{d(fx, gy) + d(fy, gx)}{2}\},$$

whenever the right hand side is non-zero. Suppose f and g satisfy property E.A. If f and g are g-compatible then f and g have a unique common fixed point.

Now, we take the following common fixed point theorem for two pairs of self mappings in semi-metric space.
In 2002, M. Aamri and D. El Moutawakil [2] established the following common fixed point theorem for two pairs of self mappings in semi-metric space.

Theorem 1.3.26. *[2] Let (X, d) be a semi-metric space that satisfies W3, W4 and H.E. Let A, B, T and S be self mappings of (X, d) such that*
i) $d(Ax, By) \leq \emptyset(max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\}$ for all $(x, y) \in X \times X$,
ii)(A, S) and (B, T) are weakly compatible,
iii)(A, S)or (B, T) satisfies the property E.A, and
iv) $AX \subset TX$ and $BX \subset SX$.
If the range of the one of the mappings A,B,T or S is a complete subspace of X, then A,B,T and S have a unique common fixed point.

In 2006, A. Aliouche [6] established the following common fixed point theorem for pair of self-mappings in semi-metric space with the integral form.

Theorem 1.3.27. *[6] Let (X, d) be a semi-metric space that* *satisfies W3, W4 and H.E. Let A,B,S and T be selfmappings of (X, d) such that*

i) $\int_0^{d(Ax,By)} \psi(t)dt \leq \phi(\int_0^{maxd(Sx,Ty),d(Sx,By),d(By,Ty)} \psi(t)dt)$ *for all* $x, y \in X$ *where* $\varphi : R_+ \to R_+$ *is a Lebesgue -integral mapping which is summable, non-negative and such that*

ii) $\int_0^{\in} \psi(t)dt > 0$ *for all* $\in > 0$.

Suppose that $A(X) \subset T(X)$ and $B(X) \subset S(X)$, (A, S) and (B, T) are weakly compatible and (A, S) or (B, T) satisfies property (E.A). If the range of one of the mappings A, B, S and T is a complete subspace of X, then A, B, S and T have a unique common fixed point in X.

In 2009, I. D. Arandelovic and D. S. Petkovic [7] established the following common fixed point theorem in semi-metric space for two pairs of self mappings using W3 and H.E propereties.

Theorem 1.3.28. *[7] Let (X, d) be a semi-metric space which satisfies properties W4 and H.E. Let $\varphi \in \Phi$, $F \in \delta$ and let $A, B, S, T : X \longrightarrow X$ be self-mappings of X such that:*

i)$F(d(Ax, By) \leq \varphi(F(max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\})$ *for any* x, y,

ii)(A, T) and (B, S) are weakly compatible,

iii) (A, S) or (B, T) satisfy the property E.A, and

iv) $AX \subseteq TX$ *and* $BX \subseteq SX$.

If the range of one of the mappings A ,B, S or T is a complete subspace of X, then A ,B, S and T have unique common fixed point.

In 2008, S. H. Cho, G. Y. Lee and J. S. Bae [15] established the following common fixed point theorem in semi-metric space for two pairs of self-mappings.

Theorem 1.3.29. *[15] Let (X, d) be a semi-metric space that satisfies W3 and H.E. Let A, B, S and T be self-mappings of X such that*
i) $AX \subset TX$ and $BX \subset SX$,
ii)the pair (B,T)or (A, S) satisfies property E.A,
iii) the pairs (A, S) and (B,T) are weakly compatible,
iv) for any $x, y \in X(x \neq y)$, $d(Ax, By) < m(x, y)$, where

$$m(x,y) = max\{d(Sx,Ty), min\{d(Ax,Sx)\}, d(By,Ty), min\{d(Ax,Ty)\}, d(By,Sx)\}$$

v) SX and TX are d-closed subset of X.
Then A,B,S and T have a unique common fixed point in X.

The notion of implicit relation is defined as follows: Let F_6 be the set of all real-valued functions
$F(t_1, ...t_6) : R_+^6 \longrightarrow R$, satisfying the following conditions:
F_1: F is non-decreasing in variables t_2, t_5, t_6,
F_2: $F(t, t, 0, 0, t, t) < 0$, for each $t > 0$.
In 2009, M. Imdad and J. Ali [36] established the following common fixed point theorem in semi-metric space for two pairs of self-mappings using implicit relation.

Theorem 1.3.30. *[36] Let A,B,S and T be self mappings of a semi-metric space (X, d) such that*
i) the pairs (A, S) and (B,T) satisfies the common property E.A,

ii) SX and TX are closed subsets of X,
iii) for all $x \neq y \in X$ *and* $F \in \Phi$
$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By),$
$d(Ty, Ax)) < 0$
Then, the pair (A, S) and (B, T) have a point of coincidence. Moreover, if the pairs (A, S) and (B, T) are weakly compatible then A, B, S and T have a unique common fixed point in X.

In 2010, A. H. Soliman and M. Imdad [37] established the following common fixed point theorem in semi-metric space for two pairs of self-mappings using S-continuous and T-continuous
.

Theorem 1.3.31. *[37] Let Y be an arbitrary non-empty set whereas X be another non-empty set with semi-metric space* (X, d) *which satisfies W3 and H.E. Let* $A, B, S, T : Y \longrightarrow X$ *be four mappings which satisfies the following conditions*
i) A is S-continuous and B is T-continuous,
ii) the pair (A, S)and (B, T) satisfy the common property E.A,
iii) SX and TX are d-closed subset of X,
then there exists $u, w \in X$ *such that* $Au = Su = Tw = Bw$*. Moreover, if* $Y = X$ *along with*
iv) the pairs (A, S)and (B, T) are weakly compatible and
v) $d(Ax, BAx) \neq max\{d(Sx, TAx), d(BAx, TAx), d(Ax, TAx),$
$d(Ax, Sx), d(BAx, Sx)\}$ *whenever the right hand side is non-zero. Then A, B, S and T have a common fixed point in X.*

In 2011, H. K. Pathak and R. K. Verma [90] established the following common fixed point theorem in semi-metric space for two pairs of self-mappings using occasionally converse commuting.

Theorem 1.3.32. *[90] Let A, B, S and T be self mappings of a semi-metric space (X, d) satisfying*
$\emptyset(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(By, SX),$
$d(Ax, Ty)) \geqslant 0$, *where* $x, y \in X$ *and* $\emptyset \in F_6$. *If one of the follwoing conditions holds*
i) the pairs (A,S) is occasionally converse commuting and the pairs (B,T) s occasionally weakly compatible or
ii) the pairs (B,T) is occasionally converse commuting and the pairs (A,S) is occasionally weakly compatible,
then A,B,S and T have a unique common fixed point in X.

In 2014, M. Imdad, A. Sharma and S. Chauhan [38] established the following theorem in semi-metric space using common limit range property.

Theorem 1.3.33. *[38] Let (X, d) be a semi-metric space wherein d satisfies the conditions (CC) and $(H.E)$ and Y is an arbitrary non-empty set with $A, B, S, T : Y \to X$. Suppose that the inequality $forall$x,y$\in Y$ and $\psi \in \Psi$,*
$\psi(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By),$
$d(Ty, Ax)) \leq 0$ *holds.*
If the pairs (A, S) and (B, T) enjoys the common limit range with respect to mappings S and T (CLR_{ST}) property then (A, S) and (B, T) have a coincidence point each. Moreover, if $Y = X$,

then A, B, S *and* T *have a unique common fixed point provided both the pairs* (A, S) *and* (B, T) *are weakly compatible.*

Chapter 2

Common Fixed Point Theorems in Semi-metric Space using E.A Property

In this chapter, some newly established common fixed point theorems have been presented in semi-metric space using E.A property with associated corollaries and examples.

2.1 Introduction

Polish mathematician S. Banach published his contraction principle in metric space in 1922. Since then, this principle has been extended and generalized in several ways. In 1928, K. Menger [70] introduced the concept of semi-metric space as a generalization of metric space. In 1997, T.L. Hicks and B. E. Rhoades [29] generalized Banach Contraction Principle in semi-metric space. Besides these results, there have been interensting generalized and formulated results in semi-metric space initiated by M. Frechet[19], K. Menger[70] and W. A. Wilson [110]. Also, in 1976, M. Cicchese[17] introduced the notion of a contractive mapping in semi-metric space. Further, fixed point results for the class of spaces are obtained by T. L. Hicks and B. E. Rhoades, M. Aamri and D. El Moutawakil[2], M. Imdad, J. Ali and L. Khan [35].

In 1986, G. Jungck introduced the concept of compatible mappings in metric space. This concept has been frequently used to prove existence theorem in common fixed point theory. However, the study of common fixed point theorems for non-compatible mappings has also become interesting notions. S. H. Cho, G. Y. Lee and J. S. Bae [15] intially proved some common fixed point theorems for non-compatible mappings and gave a notion of property E.A. In 2002, M. Aamri and D. El Moutawakil[2] established some common fixed point theorems for self mappings under a contractive condition. In 2008, S. H. Cho and D. J. Kim [16] generalized the paper of M. Aamri and Moutawakil replacing W4 property by C.C property with different contractive conditions.

In this chapter, some common fixed point theorems have been established using weakly compatible and E.A property which extends the results of S. H. Cho and D. J. Kim [16] and other similar results.

In 1998, G. Jungck and B. E. Rhoades [55] introduced the no-

tion of weakly compatible mappings and showed that compatible mappings are weakly compatible but not conversely. In 2009, M. Imdad and J. Ali[36] introduced the new class of implicit function and proved some common fixed point theorems.The significance of this type of implicit function is to relax the notion of triangle inequality.
The notion of new class of implicit function given by M. Imdad and J. Ali [36] which is different from the one considered in V. Popa[91] with example.

Definition 2.1.1. *[36] Let Φ be the family of lower semi-continuous function as a new class of* **implicit functions** $F : R^6 \to R$ *satisfying the following conditions.*
(F_1) : $F(t, 0, 0, t, t, 0) > 0$, *for all* $t > 0$.
(F_2) : $F(t, 0, t, 0, 0, t) > 0$, *for all* $t > 0$.
(F_3) : $F(t, t, 0, 0, t, t) > 0$, *for all* $t > 0$.

Example 2.1.2. $F(t_1, ...t_6) = t_1 - max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}\}$
(F_1) : $F(t, 0, 0, t, t, 0) = \frac{t}{2} > 0$, *for all* $t > 0$.
(F_2) : $F(t, 0, t, 0, 0, t) = \frac{t}{2} > 0$, *for all* $t > 0$.
(F_3) : $F(t, t, 0, 0, t, t) = 0$, *for all* $t > 0$.

In this chapter, we consider the some definitions of semi-metric space, properties W3, W4, W5, H.E, C.C.,E.A, commuting, coincidence point, common fixed point, compatible, weakly compatible and occasionally weakly compatible as defined in the first chapter.

Definition 2.1.3. *[52] Let A and B be two self-mappings of a semi-metric space (X, d). Then A and B are said to be* **compatible mapping of type (E)***,*
if $lim_{n\to\infty}AAx_n = lim_{n\to\infty}ABx_n = B(t)$
and $lim_{n\to\infty}BBx_n = lim_{n\to\infty}BAx_n = A(t)$*;*
whenever $\{x_n\}$ *is a sequence in X such that*
$lim_{n\to\infty}d(Ax_n, t) = lim_{n\to\infty}d(Bx_n, t) = 0$, *for some* $t \in X$.

We take the following theorem as a source of our fixed point result. In 2002, M. Aamri and D. El Moutawakil[2] established the following common fixed point theorem for two pairs of self mappings in semi-metric space using weakly compatibility and E.A property.

Theorem 2.1.4. *[2] Let* (X, d) *be a semi-metric space that satisfies W3, W4 and HE. Let* A, B, T *and* S *be self mappings of* (X, d) *such that*
i) $\alpha(d(Ax, By)) \leq \emptyset(max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\}$ *for all* $(x, y) \in X \times X$,
ii)(A, S) *and* (B, T) *are weakly compatibles,*
iii)(A, S)*or* (B, T) *satisfies the property E.A, and*
iv) $AX \subset TX$ *and* $BX \subset SX$.
If the range of the one of the mappings A,B,T or S is a complete subspace of X, then A,B,T and S have a unique common fixed point.

2.2 Common Fixed Point Theorems in Semi-metric Space

In 2014, K. Jha, M. Imdad and U. Rajopadhyaya [53] established the following common fixed point theorem for three pairs of mappings using weakly compatible and E.A property, that extends the result of M. Aamri and D. El Moutawakil[2].
This result has been accepted to publish in Application and Applied Mathematics An International Journal.

Theorem 2.2.1. *[53] Let (X, d) be a semi-metric space that satisfies W4 and H.E. Let A, B, T, S, P and Q be self-mappings of X such that*
i) $ABX \subset PX$ and $TSX \subset QX$,
ii) $d(ABx, TSy) \leq \emptyset(max\{d(Qx, Py), d(Qx, TSy), d(Py, TSy)\})$ for all $x, y \in X \times X$,
iii) (AB,Q) or (TS,P) satisfies the property E.A.,and
iv) (AB,Q) and (TS,P) are weakly compatibles
If the range of the one of the mappings AB, TS, P and Q is a complete subspace of X then AB, TS, P and Q have a unique common fixed point. Furthermore, if the pairs (A,B), (A,P), (B,P), (S,T),(S,Q) and (T,Q) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

Proof: Suppose that (TS, P) satisfies the property E.A, then there exists a sequence $\{x_n\}$ in X such that
$lim_{n\to\infty}d(TSx_n, t) = lim_{n\to\infty}d(Px_n, t) = 0$

for some $t \in X$. Hence, by the property H.E, we get $lim_{n\to\infty} d(TSx_n, Px_n)$. Since $TSX \subset QX$ there exists a sequence $\{y_n\}$ in X such that $TSx_n = Qy_n$.

Hence, we get $lim_{n\to\infty} d(Qy_n, t) = 0$.

We prove that $lim_{n\to\infty} d(ABy_n, t) = 0$.

Using condition (ii), we get

$$
\begin{aligned}
d(ABy_n, TSx_n)& \\
&\leq \emptyset(max\{d(Qy_n, Px_n), d(Qy_n, TSx_n), d(Px_n, TSx_n)\}) \\
&=(\emptyset(max\{d(TSx_n, Px_n), d(Qy_n, Qy_n), d(Px_n, TSx_n)\}) \\
&= \emptyset(max\{d(TSx_n, Px_n), 0, d(Px_n, TSx_n)\}) \\
&= \emptyset(d(TSx_n, Px_n)) \\
&< d(TSx_n, Px_n).
\end{aligned}
$$

Letting $n \to \infty$, we have $lim_{n\to\infty} d(ABy_n, TSx_n) = 0$.

Also, by the property W4, we have $lim_{n\to\infty} d(ABy_n, t) = 0$.

Suppose QX is a complete subspace of X.

Then $Qu = t$ for some $u \in X$. Also, we have

$$
\begin{aligned}
&lim_{n\to\infty} d(ABy_n, Qu) = lim_{n\to\infty} d(TSx_n, Qu) \\
&\quad = lim_{n\to\infty} d(Px_n, Qu) = lim_{n\to\infty} d(Qy_n, Qu) = 0.
\end{aligned}
$$

Using condition (ii), it follows that

$d(ABu, TSx_n) \leq \emptyset(max\{d(Qu, Px_n), d(Qu, TSx_n), d(Px_n, TSx_n)\})$.

Letting $n \to \infty$, we have $lim_{n\to\infty} d(ABu, TSx_n) = 0$.

By the property W4, we have $lim_{n\to\infty} d(ABy_n, ABu) = 0$. This implies $lim_{n\to\infty} d(ABu, TSx_n) = 0$.

By the property W4, we have $lim_{n\to\infty} d(ABy_n, ABu) = 0$.

This implies $lim_{n\to\infty} d(t, ABu) = 0$ and hence $lim_{n\to\infty} d(Qu, ABu) = 0$.

Therefore, we have $Qu = ABu$.

The weak compatibility of AB and Q implies that

$ABQu = QABu$, then we have

$ABABu = ABQu = QABu = QQu$.

Again, since $ABX \subset PX$, so there exists $z \in X$, such that $ABu = Pz$.

This implies $ABu = Qu = Pz$. We claim that $Pz = TSz$.

If not, condition (ii) gives,

$$\begin{aligned}
&d(ABu, TSz)\\
&\quad \leq \emptyset(max\{d(Qu, Pz), d(Qu, TSz), d(Pz, TSz)\})\\
&\quad = \emptyset(max\{d(ABu, ABu), d(ABu, TSz), d(ABu, TSz)\})\\
&\quad = \emptyset(d(ABu, TSz))\\
&\quad < d(ABu, TSz), \text{which is contradiction.}
\end{aligned}$$

Therefore, we get $Pz = TSz$. Hence, we get

$ABu = Qu = Pz = TSz$.

The weak compatibility of TS and P imply that

$TSPz = PTSz$ and $PPz = PTSz = TSPz = TSTSz$.

Now, we prove that ABu is a common fixed point of AB, TS, P and Q.

Suppose that $AB(ABu) \neq ABu$. Then, using condition (ii), we have

$$\begin{aligned}
&d(ABu, AB(ABu)\\
&\quad = d(AB(ABu), TSz)\\
&\quad \leq \emptyset(max\{d(QABu, Pz), d(QABu, TSz), d(Pz, TSz)\})
\end{aligned}$$

$= \emptyset(max\{d(ABABu, ABu), d(ABABu, ABu), d(Pz, Pz)\})$
$= \emptyset(d(ABABu, ABu))$
$< d(ABABu, ABu)$, which is contradiction.

Therefore, we get $ABu = AB(ABu) = Q(ABu)$.

Hence, ABu is a common fixed point of AB and Q. Similarly, we can prove that TSz is a common fixed point of TS and P. Since $ABu = TSz$, we conclude that ABu is a common fixed point of AB, TS, P and Q.

The proof is similar when PX is assumed to be a complete subspace of X. The case in which ABX or TSX is a complete subspace of X are similar to the case in which PX or QX respectively is complete since $ABX \subset PX$ and $TSX \subset QX$.

Since ABu is a common fixed point of AB, TS, P and Q. We can write

$AB(ABu) = TS(ABu) = P(ABu) = Q(ABu) = ABu$.

If v is another common fixed point of AB, TS, P and Q, then for $p \in X$ and $v \neq ABu$, we can write

$AB(v) = TS(v) = P(v) = Q(v) = v$.

Therefore, we have

$d(ABu, v)$
$= d(AB(ABu), TSv)$
$\leq \emptyset(max\{d(Q(ABu), Pv), d(Q(ABu), TSv), d(Pv, TSv)\})$
$= \emptyset(max\{d(Q(ABu), Pv), d(Q(ABu), Pv), d(Pv, Pv)\})$
$= \emptyset(d(Q(ABu), Pv))$
$= \emptyset(d(ABu, v))$

$< d(ABu, v)$, which is contradiction.
Therefore, we have $ABu = v$. Hence AB, TS, P and Q have common fixed point.
We need to show that v is only the common fixed point of the family $F = \{A, B, T, S, P, Q\}$ when the pairs $(A, B), (A, P), (B, P)$, $(S, T), (S, Q)$ and (T, Q) are commuting mappings. For this, we can write,
$Av = A(ABv) = A(BAv) = AB(Av)$,
Av = A(Pv) = P(Av),
$Bv = B(ABv) = BA(Bv) = AB(Bv)$, and
$Bv = B(Pv) = P(Bv)$.
This shows that Av and Bv are common fixed point of (AB, P).
This implies that
$Av = v = Bv = Pv = ABv$. Similarly, we have
$Tv = v = Sv = Qv = TSv$.
Thus A, B, T, S, P and Q have a unique common fixed point T.
This completes the proof.

The following example justifies the Theorem (2.2.1)

Example 2.2.2. *[53] Consider $X = [0, 1]$ with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define a self maps A, B, T, S, P and Q as*
$Ax = \frac{3x}{4}$, $Bx = \frac{4x}{5}$, $S(x) = \frac{2x}{5}$, $Tx = \frac{5x}{6}$, $Px = \frac{2x}{3}$ and $Qx = \frac{9x}{10}$. Then d satisfies W4 and H.E for the sequence $x_n = \frac{1}{n}$. The mappings satisfy all the conditions of above Theorem (2.2.1) and hence they have a unique common fixed point $x = 0$. It is noted

that the above theorem holds true if condition (iii) is replaced by (AB, Q) *and* (TS, P) *that satisfies the property E.A.*
In Theorem (2.2.1), *if we take* $A = B$ *and* $T = S$, *then we have the following corollary.*

Corollary 2.2.3. *[53] Let* (X, d) *be a semi-space that satisfies W4 and H.E. Let* A, T, P *and* Q *be self mappings of* X *such that*
i) $AX \subset PX$ *and* $TX \subset QX$,
ii) $d(Ax, Ty) \leq \emptyset(max\{d(Qx, Py), d(Qx, Ty), d(Py, Ty)\})$ *for all* $x, y \in X \times X$,
iii) (A, Q) *or* (T, P) *satisfies the property E.A and*
iv) (A, Q) *and* (T, P) *are weakly compatibles.*
If the range of the one of the mapping A, T, P *and* Q *is complete subspace of* X *then* A, T, P *and* Q *have a unique common fixed point.*

In Theorem (2.2.1), if we take $A = B = P$ and $T = S = Q$, we have the following Corollary.

Corollary 2.2.4. *[53] Let* (X, d) *be a semi-metric space that satisfies W4 and H.E. Let* A *and* T *be self mappings of* X *such that*
i) $AX \subset TX$,
ii) $d(Ax, Ty) \leq \emptyset(max\{d(Tx, Ay), d(Tx, Ty), d(Ay, Ty)\})$ *for all* $x, y \in X \times X$,
iii) A *and* T *satisfy the property E.A, and*
iv) A *and* T *are weakly compatibles.*
If the range of the one of the mapping A, T, P *and* Q *is complete*

subspace of X then A and T have a unique common fixed point. These corollaries (2.2.3) *and* (2.2.4) *are the result of M.Aamri and D.El Moutawakil [2].*

Now, a common fixed point theorem has been proved for six mappings using weakly compatible and the property E.A , that extends the result of M. Aamri and D. El Moutawakil[2] by using the property C.C and H.E only under different contraction. The result has been published in International Journal of Mathematical Sciences and Engineering Applications **1**(8)(2014), 139-146.
We denote Λ by the class of non-decreasing continuous function $\alpha : R^+ \to R^+$ such that (α_1) $\alpha(0) = 0$ and (α_2) $\alpha(S) > 0$ for all $S > 0$.

Theorem 2.2.5. *[51] Let (X, d) be a semi-metric space that satisfies H.E. and C.C. Let A, B, T, S, P and Q be self-mappings of X such that*
i)$ABX \subset QX$ and $SX \subset PX$,
ii) $\alpha(d(ABx, TSy))$
$$\leq \emptyset(\alpha(max\{d(Px, Qy), d(ABx, Px), d(TSy, Qy), d(ABx, Qy), d(TSy, Px)\})) \text{ for all } x, y \in X \times X,$$
iii) the pairs (TS,Q) satisfies the property E.A.,
iv) (AB, P) and (TS, Q) are weakly compatibles, and
v) PX is d-closed subset of X.
Then AB, TS, P and Q have a unique common fixed point. Furthermore, if the pairs (A,B), (A,P), (B,P), (S,T), (S,Q) and

(T,Q) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

Proof: Suppose that (TS, Q) satisfies the property E.A, then there exists a sequence $\{x_n\}$ in X such that
$lim_{n\to\infty}\ d(TSx_n, t) = lim_{n\to\infty}\ d(Qx_n, t) = 0$ for some $t \in X$.
Since $TSX \subset PX$, there exists a sequence $\{y_n\}$ in X such that $Sx_n = Py_n$. Hence, we get $lim_{n\to\infty}\ d(Py_n, t) = 0$.
Hence, using the property H.E we get
$lim_{n\to\infty} d(TSx_n, Qx_n) = lim_{n\to\infty} d(Py_n, Qx_n) = 0$.
Since PX is d-closed subset of X, there exists a point $u \in X$ such that $Pu = t$.
We show that $ABu = Pu$. Using condition (ii), it follows that

$$\alpha(d(ABu, TSx_n)) \leq \emptyset(\alpha(max\{d(Pu, Qx_n), d(ABu, Pu), d(TSx_n, Qx_n), d(ABu, Qx_n), d(TSx_n, Pu)\})).$$

In the above inequality, we take $n \to \infty$ and using the properties C.C. and H.E, we have

$$\alpha(d(ABu, Pu)) \leq (\alpha(max\{0, d(ABu, Pu), 0, d(ABu, Pu, 0)\})) = \emptyset(\alpha(d(ABu, Pu)))$$

which implies
$\alpha(d(ABu, Pu)) = 0$. By (α_1), we have $d(ABu, Pu) = 0$. Hence, we get $ABu = Pu$. Since $ABX \subset QX$, so there exists $w \in X$ such that $ABu = Qw$. Thus, we get $ABu = Pu = Qw$. Also, we show that $Qw = TSw$. From condition (ii), we have

$\alpha(d(Qw, TSw))$

$= \alpha(d(ABu, TSw))$

$\leq \emptyset(\alpha(max\{d(Pu, Qw), d(ABu, Pu), d(TSw, Qw), d(ABu, Qw), d(TSw, Pu)\}))$

$= \emptyset(\alpha(max\{d(Qw, Qw), d(ABu, ABu), d(TSw, Qw), d(Qw, Qw), d(TSw, Qw)\}))$

$= \emptyset(\alpha(max\{0, 0, d(TSw, Qw), 0, d(TSw, Qu)\}))$

$= \emptyset(\alpha(d(TSw, Qw)))$.

This implies $\alpha(d(TSw, Qw)) = 0$. Using (α_1), we get $d(TSw, Qw) = 0$, we have $TSw = Qw$. Therefore, we have $z = ABu = Pu = Qw = TSw$ ...(2.1)

From condition (iv), (AB, P) are weakly compatible and we have $ABABu = ABPu = PABu = PPu$ ...(2.2)

Similarly, Since (TS, Q) are weakly compatible, so we have

$TSTSw = TSQw = QTSw = QQw$ ...(2.3)

Now we claim that $AB = z$.

Using relations $(2.1), (2, 2)$ and condition (ii), we get

$\alpha(d(z, ABz))$

$= \alpha(d(ABu, ABABu))$

$= \alpha(d(ABABu, TSw))$

$\leq \emptyset(\alpha(max\{d(PABu, Qw), d(ABABu, PABu), d(TSw, Qw), d(ABABu, Qw), d(TSw, PABu)\}))$

$= \emptyset(\alpha(max\{d(ABABu, Qw), d(ABABu, ABABu), d(TSw, TSw), d(ABABu, Qw), d(ABABu, Qw)\}))$

$= \emptyset(\alpha(max\{d(ABABu, Qw), 0, 0, d(ABABu, Qw),$

$d(ABABu, Qw)\}))$

$= \emptyset(\alpha(d(ABABu, Qw))$

$= \emptyset(\alpha(d(ABz, z))$

which implies $\alpha(d(ABz, z) = 0$. So, by (α_1), we have $d(ABz, z) = 0$, and hence $ABz = z$.

From (2.1) and (2.2), we have $ABz = Pz = z$.

Again, we show that $TSz = z$. Using condition (ii), we get

$\alpha(d(z, TSz))$

$= \alpha(d(TSw, TSTSw))$

$= \alpha(d(ABu, TSTSw))$

$\leq \emptyset(\alpha(max\{d(Pu, QTSw), d(ABu, Pu), d(TSw, Qw),$

$d(TSTSw, QTSw), d(ABu, QTSw), d(TSTSw, Pu)\}))$

$= \emptyset(\alpha(max\{d(Pu, TSTSw), d(TSw, Pu), d(TSTSw, TSTSw),$

$d(TSw, TSTSw), d(TSTSw, TSw)\}))$

$= \emptyset(\alpha(d(TSTSw, TSw))$

$= \emptyset(\alpha(d(TSz, z))$

which implies that $\alpha(d(TSz, z)) = 0$.

By (α_1), we have $d(TSz, z) = 0$.

Therefore, we have $ABz = Pz = Qz = TSz = z$.

Therefore, z is a common fixed point of AB, TS, P and Q.

For uniqueness, let v be another fixed point, $v \neq z$. Then, using condition (ii), we get

$\alpha(d(z, v)$

$= \alpha(d(ABz, TSv))$

$\leq \emptyset(\alpha(max\{d(Pz, Qv), d(ABz, Pz), d(TSv, Qv), d(ABz, Qv),$

$d(TSv, Pz)\}))$.

$= \emptyset(\alpha(max\{d(z, v), d(z, z), d(v, v), d(z, v), d(v, z)\}))$

$= \emptyset(\alpha(d(z, v)))$

which implies that $\alpha(d(z, v)) = 0$. By (α_1), we get $d(z, v) = 0$. This implies that $z = v$. Hence AB, TS, P and Q have a unique common fixed point. Finally, we need to show that z is only the common fixed point of the family $F = \{A, B, T, S, P, Q\}$ when the pairs $(A, B), (A, P), (B, P), (S, T), (S, Q)$ and (T, Q) are commuting mappings. For this, we can write

$Az = A(ABz) = A(BAz) = AB(Az)$, $Az = A(Pz) = P(Az)$,

$Bz = B(ABz) = BA(Bz) = AB(Bz)$ and

$Bz = B(Pz) = P(Bz)$.

This shows that A, B, T, S, P and Q have a unique common fixed point. This completes the proof.

We have given the following example to justify the Theorem (2.2.5)

Example 2.2.6. *[51] Consider* $X = [0, 1]$ *with the semi-metric space* (X, d) *defined by* $d(x, y) = (x - y)^2$. *Define a self map* A, B, T, S, P *and* Q *as* $Ax = \frac{3x}{8}$, $Bx = \frac{4x}{10}$, $Sx = \frac{x}{5}$, $Tx = \frac{5x}{12}$, $Px = \frac{9x}{20}$ *and* $Qx = \frac{x}{3}$. *Then,* d *satisfies C.C and H.E properties for the sequences* $x_n = \frac{1}{n}$ *and* $y_n = \frac{1}{n} + 1$. *Also, the mappings satisfy all the conditions of above Theorem* (2.2.5) *and hence have a unique common fixed point* $x = 0$.

It is noted that the above theorem holds true if condition (iii) *is replaced by* (TS, Q) *and* (AB, P) *satisfies the property E.A. In the Theorem* (2.2.5)*, If we take* $A = B$ *and* $T = S$*, then we have the following corollary.*

Corollary 2.2.7. *[51] Let* (X, d) *be a semi-metrics space that satisfies the properties H.E. and C.C. Let A,T,P and Q be self-mappings of X such that*
i)$AX \subset QX$ *and* $TX \subset PX$*,*
ii) $\alpha(d(Ax, Ty)) \leq \emptyset(\alpha(max\{d(Px, Qy), d(Ax, Px), d(Ty, Qy),$
$d(Ax, Qy)$*,d(Ty,Px)*$\}$*))*$forallx$*,y*$\in X \times X$*,*
iii) the pairs (T,Q) satisfies the property E.A,
iv) (A,P) and (T,Q) are weakly compatibles and
v) PX is d-closed subset of X.
Then A, T, P and Q have a unique common fixed point.

This is the result of S. H. Cho and D. j. Kim [16].
In Theorem (2.2.5), if we take $A = B = Q$ and $T = S = P$ then we have the following corollary.

Corollary 2.2.8. *[51] Let* (X, d) *be a semi-metrics space that satisfies the properties H.E. and C.C. Let A, and T be self-mappings of X such that*
i)$AX \subset TX$ *,*
ii) $\alpha(d(Ax, Ty)) \leq \emptyset(\alpha(max\{d(Tx, Ay), d(Ax, Tx), d(Ty, Ay),$
$d(Ax, Ay), d(Ty, Tx)\}))$ *for all* $x, y \in X \times X$*,*
iii) the pairs (T,A) satisfies the property E.A,
iv) A and T are weakly compatibles and

v) TX is d-closed subset of X.
Then A and T have a unique common fixed point.

In 2008, M. Imdad and J. Ali [36] established the following common fixed point theorem for two pair of mappings in semi-metric space using a new implicit function and E.A property.

Theorem 2.2.9. *[36] Let A, B, S and T be self mappings of a semi-metric space (X, d) which satisfy the inequality*
$F(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty),$
$\quad d(Sx, By), d(Ty, Ax)) < 0.$
Suppose that
i) the pairs (A, S) and (B, T) share the common property E.A,
ii) $S(X)$ and $T(X)$ are closed subsets of X.
Then the pair (A, S) as well as (B, T) has a point of coincidence. Moreover,if the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

In 2015, U. Rajopadhyaya, K. Jha and R. P. Pant[97] established the following common fixed point theorem for three pairs of mappings using weakly compatible and E.A property that extends the result of M.Imdad and J.Ali [36].
This result has been published in Electronic Journal of Mathematical Analysis and Applications, **1**(3)(2015), 19 - 23.

Theorem 2.2.10. *[97] Let A, B, T, S, P and Q be self mappings of semi-metric space (X, d) which satisfy the inequality*
(i) $F((d(ABx, TSy), d(Px, Qy), d(TSy, Qy), d(Px, TSy),$

$d(Qy, ABx)) < 0$.

(ii) $AB(X) \subset QX$ *or* $TS(X) \subset P(X)$

(iii) the pairs (AB, P) *and* (TS, Q) *satisfy the property E.A, and*

(iv) the pairs (AB, P) *and* (TS, Q) *are weakly compatible mappings.*

Then AB, TS, P *and* Q *have a unique fixed point. Furthermore, if the pairs* (A, B) *and* (T, S) *are commuting pair of mappings then* A, B, T, S, P *and* Q *have a unique common fixed point.*

Proof: If (AB, P) satisfy the property E.A then there exists a sequence $\{x_n\}$ in X such that

$lim_{n\to\infty}\ ABx_n = lim_{n\to\infty}\ Px_n = t$ for some $t \in X$.

Since $AB(X) \subset Q(X)$, hence for each $\{x_n\}$ there exists $\{y_n\}$ in X such that $ABx_n = Qy_n$.

Letting $n \to \infty$, we have

$lim_{n\to\infty} Qy_n = lim_{n\to\infty} ABx_n = t$.

Therefore, we have

$lim_{n\to\infty} ABx_n = lim_{n\to\infty} Qy_n = lim_{n\to\infty} = t$.

We assert that $lim_{n\to\infty} TSy_n = t$.

If not, we put $x = x_n$ and $y = y_n$ in inequality (i), we get

$$F(d(ABx_n, TSy_n), d(Px_n, Qy_n), d(ABx_n, Px_n), d(TSy_n, Qy_n),$$
$$d(Px_n, TSy_n), d(Qy_n, ABx_n)) < 0.$$

Letting $n \to \infty$, we have

$$F(d(t, TSy_n), 0, 0, d(TSy_n, t), d(t, TSy_n), 0) < 0.$$

This is a contradiction. Therefore, we get $lim_{n\to\infty}TSy_n = t$.

Hence, we have

$lim_{n\to\infty}ABx_n = lim_{n\to\infty}Qy_n = lim_{n\to\infty}Px_n = lim_{n\to\infty}TSy_n = t$.

Since $P(X)$ is closed subset of X, hence

$lim_{n\to\infty}Px_n = t \in P(X)$. Therefore, there exists $u \in X$ such that $Pu = t$.

We assume that $Pu = ABu$. If not, we put $x = u$ and $y = y_n$ in inequality (i), we get

$$F(d(ABu, TSy_n), d(Pu, Qy_n), d(ABu, Pu), d(TSy_n, Qy_n), \\ d(Pu, TSy_n), d(Qy_n, ABu)) < 0.$$

Letting $n \to \infty$, we have

$$F(d(ABu, t), d(Pu, t), d(ABu, Pu), 0, 0, d(t, ABu)) < 0.$$

This is a contradiction. Therefore, we get $ABu = Pu$. Hence, u is a coincidence point of (AB, P).

Also, we have $Q(X)$ is closed subset of X.

Therefore, we get $lim_{n\to\infty}Qy_n = t \in Q(X)$.

Hence, we have $Qw = t$ for some $w \in X$. Suppose $Qw = TSw$. If not we put $x = x_n$ and $y = w$ in inequality (i), we get

$$F(d(ABx_n, TSw), d(Px_n, Qw), d(ABx_n, Px_n), d(TSw, Qw), \\ d(Px_n, TSw), d(Qw, ABx_n)) < 0.$$

Letting $n \to \infty$, we have

$$F(d(t, TSw), d(t, Qw), d(t, t), d(TSw, Qw), d(t, TSw), \\ d(Qw, t)) < 0.$$

$$F(d(Qw, TSw), 0, 0, d(TSw, Qw), d(Qw, TSw), d(Qw, t)) < 0.$$

This is a contradiction. Therefore, we get $Qw = TSw$. Hence w is a coincidence point of (TS, Q).
Since, (AB, P) and (TS, Q) are weakly compatible mappings, we write

$ABt = ABPu = PABu = Pt$ and
$TSt = TSQw = QTSw = Qt.$

we consider $ABt = t$. If not we put $x = t$ and $y = y_n$ in inequality (i), we get

$$F(d(ABt, TSy_n), d(Pt, Qy_n), d(ABt, Pt), d(TSy_n, Qy_n),$$
$$d(Pt, TSy_n), d(Qy_n, ABt)) < 0.$$

Letting $n \to \infty$, we have

$$F(d(ABt, t), d(ABt, t), 0, 0, d(AB, t), d(t, ABt)) < 0.$$

This is a contradiction. Therefore, we get, $AB = t$. Therefore t is a common fixed point of AB and P.
Similarly, we can show that t is a common fixed point of TS and Q. Hence t is a common fixed point of AB, TS,P and Q.
Therefore, we get $ABt = TSt = Pt = Qt = t$.

For Uniqueness, let z be another fixed point of AB, TS, P and Q. Then by definition

we get $ABz = TSz = Pz = Qz = z$.
If we put $x = t$ and $y = z$ in inequality (i), we get

$$F(d(ABt, TSz), d(Pt, Qz), d(ABt, Pt), d(TSz, Qz),$$
$$d(Pt, TSz), d(Qz, ABt)) < 0.$$

$F(d(t,z), d(t,z), d(t,t), d(z,z), d(t,z), d(z,t)) < 0$

or $F(d(t,z), d(t,z), 0, 0, d(t,z), d(z,t)) < 0.$

This is a contradiction. Therefore, we get $t = z$.
Hence, we have AB, TS, P and Q have unique common fixed point. If the pairs (A, B) and (T, S) are commuting pair of mappings then

$A(t) = A(ABt) = A(BAt) = AB(At)$. This implies $At = t$.

$B(t) = B(ABt) = BA(Bt) = AB(Bt)$. This implies $Bt = t$.

Similarly, we get $T(t) = t$ and $S(t) = t$.

This shows that A, B, T, S, P and Q have a unique common fixed point. This completes the proof. The proof assuming that $TS(X)$ is subset of $P(X)$ is similar to the above proof.

We have given the following example to justify the Theorem (2.2.10)

Example 2.2.11. *[97] Consider $X = R$ with the semi-metric space (X, d) defined by $d(x, y) = (x - y)^2$. Define a self map A, B, T, S, P and Q as $Ax = \frac{x+2}{3}$, $Bx = 4 - 3x$, $S(x) = 3 - 2x$, $T(x) = \frac{x+1}{2}$, $P(x) = 2 - x$ and $Q(x) = 2x - 1$.*

For the sequence $x_n = \frac{1}{n}$ and $y_n = \frac{1}{n} + 1$, the mappings satisfy all the conditions of above Theorem (2.2.10) and hence they have a unique common fixed point $x = 1$.

In the Theorem (2.2.10), if B and S are identity mapping, then we have the following corollary.

Corollary 2.2.12. *[97] Let A, T, P and Q be self-mappings of (X, d) which satisfy the inequality*

(i)$F((d(Ax, Ty), d(Px, Qy), d(Ax, Px), d(Ty, Qy),$
$d(Px, Ty), d(Qy, Ax)) < 0$.

(ii) $A(X) \subset Q(X)$ or $T(X) \subset P(X)$

(iii) $P(X)$ and $Q(X)$ are closed subset of X, and

(iv) the pairs (A, P) and (T, Q) are weakly compatible mappings

Then A,T,P and Q have a unique common fixed point.

This is the result of M. Imdad and J. Ali [2].

Among various type of compatible mappings, M. R. Singh and Y. M. Singh [108] introduced the concept of compatible mappings of type (E) in 2007. We prove a common fixed point theorem for two pairs of self mappings using compatible mapping of type (E) in semi-metric space that extends the results of M. Aamri and D. El Moutawakil [2] and other similar results in semi-metric space.

Proposition 2.2.13. *[52] Let A and B be two compatible mappings of type(E). If one of the function is continuous, then*

i) $A(t) = B(t)$ and

$lim_{n\to\infty} AAx_n = lim_{n\to\infty} BBx_n = lim_{n\to\infty} ABx_n = lim_{n\to\infty} BAx_n$,

where $lim_{n\to\infty} Ax_n = t$ and $lim_{n\to\infty} Bx_n = t$.

ii) If there exists $u \in X$ such that $Au = Bu = t$,

then $ABu = BAu$.

In order to establish our result, we need a function $\emptyset : R^+ \to R^+$ *satisfying* $0 < \emptyset(t) < t$, $t > 0$.

In 2014, U. Rajopadhyaya, K. Jha and M. Imdad [94] established the following common fixed point theorem for two pairs of mappings using compatible mappings of type (E).
This result has been published in Bulletin of Mathematical Sciences and Applications, **10**(2014), 141 - 147.

Theorem 2.2.14. *[94] Let (X,d) be a semi-metric space that satisfies W4 and H.E. Let A,B,T and S be self mappings of X such that*
i) $AX \subset TX$ *and* $BX \subset SX$,
ii) $d(Ax, By) \leq \emptyset(max\{d(Sx, Ty), d(Sx, By), d(Ty, By)\})$ *for all* $(x, y) \in X \times X$,
iii) The pair (B,T) or (A,S) satisfies E.A property,
iv) The pair (B,T) and (A,S) are compatible mapping of type (E), and
v) SX or TX is a d-closed subset of X.
If one of the mapping A,B,T and S is continuous then A,B,T and S have a unique common fixed point.

Proof: Since (B, T) satisfies the property E.A then there exists a sequence $\{x_n\}$ in X Such that
$lim_{n\to\infty}d(Bx_n, t) = lim_{n\to\infty}d(Tx_n, t) = 0$, for some $t \in X$.
Since $BX \subset SX$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$ and hence $lim_{n\to\infty}d(Sy_n, t) = 0$. By the property H.E, we get

$lim_{n\to\infty}d(Bx_n, Tx_n) = lim_{n\to\infty}d(Sy_n, Tx_n) = 0.$

If SX is a d-closed subset of X there exists a point $u \in X$ such that $Su = t$. Also, we have

$lim_{n\to\infty}d(Bx_n, Su) = lim_{n\to\infty}d(Tx_n, Su) = d(Sy_n, Su) = 0.$

Now, using condition (ii), we get

$d(Au, Bx_n) \leq \emptyset(max\{d(Su, Tx_n), d(Su, Bx_n), d(Tx_n, Bx_n))$

Letting $n \to \infty$, we have $lim_{n\to\infty}d(Au, Bx_n) = 0.$

By the property W4, we get

$lim_{n\to\infty}d(Tx_n, Au) = 0.$ This implies $Au = Su.$

Since $AX \subset TX$, there exists a point $w \in X$ such that $Au = Tw$. This implies $Au = Su = Tw$.

We claim that $Bw = Tw$. Using condition (ii), we get

$$\begin{aligned} d(Au, Bw) &\leq \emptyset(max\{d(Su, Tw), d(su, Bw), d(Tw, Bw)\}) \\ &\leq \emptyset(max\{d(Su, Tw), d(Su, Bw), d(Tw, Bw)\}) \\ &\leq \emptyset(max\{0, d(Tw, Bw), d(Tw, Bw)\}), \end{aligned}$$

that is, $d(Au, Bw) < d(Au, Bw)$ which is contradiction. Therefore, we have $Au = Bw$.

Hence, we have $Au = Su = Tw = Bw$.

Suppose S is continuous and (A, S) is compatible mapping of type(E), then by the proposition (2.2.13), we have $Au = Su$ and then $ASu = SAu$.

Now, we have $AAu = ASu = SAu = SSu$.

We claim that Au is a common fixed point of A and S, if $AAu \neq Au$.

Using condition (ii), we get

$$
\begin{aligned}
d(Au, AAu) &= d(AAu, Bw) \\
&\leq \emptyset(max\{d(SAu, Tw), d(SAu, Bw), d(Tw, Bw)\}) \\
&\leq \emptyset(max\{d(AAu, Au), d(AAu, Au), 0\}),
\end{aligned}
$$

that is $d(Au, AAu) < d(Au, AAu)$, which is contradiction.

Therefore, we have $AAu = Au$ which implies $AAu = SAu = Au$. Hence Au is common fixed point of A and S. By the proposition (2.2.13), we have $BTw = TBw$.

Thus, we have $TTw = TBw = BTw = BBw$.

Similarly, we can show that Bw is common fixed point of B and T.

Since $Au = Bw$, Au is a common fixed point of A, B, T and S. If z is another common fixed point of A, B, T and S, $z \neq Au$ such that $Az = Bz = Tz = Sz = z$.

Using condition (ii), we get

$$
\begin{aligned}
d(Au, z) &= d(AAu, Bz) \\
&\leq \emptyset(max\{d(SAu, Tz), d(SAu, Bz), d(TAu, Bz)\}) \\
&\leq \emptyset(max\{d(Au, z), d(Au, z), d(Au, z)\}), \text{ that is,}
\end{aligned}
$$

$d(Au, z) < d(Au, z)$,

which is contradiction. So, we get $Au = z$. Hence A, B, T and S have a unique common fixed point.

We have given the following example to justify the Theorem (2.2.14)

Example 2.2.15. *[94] Let X=(0,5] with semi-metric space (X,d), define d on X by* $d(x, y) = (x - y)^2$*.Define self-mappings* A, B, T *and* S *as*

$Ax = 1$ for all x.
$Bx = 1$ if $x \leq 2$ and $x = 3$, $Bx = 1 + x$ if $2 < x < 3$
$Sx = x$ if $x < 4$, $Sx = 4$ if $x > 4$
$Tx = 1$ if $x \leq 2$, $Tx = x - 1$ if $2 < x \leq 5$.
Then A, B, T and S satisfy all the conditions of the above theorem and have a unique common fixed point at $x = 1$.

In Theorem (2.2.14), if we take $A = B$ and $T = S$ we have the following corollary.

Corollary 2.2.16. *[94] Let (X,d) be a semi-metric space that satisfies W4 and H.E. Let A and T be self mappings of X, such that*
i) $AX \subset TX$
ii) $d(Ax, Ay) \leq \emptyset(max\{d(Tx, Ty), d(Tx, Ay), d(Ty, Ay)\})$ for all $(x, y) \in X \times X$,
iii) The pair (A,T) satisfies E.A property,
iv) The pair (A,T) are compatible mapping of type (E), and
v) TX is a d-closed subset of X.
If one of the mapping A or T is continuous then A and T have a unique common fixed point.

Proof: It is the direct consequence of proof of Theorem 2.2.14

Chapter 3

Some Common Fixed Point Theorems in Semi-metric Space Using Weakly Commuting Mappings

This chapter deals with the introduction of occasionally weakly compatible mapping and occasionally converse commuting mappings in semi-metric space and have established some common fixed point theorems in semi-metric space using weakly commuting mappings.

3.1 Introduction

In 1998, G. Jungck and B. E. Rhoades [55] introduced the notion of weakly compatible mappings and showed that the compatible mappings are weakly compatible but not conversely. Recently in 2006, G. Jungck and B. E. Rhoades [56] introduced occasionally weakly compatible mappings which is more general among the commutativity concepts. G. Jungck and B. E. Rhoades [56] obtained several common fixed point theorems using the idea of occasionally weakly compatible mappings. Several interesting and elegant results have been obtained by various authors in this direction.

In 2002, the concept of converse commuting mappings was introduced by Z. Lu [67] as a reverse process of weakly compatible mappings.In 2011 H. K. Pathak and R.K.Verma [90] introduced the concept of occasionally converse commuting(shortly occ) mappings in semi-metric space as a reverse process of occasionally weakly compatible mappings in semi-metric space.

Now, we start with the following definitions, lemmas and theorems.

Definition 3.1.1. *[56] Let A and B be two self mappings of a semi-metric space (X,d).Then A and B are said to be* **occasionally weakly compatible** *(owc) if there is a point $x \in X$ which is coincidence point of A and B at which A and B commute.*

It is important to note that weakly compatible mappings are occasionally weakly compatible mappings but not the converse. This is clear from the example (1.3.14)

Definition 3.1.2. *[90] Let A and B be two self-mappings of a semi-metric space (X, d). Then A and B are said to be* **converse commuting** *it for all $x \in X$ $ABx = BAx$ implies $Bx = Ax$.*

Definition 3.1.3. *[90] Let A and B be two self-mappings of a semi-metric space (X, d). Then A and B are said to be* **occasionally converse commuting** *(occ) if for some $x \in X$*
$ABx = BAx$ implies Ax=Bx.

The coincidence point implies commutative in occasionally weakly compatible (owc) however commutative point implies coincidence point in occasionally converse commuting (occ). We have given the follwing example to verify the above fact.

Example 3.1.4. *[95] Consider $X = R$ the set of real numbers, with the semi-metric space (X, d) defined by*
$d(x, y) = (x - y)^2$. Define self mappings A and B,
$Ax = 2x$ if $x < 1$, $\quad Bx = x^2 + 1$ if $x < 1$ and
$Ax = 2x - 1$ if $x \geq 1$, $\quad Bx = x^2$ if $x \geq 1$.
If $x = 1$, $AB(x)$=$2x^2 - 1$ and $\quad BA(x) = (2x - 1)^2$ then
$AB(1) = BA(1) = 1$ implies $A(1) = B(1) = 1$.
If x= $\frac{1}{\sqrt{2}}$, $\quad AB(x) = 2(x^2 + 1)$ and $BA(x) = 4x^2 + 1$ then
$AB(\frac{1}{\sqrt{2}}) = BA(\frac{1}{\sqrt{2}}) = 3$ but $A(\frac{2}{\sqrt{2}}) = \frac{2}{\sqrt{2}}$ and $B(\frac{1}{\sqrt{2}}) = \frac{3}{2}$,
therefore $A(\frac{1}{\sqrt{2}}) \neq B(\frac{1}{\sqrt{2}})$.
This example shows that converse commuting self mappings A and B are occasionally converse commuting but not conversely. Also this example verifies that occasionally converse commuting self mappings A and B are reverse process of occasionally weakly compatible.

Lemma 3.1.5. *[56] Let (X,d) be a semi-metric space. If the self mappings A and B on X have a unique point of coincidence*

$w = Ax = Bx$, then w is the unique common fixed point of A and B.

In order to establish the following theorem, we consider a function $\emptyset : R^+ \to R^+$ satisfying $(\emptyset_1)$ $0 < \emptyset(t)$, for $t > 0$ and $(\emptyset_2)$ for each $t > 0$ $lim_{n\to\infty}\emptyset^n(t) = 0$.

3.2 Common Fixed Point Theorems in Semi-metric Space Using Occasionally Weakly Compatible Mappings

In 2014, K. Jha, M. Imdad and U. Rajopadhyaya [49] established the following common fixed point theorem in semi-metric space for three pair of mappings using occasionally weakly compatible mapping.

This result has been published in the Annals of Pure and Applied Mathematics, **2**(5) (2014), 153-157.

Theorem 3.2.1. *[49] Let (X,d) be a semi-metric space. Let A, B, T, S, P and Q be self mappings of X such that*

(i) $\{AB, P\}$ and $\{TS, Q\}$ are occasionally weakly compatible (owc),

(ii) $$d(ABx, TSy) \le \emptyset(max\{d(Px, Qy), \tfrac{1}{2}[d(ABx, Px) + d(TSy, Qy)], \tfrac{1}{2}[d(ABx, Qy) + d(TSy, Px)]\}) \text{ for all } (x, y) \in X \times X,$$

Then AB, TS, P and Q have a unique common fixed point. Furthermore, if the pairs (A,B) and (T,S) are commuting pair of *mappings then A, B, T, S, P and Q have a unique common fixed point.*

Proof: *Since $\{AB, P\}$ and $\{TS, Q\}$ are owc, then there exists $x, y \in X$ such that $ABx = Px$ and $TSy = Qy$. We claim that $ABx = TSy$. Using condition (ii), we get*

$$\begin{aligned}
d(ABx, TSy) & \\
&\leq \emptyset(max\{d(Px, Qy), \tfrac{1}{2}[d(ABx, Px) + d(TSy, Qy)], \\
&\quad \tfrac{1}{2}[d(ABx, Qy) + d(TSy, Px)]\}) \\
&= \emptyset(max\{d(ABx, TSy), \tfrac{1}{2}[d(ABx, ABx) + d(TSy, TSy)], \\
&\quad \tfrac{1}{2}[d(ABx, TSy) + d(TSy, ABx)]\}) \\
&= \emptyset(max\{d(ABx, TSy), 0, d(ABx, TSy)\}) \\
&= \emptyset(max\{d(ABx, TSy)\}) \\
&= \emptyset(d(ABx, TSy)) \\
&< d(ABx, TSy)
\end{aligned}$$

which is contradiction. So, we have $ABx = TSy$.

Therefore $ABx = Px = TSy = Qy$ ...(3.1)

Moreover, if there is another point of coincidence z such that $ABz = Pz$, then, using condition (ii), we get

$ABz = Pz = TSy = Qy$ ...(3.2)

Also, from (3.1) and (3.2), it follows that $ABz = ABx$. This implies that $z = x$. Hence, we get $w = ABx = Px$, for $w \in X$, is the unique point of coincidence of AB and P.

By lemma (3.1.5), w is the unique common fixed point of AB and P. Hence $ABw = Pw = w$. Similarly, there is a unique common fixed point $u \in X$ such that $u = TSu = Qu$. Suppose *that $u \neq w$. Then using condition (ii), we get,*

$$
\begin{aligned}
d(w,u) &= d(ABw, TSu)\\
&\leq \emptyset(max\{d(Pw,Qu), \tfrac{1}{2}[d(ABw,Pw)+d(TSu,Qu)],\\
&\quad \tfrac{1}{2}[d(ABw,Qu)+d(TSu,Pw)]\})\\
&= \emptyset(max\{d(w,u), \tfrac{1}{2}[d(w,w)+d(u,u)], \tfrac{1}{2}[d(w,u)+d(u,w)]\})\\
&= \emptyset(max\{d(w,u), 0, d(w,u)\}\\
&= \emptyset(d(w,u))\\
&< d(w,u).
\end{aligned}
$$

This is a contradiction. Therefore, we have $w = u$. Hence w is the unique common fixed point of AB, TS, P and Q. Finally, we need to show that w is the only common fixed point of mappings A, B, T, S, P and Q. If the pairs (A, B) and (T, S) are commuting pairs then for this, we can write

$Aw = A(ABw) = A(BAw) = AB(Aw)$.

This implies that $Aw = w$.

Also, $Bw = B(ABw) = BA(Bw) = AB(Bw)$. This implies that $Bw = w$. Similarly, we have $Tw = w$ and $Sw = w$. Hence A, B, T, S, P and Q have a unique common fixed point.

We have given the following example to justify the above theorem (3.2.1)

Example 3.2.2. *[49] Consider $X = [0,1]$ with the semi-metric space (X, d) defined by $d(x,y) = (x-y)^2$. Define self mappings A, B, T, S, P and Q as $Ax = \frac{x+1}{2}$, $Bx = \frac{2+3x}{5}$, $Tx = \frac{2x+1}{3}$, $Sx = \frac{x+3}{4}$, $P(x) = \frac{3x+1}{4}$ and $Qx = \frac{2x+3}{5}$. Also, the mappings satisfy all the conditions of above Theorem* (3.2.1) *and*

hence have a unique common fixed point at $x = 1$.
On the basis of the Theorem (3.2.1), *we have the following corollary.*

Corollary 3.2.3. *[49] Let (X,d) be a semi-metric space. Let* A, T, P *and* Q *be self mappings of* X *such that*
(i) $\{A, P\}$ *and* $\{T, Q\}$ *are occasionally weakly compatible (owc),*
(ii) $d(Ax, Ty) \leq \emptyset(max\{d(Px, Qy), \frac{1}{2}[d(Ax, Px) + d(Ty, Qy)],$
$\frac{1}{2}[d(Ax, Qy) + d(Ty, Px)]\})$ *for all* $(x, y) \in X \times X$,
Then A, T, P *and* Q *have a unique common fixed point.*
In the Theorem (3.2.1), *if we take* $A = B = Q$ *and* $T = S = P$, *then we have the following corollary.*

Corollary 3.2.4. *[49] Let (X,d) be a semi-metric space. Let* A *and* T *be self mappings of* X *such that*
(i) A *and* T *are occasionally weakly compatible (owc),*
(ii) $d(Ax, Ty) \leq \emptyset(max\{d(Tx, Ay), \frac{1}{2}[d(Ax, Tx) + d(Ty, Ay)],$
$\frac{1}{2}[d(Ax, Ay) + d(Ty, Tx)]\})$ *for all* $(x, y) \in X \times X$,
Then A *and* T *have a unique common fixed point*

Our result generalizes the results of Jungck and Rhoades [56], Manro [69], Pant and Chauhan [86] and other similar results in the semi-metric spaces.

3.3 Common Fixed Point Theorems in Semi-metric Space Using Occasionally Converse Commuting Mappings

In 2014, U. Rajopadhyaya, K. Jha and Y. J. Cho [95] established the following common fixed point theorem in semi-metric space for three pair of mappings using occasionally converse commuting mappings.
This result has been published in the International Journal of Mathmatical Analysis, **13** (8),2014, 627-634.

Theorem 3.3.1. *[95] Let (X,d) be a semi-metric space. Let A, B, T, S, P and Q be self mappings of X such that*
(i) $\{AB, P\}$ and $\{TS, Q\}$ are occasionally converse commuting (occ),
(ii) $d(ABx, TSy)$

$$\leq \emptyset(max\{d(Px, Qy), \tfrac{1}{2}[d(ABx, Px) + d(TSy, Qy)], \tfrac{1}{2}[d(ABx, Qy) + d(TSy, Px)]\}) \text{ for all } (x, y) \in X \times X,$$

Then AB, TS, P and Q have a unique common fixed point. Furthermore, if the pairs (A,B) and (T,S) are commuting pair of mappings then A, B, T, S, P and Q have a unique common fixed point.

Proof: *Suppose $\{TS, Q\}$ is occasionally converse commuting, then there exists any $v \in X$ such that $TSQv = QTSv$ implies $TSv = Qv$ =z (say). So that for a given v, we have $TSz = Qz$ whenever $TSv = Qv = z$.*

Also, if $\{AB, P\}$ is occasionally converse commuting then there exists any $u \in X$ such that $ABPu = PABu$ implies $ABu = Pu = w$ (say). So that for a given u, we have ABw=Pw whenever $ABu = Pu = w$.

We claim that $ABABu = TSz$. If not, then putting $x = ABu$ and $y = z$ in condition (ii), we get

$$
\begin{aligned}
&d(ABABu, TSz)\\
&\quad \leq \emptyset(max\{d(PABu, Qz), \tfrac{1}{2}[d(ABABu, PABu) + d(TSz, Qz)],\\
&\qquad \tfrac{1}{2}[d(ABABu, Qz) + d(TSz, PABu)]\})\\
&\quad = \emptyset(max\{d(PABu, TSz), \tfrac{1}{2}[d(ABABu, ABPu) + d(Qz, Qz)],\\
&\qquad \tfrac{1}{2}[d(ABABu, TSz) + d(TSz, ABABu)]\})\\
&\quad = \emptyset(max\{d(ABABu, TSz), 0, d(ABABu, TSz)\})\\
&\quad = \emptyset(d(ABABu, TSz))\\
&\quad < d(ABABu, TSz).
\end{aligned}
$$

This is a contradiction. So, we get $ABABu = TSz$. Therefore, we have $ABw = Pw = TSz = Qz$.

We claim that $ABu = TSz$. If not, putting $x = u$ and $y = z$ in condition (ii), we get

$$
\begin{aligned}
&d(ABu, TSz)\\
&\quad \leq \emptyset(max\{d(Pu, Qz), \tfrac{1}{2}[d(ABu, Pu) + d(TSz, Qz)],\\
&\qquad \tfrac{1}{2}[d(ABu, Qz) + d(TSz, Pu)]\})\\
&\quad = \emptyset(max\{d(ABu, Qz), \tfrac{1}{2}[d(Pu, Pu) + d(Qz, Qz)],\\
&\qquad \tfrac{1}{2}[d(ABu, TSz) + d(TSz, ABu)]\})\\
&\quad = \emptyset(max\{d(ABu, Qz), 0, d(ABu, TSx)\})\\
&\quad = \emptyset(d(ABu, TSz))
\end{aligned}
$$

$$< d(ABu, TSz).$$

This is a contradiction. Therefore, we get $ABu = TSz$.

Hence, we have

$ABu = Tz = Qz = Pu = ABw = ABPu = PABu = ABABu.$

It follows that ABu *is a common fixed point of* AB *and* P.

We claim that $TSz = z$. *If not, then putting* $x = u$ *and* $y = v$ *in condition (ii), we get*

$$\begin{aligned}
d(TSz, z) &= d(ABu, TSv) \\
&\leq \emptyset(max\{d(Pu, Qv), \tfrac{1}{2}[d(ABu, Pu) + d(TSv, Qv)], \\
&\quad \tfrac{1}{2}[d(ABu, Qv) + d(TSv, Pu)]\}) \\
&= \emptyset(max\{d(ABu, Qv), \tfrac{1}{2}[d(Pu, Pu) + d(Qv, Qv)], \\
&\quad \tfrac{1}{2}[d(ABu, TSv) + d(TSv, ABu)]\}) \\
&= \emptyset(max\{d(ABu, z), 0, d(ABu, z) \\
&= \emptyset(d(ABu, z)) \\
&< d(TSz, z).
\end{aligned}$$

This is a contradiction. Therefore, we get $TSz = z$. *Thus we have* $z = ABu = TSz = Qz = Pu$. *It follows that* z *is a common fixed point of* TS *and* Q. *Since* $z = ABu$, z *is common fixed point of* AB, TS, P *and* Q. *For uniqueness, let* z_0 *be another common fixed point of* AB, TS, P *and* Q. *Then by putting* $x = z$ *and* $y = z_0$ *in condition (ii), we get*

$$\begin{aligned}
d(z, z_0) &= d(ABz, TSz_0) \\
&\leq \emptyset(max\{d(Pz, Qz_0), \tfrac{1}{2}[d(ABz, Pz) + d(TSz_0, Qz_0)], \\
&\quad \tfrac{1}{2}[d(ABz, Qz_0) + d(TSz_o, Pz)]\}) \\
&= \emptyset(max\{d(z, z_0), \tfrac{1}{2}[d(z, z) + d(z_0, z_0)], \tfrac{1}{2}[d(z, z_0) + d(z_0, z_0)]\})
\end{aligned}$$

$= \emptyset(max\{d(z, z_0), 0, d(z, z_0)$
$= \emptyset(d(z, z_0))$
$< d(z, z_0).$

This is a contradiction. Therefore, we get $z = z_0$. *Thus* AB, TS, P *and* Q *have unique common fixed point. Finally, we need to show that* z *is only the common fixed point of mappings* A, B, T, S, P *and* Q. *Suppose the pairs* (A, B) *and* (T, S) *are commuting pair. For this, we can write*

$Az = A(ABz) = A(BAz) = AB(Az)$. *This implies* $Az = z$. *Also, we have* $Bz = B(ABz) = BA(Bz) = AB(Bz)$.

This implies that $Bz = z$.

Similarly, we get $Tz = z$ *and* $Sz = z$.

Hence A, B, T, S, P *and* Q *have a unique common fixed point.*

We have given the following example to verify the Theorem (3.3.1)

Example 3.3.2. *[95] Consider* $X = [0, 1]$ *with the semi-metric space* (X, d) *defined by* $d(x, y) = (x - y)^2$. *Define self mappings* A, B, T, S, P *and* Q *as* $Ax = \frac{x+1}{2}$, $Bx = \frac{2+3x}{5}$, $Tx = \frac{2x+1}{3}$, $Sx = \frac{x+3}{4}$, $P(x) = \frac{3x+1}{4}$ *and* $Qx = \frac{2x+3}{5}$. *Then, the mappings satisfy all the conditions of above Theorem* (3.3.1) *and hence have a unique common fixed point at* $x = 1$.

Corollary 3.3.3. *[95] Let (X,d) be a semi-metric space. Let* A, B, T, S, P *and* Q *be self mappings of* X *such that*

(i) $\{AB, P\}$ *and* $\{TS, Q\}$ *are occasionally converse commuting (occ),*

(ii) $d(ABx, TSy) \leq \emptyset(max\{d(Px, Qy), d(ABx, Qy), d(TSy, Px),$
$\frac{1}{2}[d(ABx, Px) + d(TSy, Qy)]\})$
for all $(x, y) \in X \times X$,
Then AB, TS, P *and* Q *have a unique common fixed point. Furthermore, if the pairs (A,B) and (T,S) are commuting pair of mappings then* A, B, T, S, P *and* Q *have a unique common fixed point.*

In the above Theorem (3.3.1), if we take $A = B$ and $T = S$, then we have the follwoing corollary. This is the result of H. K. Pathak and R. K. Verma [90] and T. K. Samanta and S.Mohinta [101].

Corollary 3.3.4. *[95] Let (X,d) be a semi-metric space. Let* A, T, P *and* Q *be self mappings of* X *such that*
(i) $\{A, P\}$ *and* $\{T, Q\}$ *are occasionally converse commuting (occ),*
(ii) $d(Ax, Ty) \leq \emptyset(max\{d(Px, Qy), d(Ax, Qy), d(Ty, Px),$
$\frac{1}{2}[d(Ax, Px) + d(Ty, Qy)]\})$ *for all* $(x, y) \in X \times X$,
Then A, T, P *and* Q *have a unique common fixed point.*

In the above Theorem (3.3.1), if we take $A = B = Q$ and $T = S = P$, then we have the following corollary.

Corollary 3.3.5. *[95] Let (X,d) be a semi-metric space. Let* A *and* T *be self mappings of* X *such that*
(i) A *and* T *are occasionally converse commuting (occ),*
(ii) $d(Ax, Ty) \leq \emptyset(max\{d(Tx, Ay), d(Ax, Ay), d(Ty, Tx),$

$\frac{1}{2}[d(Ax, Tx) + d(Ty, Ay)]\})$ *for all* $(x, y) \in X \times X$,
Then A *and* T *have a unique common fixed point.*

Our result generalizes the results of H. K. Pathak and R. K. Verma [90], T. K. Samanta and S. Mohinta [101] and Q. Liu and X. Hu [65], V. Popa [91] and improves other similar results in the semi-metric space.

Chapter 4

Common Fixed Point Result in Fuzzy Semi-metric Space for Weakly compatible Mappings

In this chapter, fuzzy semi-metric space and some associated properties related to fuzzy semi-metric space have been introduced. Also, a common fixed point theorem in fuzzy semi-metric space has been established using weakly compatible mappings.

4.1 Introduction

The concept of fuzzy set was introduced by Iranian-American Engineer A. L. Zadeh [111] in 1965 as a new way to represent vagueness in our everyday life. Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. But in real life situation, the problem in economics, engineering, environment, social science, medical science etc does not always involve crisp data.

Consequently, the last three decades were very productive for fuzzy mathematics and the recent literature has observed the fuzzification in almost every direction of mathematics such as arithmetic, topology, graph theory, probability theory, logic etc. Fuzzy set theory has applications in applied sciences such as neural network theory, stability theory, mathematical programming, modeling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication. No wonder that fuzzy fixed point theory has become an area of interest for specialists in fixed point theory, with new possibilities for fixed point results.

O. Kramosil and J. Michalek [63] introduced the concept of fuzzy metric space (briefly FM Space) in 1975, which opened an avenue for further development of analysis in such spaces. In 1994, A. George and P. Veeramani [25] modified the notion of fuzzy metric spaces with the help of continuous t-norms and have generalized several fixed point theorems. S. N. Mishra, N. Sharma and S. L. Singh[73] in 1994 introduced the notion of compatible mappings under the name of asymptotically commuting maps in FM space. B. Singh and S. Jain[104] introduced the concept of weak compatibility in fuzzy metric space in 2005. V. Pant and R. P. Pant [82] in 2007 introduced the notion non-compatible maps in fuzzy metric space. In 2012, T. K. Samanta et. al.[100] introduced the notion of fuzzy semi-metric space and established the common fixed point theorem using various contractive conditions.

Definition 4.1.1. *[100]* $*: [0,1]\times[0,1] \to [0,1]$, *a binary operation is called a* **continuous t − norm** *if* $*$ *satisfies the following conditions*
i) $*$ *is commutative and associative.*
ii) $*$ *is continuous.*
iii) $a * 1 = a$ *for all* $a \in [0,1]$, *and*
iv) $a * b \leq c * d$ *whenever* $a \leq c$, $b \leq d$ *and* $a, b, c, d \in [0,1]$.

Example 4.1.2. $a * b = min(a, b)$ *is a continuous* $t - norm$ *for all* $a, b \in [0,1]$, *where* $* : [0,1] \times [0,1] \to [0,1]$ *is a binary operation.*

Definition 4.1.3. *[25] The 3-tuple* $(X, M, *)$ *is called* **fuzzy metric space** *in* X *is an arbitrary non-empty set,* $*$ *is continuous t-norm and* M *is a fuzzy set in* $X^2 \times (0, \infty)$ *satisfying the following conditions:*
i) $M(x, y, t) > 0$
ii) $M(x, y, t) = 1$ *if and only if* $x = y$
iii) $M(x, y, t) = M(y, x, t)$
iv) $M(x, y, s) * M(y, z, t) \leq M(x, z, s + t)$,*and*
v) $M(x, y, \cdot) : (0, \infty) \to (0, 1]$ *is continuous, for all* $x, y, z \in X$ *and* $t, s > 0$.

Definition 4.1.4. *[102] The (X, M) is called a* **fuzzy semi-metric space** *if $X^2 \times (0, \infty)$ satisfying the following conditions*
i) $M(x, y, t) > 0$
ii) $M(x, y, t) = 1$ *if and only if* $x = y$, *and*
iii) $M(x, y, t) = M(y, x, t)$.

Remark 4.1.5. *[102] Every fuzzy metric space is fuzzy semi-metric space but the converse is not necessarily true.*

Example 4.1.6. *[100] Consider* $X = (0, \infty)$ *and*
$M(x, y, t) = \frac{t}{t+|x-y|}$ *if* $x \neq 0, y \neq 0$. *and* $M(x, y, t) = \frac{t}{t+\frac{1}{x}}$ *if* $x \neq 0$. *Then,* (X, M) *is fuzzy semi-metric space. Also for* $x = 1$, $y = \frac{1}{2}$, $z = 0$, $s = 1$, $t = 0$ *and* $a * b = max\{a, b\}$.
Then condition (iv) of definition 4.1.3 is not satisfied. Hence (X, M) *is not a fuzzy metric space.*

We have following useful conditions W3, W4, H.E, introduced by Samanta and Mohinta [102] to establish fixed point results in fuzzy semi-metric space replacing triangle inequality.
Let (X, M) be a fuzzy semi-metric space. Then, for sequences $\{x_n\}$ and $\{y_n\}$, we have
(**W3**) For a sequence $\{x_n\}$ and $x, y \in X$, the relations
$lim_{n\to\infty} M(x_n, x, t) = 1$ and $lim_{n\to\infty} M(x_n, y, t) = 1$ imply $x = y$.
(**W4**) For sequences $\{x_n\}$, $\{y_n\}$ and $x, y \in X$, the relations
$lim_{n\to\infty} M(x_n, x, t) = 1$ and $lim_{n\to\infty} M(x_n, y_n, t) = 1$ imply
$M(y_n, x, t) = 1$.
(**H.E**) For sequences $\{x_n\}$, $\{y_n\}$ and $x, y \in X$, the relations

$lim_{n\to\infty} M(x_n, x, t) = 1$ and $lim_{n\to\infty} M(x_n, y_n, t) = 1$ imply $M(x_n, y_n, t) = 1$.

Proposition 4.1.7. *[102] For axioms in fuzzy semi-metric space (X, M), W4 implies W3. But the converse is not true.*

Definition 4.1.8. *[102] Let A and B be two self mappings of a fuzzy semi-metric space (X, M). Then a pair of self mappings A and B satisfy the* **property E.A** *if there exists a sequence $\{x_n\}$ in X such that $lim_{n\to\infty} M(Ax_n, r, t) = lim_{n\to\infty} M(Bx_n, r, t) = 1$.*

Definition 4.1.9. *[102] Let A and B be two self mappings of a fuzzy semi-metric space (X, M). Then, self mappings A and B are said to be* **weakly compatible** *if they commute at their coincidence points, that is $Az = Bz$ implies that $ABz = BAz$.*

We denote Φ by the class of continuous function
$\phi : [0, 1] \to [0, 1]$ satisfying :
(ϕ_1) $\phi(r) > r$ for all $r \in [0, 1)$, and
(ϕ_2) $\phi(1) = 1$.

4.2 Common Fixed Point Theorem in Fuzzy Semi-metric Space

In 2012, T. K. Samanta et.al.[100] established the following common fixed point theorem for two pairs of self mappings in fuzzy semi-metri space using weakly compatibility and E.A property.

Theorem 4.2.1. *[100] Let (X, μ) be a fuzzy semi-metric space that satisfies $(W3)$ and $(H.E)$. Let A, B, S and T be self-mappings of X such that*
i) $AX \subset TX$ and $BX \subset SX$,
ii) the pairs (B, T) and (A, S) satisfy property $(E.A)$,
iii) the pairs (A, S) and (B, T) are weakly compatible,
iv) for any $x, y \in X$ $(x \neq y)$, $\mu(Ax, By, t) > v(x, y, t)$, where
$$v(x,y,t) = min\{\mu(Sx,Ty,t), max\{\mu(Ax,Sx,t), \mu(By,Ty,t)\}, max\{\mu(Ax,Ty,t), \mu(By,Sx,t)\}\}, \text{ and}$$
v) SX is a $\mu-$closed subset of X (resp.TX is a $\mu-$closed subset of X.'
Then A, B, S and T have a unique common fixed point in X.

In 2014, U. Rajopadhyaya, K. Jha and P. Kumam [96] established the common fixed point theorem in fuzzy semi-metric space for three pair of mappings using weakly compatible mappings.
This result has been published in Journal of Mathematics and System Science,**(4)**(2014), 720 - 724.

Theorem 4.2.2. *[96] Let (X, M) be a fuzzy semi-metric space that satisfies (W4) and (H.E). Let A,B,T,S,P and Q be self-mappings of X such that*
i) $ABX \subset PX$ and $TSX \subset QX$,
$$ii)\ M(ABx, TSy, t) \geq (\phi(min\{M(Qx, Py, t), M(Qx, TSy, t), M(Py, TSy, t)\})) \text{ for all } (x, y) \in X \times X,$$
iii) (AB, Q) or (TS, P) satisfies the property E.A, and

iv) (AB, Q) *and* (TS, P) *are weakly compatible.*
If the range of of the one of the mappings AB, TS,P and Q is complete subspace of X then AB, TS,P and Q have a unique common fixed point. Furthermore, if the pairs (A,B),(A,P),(B,P),(S,T),(S,J) and (T,Q) are commuting pair of mappings then A,B,T,S,P and Q have a unique common fixed point.

Proof: Suppose that (TS,P) satisfies the property E.A, then there exists a sequence $\{x_n\}$ in X such that
$M(TSx_n, r, t) = lim_{n\to\infty} M(Px_n, r, t) = 1.$ for some $r \in X$.
Hence, by the property H.E, we get $M(TSx_n, Px_n, t) = 1$. Since $TSX \subset QX$ there exists a sequence $\{y_n\}$ in X such that
$TSx_n = Qy_n$. Hence, we get $M(ABy_n, r, t) = 1$.
Using condition (ii), we get

$$\begin{aligned}
M(ABy_n, TSx_n, t) &\geq \phi(min\{M(Qy_n, Px_n, t), M(Qy_n, TSx_n, t), t\}) \\
&= \phi(min\{M(TSx_n, Px_n, t), M(Qy_n, Qy_n, t), M(Px_n, TSx_n, t)\}) \\
&= \phi(min\{M(TSx_n, Px_n, t), 1, M(Px_n, TSx_n, t)\}) \\
&= \phi(M(TSx_n, Px_n, t)) \\
&< M(TSx_n, Px_n, t)
\end{aligned}$$

Letting $n \to \infty$, we have $M(ABy_n, TSx_n, t) = 1$.
By the property (W4),we have $M(ABy_n, r, t) = 1$.
Suppose QX is a complete subspace of X. Then we have $Qu = r$ for some $u \in X$. Also, we have
$M(ABy_n, Qu, t) = M(TSx_n, Qu, t) = M(Px_n, Qu, t)$
$= M(Qy_n, Qu, t) = 1$.
Using condition (ii), it follows that

$$M(ABu, TSx_n, t) \geq \phi\,(min\{M(Qu, Px_n, t), M(Qu, TSx_n, t), t),$$
$$M(Px_n, TSx_n, t)\})$$

Leting $n \to \infty$, we have $M(ABu, TSx_n = 1$.

Also, by the property W4, we have $Qu = ABu$. The weak compatiblity of AB and Q implies that $ABQu = QABu$. Then, we have $ABABu = ABQu = QABu = QQu$.

Again, since $ABX \subset PX$, so there exists $z \in X$ such that $ABu = Pz$. This implies $ABu = Qu = Pz$. We claim that $ABu = TSz$. If not, the condition (ii) gives

$$\begin{aligned} M(ABu, TSz, t) & \\ &\geq \phi(min\{M(Qu, Pz, t), M(Qu, TSz, t), M(Pz, TSz, t)\}) \\ &= \phi(min\{M(ABu, ABu, t), M(ABu, TSz, t), M(ABu, TSz, t)\}) \\ &= \phi(min\{1, M(ABu, TSz, t), M(ABu, TSz, t)\}) \\ &= \phi(M(ABu, TSz, t)) \\ &> M(ABu, TSz, t), \end{aligned}$$

which is a contradiction.

Hence, we get $ABu = Qu = Pz = TSz$.

The weak compatibility of TS and P imply that $TSPz = PTSz$ and $PPz = PTSz = TSPz = TSTSz$.

Now, we prove that ABu is a common fixed point of AB,TS,P and Q. Suppose that $AB(ABu) \neq ABu$. Then, using condition (ii), we get

$$\begin{aligned} M(ABu, AB(ABu), t) &= M(AB(ABu), TSz, t) \\ &\geq \phi(min\{M(QABu, Pz, t), M(QABu, TSz, t), M(Pz, TSz, t)\}) \\ &= \phi(min\{M(ABABu, ABu, t), M(ABABu, ABu, t), M(Pz, Pz, t)\}) \end{aligned}$$

$$= \phi(min\{M(ABABu, ABu, t), M(ABABu, ABu, t), 1\})$$
$$= \phi(M(ABABu, ABu, t))$$
$$> M(ABABu, ABu, t),$$

which is a contradiction.

Therefore, we get $ABu = AB(ABu) = Q(ABu)$.

Hence, ABu is a common fixed point of AB and Q. Similarly, we can prove that TSz is a common fixed point of TS and P. Since $ABu = TSz$, we conclude that ABu is a common fixed point of AB, TS, P and Q.

The proof is similar when PX is assumed to be a complete subspace of X. The case in which ABX or TSX is a complete subspace of X are similar to the case in which PX or QX respectively is complete since $ABX \subset PX$ and $TSX \subset QX$.

Since ABu is a common fixed point of AB,TS,P and Q, we can write $AB(ABu) = TS(ABu) = P(ABu) = Q(ABu) = ABu$.

If v is another common fixed point of AB,TS,P and Q, then for $p \in X$ and $v \neq ABu$, we can write

$AB(v) = TS(v) = P(v) = Q(v) = v$.

Using condition (ii), we get,

$$M(ABu, v, t) = M(AB(ABu), TSv, t)$$
$$\geq \phi(min\{M(Q(ABu), Pv, t), M(Q(ABu), TSv, t), M(Pv, TSv, t)\})$$
$$= \phi(min\{M(Q(ABu), Pv, t), M(Q(ABu), Pv, t), M(Pv, Pv, t)\})$$
$$= \phi(min\{M(Q(ABu), Pv, t), M(Q(ABu), Pv, t), 1\})$$
$$= \phi(M(Q(ABu), Pv, t))$$
$$= \phi(M(ABu, v, t)$$

$> M(ABu, v, t),$

which is a contradiction. Therefore, we have $ABu = v$.

Hence AB, TS, P and Q have unique common fixed point. We need to show that v is only the common fixed point of the family $F = \{A, B, T, S, P, Q\}$ when the pairs (A,B),(A,P),(B,P),(S,T),(S,Q)and (T,Q) are commuting mappings. For this, we can write,

$Av = A(ABv) = A(BAv) = AB(Av)$, $Av = A(Pv) = P(Av)$,

and

$Bv = B(ABv) = BA(Bv) = AB(Bv)$, $Bv = B(Pv) = P(Bv)$.

This shows that Av and Bv are common fixed point of (AB,P).

This implies that $Av = v = Bv = Pv = ABv$.

Similarly, we have $Tv = v = Sv = Qv = TSv$.

Thus, A, B, T, S, P and Q have a unique common fixed point T. This completes the proof.

We have given the following example to verify the Theorem (4.2.2)

Example 4.2.3. *[96] Consider* $X = [0, 1]$ *with the fuzzy semi-metric space* (X, M) *defined by* $M(x, y, t) = \frac{t}{t+|x-y|}$ *satisfying properties W4 and H.E. Also, define self-mappings A, B, T, S, P and Q as* $Ax = \frac{3x}{4}$, $Bx = \frac{4x}{5}$, $Sx = \frac{2x}{5}$, $Tx = \frac{5x}{6}$, $Px = \frac{2x}{3}$ *and* $Qx = \frac{9x}{10}$. *Then for the sequence* $x_n = \frac{1}{n}$ *and* $y_n = \frac{1}{n} + 1$, *the mappings satisfy all the conditions of above Theorem* (4.2.2) *and hence they have a unique common fixed point* $x = 0$.

Corollary 4.2.4. *[96] Let* (X, M) *be a fuzzy semi-metric space that satisfies (W4) and (H.E). Let A,B,T,S,P and Q be* *self-mappings of X such that*

i) $ABX \subset PX$ *and* $TSX \subset QX$,

ii) $M(ABx, TSy, t) \leq (\phi(max\{M(Qx, Py, t), M(Qx, TSy, t),$

$$M(Py, TSy, t)\})$$ *for all* $(x, y) \in X \times X$,

iii) (AB, Q) *or* (TS, P) *satisfies the property E.A, and*

iv) (AB, Q) *and* (TS, P) *are weakly compatible.*

If the range of of the one of the mappings AB, TS,P and Q is complete subspace of X then AB, TS, P and Q have a unique common fixed point. Furthermore if the pairs (A,B),(A,P),(B,P),(S,T),(S,J) and (T,Q) are commuting pair of mappings then A, B, T,S, P and Q have a unique common fixed point.

Proof: Its proof is direct consequence of proof of Theorem 4.2.2.

Remark 4.2.5. *This result extends the result of Samanta et al[102], Jha et al.[53] and other similar results in the semi-metric space.*

4.3 Conclusion with Research Scope

In 1928, K. Menger[70] introduced the notion of semi-metric space as a generalization of metric space. In metric space, if the triangle inequality property is eliminated then the metric space reduces into semi-metric space. However triangle inequality property is very much important for convergence criteria to obtain fixed point. In semi-metric space, without using triangle inequality property, the establishment of fixed point results is a challenging task. So we use associated useful properties to establish fixed point theorems as partial replacement of triangle inequality.The properties W3, W4 and W5 were introduced by W. A. Wilson[110] in 1931, H.E. by M. Aamri and D.El. Moutawakil[2] in 2003, W by D. Mihet[72] in 2005 and C.C by S. H. Cho, G. y. Lee and J. S. Bae[15] in 2008.

Some fixed point theorems in semi-metric space have been developed for three pairs of mappings using weakly compatible mappings and E.A property as an extension of the result of M.Aamri and D.El. Moutawakil[2]. Also, the fixed point theorems in semi-metric space has been obtained for three pairs of mappings using occasionally weakly compatible mappings and occasionally converse commuting mappings as an extension of our results. Since fuzzy metric space is also another important generalization of metric space and has wide application in various fields, our results have been extended in fuzzy semi-metric space for three pairs of mappings using weakly compatible mappings.

Future aspects of fixed point theorems in semi-metric space are as follows:

1. Fixed point theorems in semi-metric space is an open wide area of research activities for the establishment of fixed point theorems using various compatible and contraction mappings.

2. To study common fixed point theorems in semi-metric space for sequence of mappings.

3. To obtain a connection of fixed point theorem in semi-metric space with economic theory.

4. To find applications of obtained classical fixed points results in different fields.

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