SOME FIXED POINT RESULTS IN FUZZY METRIC SPACE

THESIS

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BY

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UNDER THE SUPERVISION OF

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2015

Dedicated

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To

My Beloved Parents(late)

Mrs. Mangal Maya Manandhar

And

Mr. Aaita Lal Manandhar

Scholar's Declaration of Originality

I **K.B. Manandhar**, here by declare that, the research work **Some Fixed Point Results in Fuzzy Metric Space** submitted here for the fulfillment of Doctor of Philosophy **(Ph.D.)** degree in mathematics to the Department of Natural Sciences, School of Science, Kathmandu University, Nepal in April 2015, is a genuine work done originally by me and has not been published or submitted elsewhere for the requirement of a degree program. Any literature, data or works done by others and cited within this thesis has been given due acknowledgment and listed in the reference section.

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CERTIFICATION

This is to certify that the thesis entitled **Some Fixed Point Results in Fuzzy Metric Space** which is being submitted by Mr. K.B. Manandhar in the fulfillment for the award of Doctor of Philosophy (**Ph.D.**) degree in mathematics of Kathmandu University, Nepal is a record of his own work carried out by him under my guidance and supervision.

The matter embodied in this thesis has not been submitted for the award of any degree.

....

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Certificate of Approval

This is to certify that the thesis entitled **Some Fixed Point Results in Fuzzy Metric Space** is submitted by ***Kanchha Bhai Manandhar*** in the fulfillment for the award of the degree of **Doctor of Philosophy (Ph.D.)** in **Mathematics** to the Department of Natural Sciences, School of Science, Kathmandu University on November, 2015.

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Preface

Analysis is the most important branch of mathematics. Among several branches of analysis, functional analysis is the most important part of analysis. Functional analysis is divided into two parts: linear and non-linear. Fixed point theory is an important part of non-linear functional analysis since 1960. Fixed point theory is one of the most dynamic areas of research from last 60 years, with lots of applications in various fields of pure and applied mathematics, as well as, in physical, economic and life sciences. It is a fully developed branch but still continues to be an active and very wide open area of research. It has emerged as one of the major links between abstracts mathematics and its applications.
It provides a powerful tool in demonstrating the existence of solutions to a large variety of problems in applied mathematics. It is used mainly in the existence theorem of differential equations and integral equations. It is also used in artificial intelligence, computer science, decision making, medical diagnosis, neural network, social science and many other related areas. It has very fruitful application in Eigen value problems and boundary value problems. The fixed point theory deals with the classical approach to find the exact solution and to check the stability of the system.

Historically, in 1883-1884, French mathematician H. Poincare announced the first fixed point theorem without proof which is now known to be the Brouwer's fixed point theorem. The Brouwer's fixed point theorem was one of the early achievements. Perhaps the most widely applied fixed-point theorem is due to Polish mathematician Stefen Banach in 1922 and it is known as Banach's contraction principle.
In 1965, the concept of fuzzy set was introduced by L.A Zadeh [200]. Then, fuzzy metric spaces have been introduced by O. Kramosil and J. Michalek

[102] in 1975. George and Veeramani [49] modified the notion of fuzzy metric spaces with the help of continuous t-norms and also many others have been introduced and generalized in different ways. Recently, many authors have studied the fixed point theory in the fuzzy metric space and number of fixed point theorems have been obtained in fuzzy metric space by using the notion of compatibility, weak compatibility, semi compatibility of self maps or by using its generalized, contractive definitions.

In this thesis, we have established some common fixed point theorems in metric space and fuzzy metric space which generalizes and improves existing similar results in the literature.

Chapter wise cameo description of the present study is as follows.

CHAPTER ONE deals with the general introduction of fixed point theory. It defines some fundamental concepts and notations relevant to the development of fixed point theory. A brief survey of the development of the fixed point theory in metric space and fuzzy metric space has been presented and some of the well-known theorems have been stated. Also, it has included some types of compatible mappings, generalized form of fuzzy metric space and some applications.

CHAPTER TWO is intended to study the fixed point theorems in different compatible mappings in metric space . It includes basic definitions and some fixed point theorems. We have obtained two common fixed point theorems in metric space using reciprocal continuous, compatible mappings of type (E). Also, we have introduced a new compatible mappings of type (K) and obtained a common fixed point theorem.

CHAPTER THREE is intended to obtain some common fixed point theorems in Fuzzy metric spaces using compatible mappings of type (E) and compatible mappings of type (K) which generalizes and improves other similar results in the literature. It includes basic definitions and those theorems specially having the relevance for the establishment of our theorems.

CHAPTER FOUR is intended to obtain some common fixed point theorems in Intutionistic Fuzzy metric spaces using compatible mappings of type (K) which generalizes and improves other similar results in the literature . It includes conclusion and some future scope.

The list of literature consulted has been placed at the end of the thesis as **Bibliography**.

Our original contributions has been contained in chapters 2, 3 and 4. A part of the research work contained in this thesis has been already published in international peer reviewed journal [79], [82],[111], [112], [113],

LIST OF PUBLICATIONS

Papers in Peer Reviewed Journals:

1. A common fixed point theorem for reciprocal continuous compatible mapping in metric space, *Annals of Pure and Applied Mathematics* , **5** (2) (2014), 120 -124.

2. A common fixed point theorem for compatible mapping of type (K) in metric space, *International. Journal of Mathematical Sciences and Engineering Applications*, **8**(2014), 383 - 391.

3. A common fixed point theorem for compatible mapping of type (E) in fuzzy metric space, *Applied Mathematical Sciences*, **8** (2014), 2007 - 2014.

4. Common fixed point theorem of compatible mappings of type (K) in fuzzy metric spaces, *Electronic Journal of Mathematical Analysis and Applications*, **2**(2) (2014), 248-253.

5. Common fixed point theorem in intuitionistic fuzzy metric space using compatible mappings of type (K), *Bulletin of Society for Mathematical Services and Standards*, **3**(2) (2014), 81-87.

6. A Common Fixed Point Theorem for Compatible Mappings of Type (K) in Intuitionistic Fuzzy Metric space, Journal Mathematical System and Science (2015) (to appear)

7. A common fixed point theorem for compatible mapping of type (E) in metric space *Applications and Applied Mathematics: An International Journal* **5** (2015) 474-479.

Papers in Proceedings of Conferences:

8. Fuzzy metric space and some fixed point results, Proceedings of National Conference of Mathematics (NCM 2012): A publication of Nepal Mathematical Society (NMS), 2012, pp.77-84.

9. Some Generalized forms of fuzzy metric space, Proceedings of National Conference of Mathematics (NCM 2014): A Publication of Nepal Mathematical Society (NMS), 2014, 48- 51.

Contents

Chapter 1

Introduction

In this chapter, we give a brief development of fixed point theory in metric and fuzzy metric space, generalized form of fuzzy metric spaces, different types of compatible mappings with some relevant fundamental concepts and applications.

1.1 Introduction

Fixed point theory is an important part of non-linear functional analysis since 1960. It is one of the most dynamic areas of research activities since last six decades, with lots of applications in various fields of pure and applied mathematics, as well as, in physical, economic and life sciences. Also, it deals with the classical approach to find the exact solution and to check the stability of the system.

Historically, in 1883-1884, French mathematician H. Poincare announced the first fixed point theorem without proof which is now known to be the Brouwer's fixed point theorem. Perhaps, the most widely applied fixed-point theorem is due to Polish mathematician Stefen Banach in 1922 and it is known as Banach's contraction principle. Fixed point theory has played vital role in the problems of non-linear functional analysis which is the blend of analysis, topology and algebra.A fixed point theorem is one which ensures the existence of a fixed point of a mapping $T : X \to X$ under suitable assumptions both on a set X and T. In fact, fixed point theorems have wide applications in non linear integral, differential equations, game theory, optimization theory and boundary value problems.

Fixed Point Theory has been classified into three major areas:

(a) Topological Fixed Point Theory

(b) Metric Fixed Point Theory

(c) Discrete Fixed Point Theory

Historically, the boundary lines between the three areas was defined by the discovery of following three major theorems:

(a) Brouwer's Fixed Point Theorem

(b) Banach's Fixed Point Theorem

(c) Tarski's Fixed Point Theorem

Apart from establishing the existence of a fixed point for self mappings, it often becomes necessary to prove the uniqueness of the fixed point. Besides, from computational point of view, an algorithm for calculating the value of the fixed point to a given degree of accuracy is desirable which involves the iterates of the given function. In essence, the question about the existence, uniqueness and approximation of fixed point provide three significant aspect of the general fixed point principle.

Banach's contraction principle which is well known to the student of mathematical analysis is perhaps one of the few most significant theorems. Not only is its proof elementary but it also answers all the three questions of existence, uniqueness and constructive algorithm convincingly. A deeper, though especial result is Brouwer's fixed point theorem. It states that any continuous function mapping a closed ball $B\overline{(a,r)}$ of $\mathbb{R}^n$ in to itself has a fixed point. In general Brouwer's fixed point theorem ensures neither the uniqueness of the fixed point nor the convergence of the iterates. While, the early proofs of Brouwer's theorem rely on algebraic-topological ideas based upon analytical arguments. A brief survey of the development of Brouwer's fixed point theorem has been presented in the paper of Jha and Panthi [80].

The well known Existence Theorem states that, If a set valued continuous function f defined on a closed interval $[a, b]$ assumes values of different signs at end points of the interval then the equation

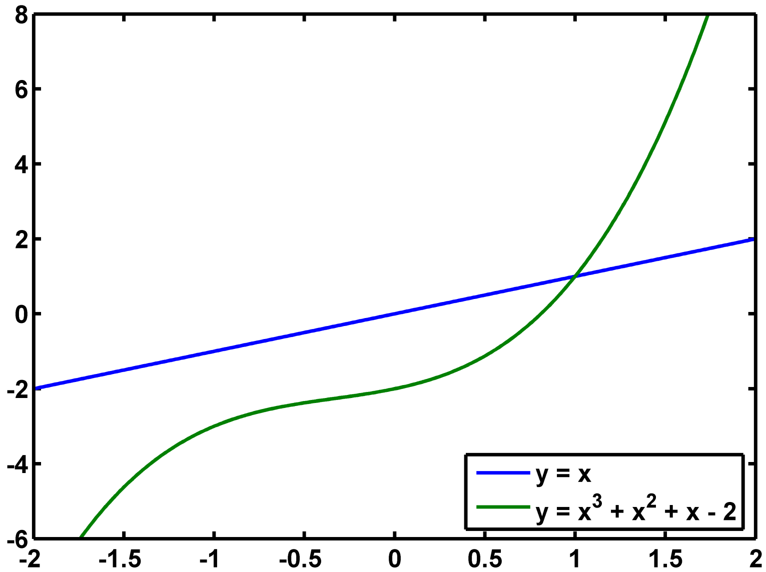
$$f(x) = 0. \tag{1.1}$$

has at least one solution inside $[a, b]$.Writing the equation (1.1) in the form $\alpha f(x) + x = x$, where α is positive parameter and denoting $(x) + x$ by T (x), we get the function equation

$$T(x) = x. \tag{1.2}$$

Now, we choose the value of α in such a way that all the values of T lie inside the interval $[a, b]$. Then, the function T is a function in $[a, b]$; that is it maps the point x from interval $[a, b]$ into the point $T(x) = y$ of the same interval and in general, it does not coincide with x. If the point $x = x_0$ is a solution of the equation (1.2) then we have $T(x_0) = x_0$ which implies that x_0 is fixed by T and so it is called the fixed point of T. Clearly, the solution of the function equation (1.1) is also a solution of equation (1.2).

Example 1.1.1. *If f is defined on the real number by $f(x) = x^3 + x^2 + x - 2$. Then, $x = 1$ is fixed point of f because $f(1) = 1$.*
The following is the graph of $f(x) = x^3 + x^2 + x - 2$ and $f(x) = x$.



Geometrically, a fixed point implies that the point is $(x, f(x))$ on the line $y = x$.

1.2 Basic Definitions

Definition 1.2.1. *Let X be a non empty set and d be a real function from $X \times X$ into $\mathbb{R}^+$ such that for all $x, y, z \in X$, we have*

(a) $d(x, y) \geq 0$,

(b) $d(x, y) = 0 \iff x = y$,

(c) $d(x, y) = d(y, x)$ *and*

(d) $d(x, z) \leq d(x, y) + d(y, z)$,

then d is called a **metric** or distance function and the pair (X, d) is called a **metric space**.

Definition 1.2.2. *A sequence $\{x_n\}$ in a metric space (X, d) is called a* **Cauchy sequence** *if for given $\epsilon > 0$, there corresponds $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.*

Definition 1.2.3. *A sequence $\{x_n\}$ in metric space is said to be* **convergent** *to a point $z \in X$ if for given $\epsilon > 0$, there exists a positive number $n_0 \in \mathbb{N}$ such that $d(x_n, z) < \epsilon$.*

In this case, z is called limit point of $\{x_n\}$ and we write $x_n \to$ z.

Definition 1.2.4. *A metric space (X, d) is called* **complete** *if every Cauchy sequence in it is convergent to a point in X.*

Definition 1.2.5. *A mapping T of a metric space X into a metric space Y is said to be* **continuous** *if*

$$\{x_n\} \to z \Rightarrow Tx_n \to Tz \quad for \quad each \quad n_0 \geq 0.$$

Definition 1.2.6. *A metric space X is said to be* **compact** *if every sequence in it has a convergent subsequence.*

Definition 1.2.7. *A subset F of metric space X is called a* **closed set** *if it contains each of its limit points.*

Definition 1.2.8. *Two self mappings T and S of a metric space X are said to be* **commuting** *if,*

$$T(S(x)) = S(T(x)) \forall x \in X.$$

Two self mappings T and S of a metric space are said to be commuting at a point z in X if $T(S(z)) = S(T(z))$.

We shall denote this by $TSz = STz$. Also, T and S are said to be non commuting if there is no such point z in X where T and S commute.

Definition 1.2.9. *[174] Two self mappings S and T of a metric space X are said to be* **weakly commuting** *if,*

d(STx, TSx) $\leq d(Sx, Tx)$ *for all* $x \in X$

The map S and T are said to be weakly commuting at a point z in X if, $d(STz, TSz) \leq d(Sz, Tz)$.

Definition 1.2.10. *[66] A mapping S of a metric space X into itself is defined to be* **Lipschitz mapping** *if, there exists a real number $k \geq 0$ such that*

$$d(Sx, Sy) \leq kd(x, y) \quad for \quad each \quad x, y \in X.$$

Definition 1.2.11. *[200] Let X be any set. A* **fuzzy set** *A of X is a function from domain X and values in [0, 1].*

Example 1.2.12. *Consider $U = \{a, b, c, d\}$ and $A : U \to [0, 1]$ define as $A(a) = 0, A(b) = 0.5, A(c) = 0.2$ and $A(d) = 1$. Then A is a fuzzy set on U. This fuzzy set also can be written as follows*
$A = \{(a, 0)(b, 0.5)(c, 0.2)(d, 1)\}$

Example 1.2.13. *A crisp interval [a, b] is represented by a fuzzy set f(x)=* $\begin{cases} {}^{0 \quad for \quad x\in(a,b)} \\ 1 \quad for \quad x \in \{a, b\} \end{cases}$

Definition 1.2.14. *[49] A binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a* **continuous t-norm** *if it satisfies the following conditions:*

- $*$*is associative and commutative,*
- $*$ *is continuous,*
- $a * 1 = a$ *for all* $a \in (0, 1)$, *and*

- $a * b \leq c * d$ *wherever* $a \leq c$ *and* $b \leq d$ *for all* $a, b, c, d \in [0, l]$.

Example 1.2.15. $a * b = ab$ *for* $a, b \in [0, 1]$ *is a continuous t-norm.*

Definition 1.2.16. *[172] A binary operation* $\diamond : [0, 1] \times$ *[0, 1]* $\rightarrow [0, 1]$ *is a* **continuous t-conorm**,*if it satisfies the following conditions:*

(a) $\diamond$ *is commutative and associative;*

(b) $\diamond$ *is continuous;*

(c) $a \diamond$ *0* $= a$ *for all* $a \in [0, 1]$;

(d) $a \diamond b \leq c \diamond d$ *whenever* $a \geq c$ *and* $b \geq d$, *for each* $a, b, c, d \in [0, 1]$.

Example 1.2.17. *A binary operation* $\diamond : [0, 1] \times [0, 1] \rightarrow [0, 1]$ *such that* $a \diamond b = min(a + b, 1)$ *is a t -conorms.*

Definition 1.2.18. *[49] A 3-tuple* $(X, M, *)$ *is said to be a* **fuzzy metric space** *if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying the following conditions: for all* $x, y, z \in X$ *and* $s, t > 0$,
(FM1) $M(x, y, t) > 0$;
(FM 2) $M(x, y, t) = 1$ *if and only if* $x = y$;
(FM 3) $M(x, y, t) = M(y, x, t)$;
(FM 4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(FM 5) $M(x, y, .) : (0, \infty) \rightarrow (0, 1]$ *is continuous.*

Then M is called a fuzzy metric on X. The function $M(x, y, t)$ denote the degree of nearness between x and y with respect to t. Also, we consider the following condition in the fuzzy metric spaces $(X, M, *)$.
(FM6) $t \overset{lim}{\rightarrow} \infty M(x, y, t) = 1$, *for all* $x, y \in X$.

Example 1.2.19. *[49] Let (X, d) be a metric space. We define $a * b = ab$ for all $a, b \in [0, 1]$ and let M be fuzzy set on $X^2 \times (0, \infty)$ defined as follows: $M(x, y, t) = \frac{t}{t+d(x,y)}$.*

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d is the standard fuzzy metric.

Definition 1.2.20. *[162] Let $(X, M, *)$ be a fuzzy metric space, and f is a self-mapping of X. Then, ξ is said to be a* **periodic point** *or an eventually fixed point, if there exists a positive integer k such that $f^k(\xi) = \xi$.*

Definition 1.2.21. *[162] Let $(X, M, *)$ be a fuzzy metric space, the mapping $f : X \to X$ is said to be a fuzzy ϵ* **- contractive** *if there exists $0 < \epsilon < 1$, such that if $1 - \epsilon < M(x, y, t) < 1$, then $M(f(x), f(y), t) > M(x, y, t)$ for all $t > 0$, and $x, y \in X$.*

Definition 1.2.22. *[49] A sequence $\{x_n\}$ in a fuzzy metric space (X, M, *) is a* **Cauchy sequence** *if and only if for each $\epsilon \in (0, 1)$ and each $t > 0$, there exists $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for all $n, m \geq n_0$.*

A fuzzy metric space in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Definition 1.2.23. *[193] Two mappings A and S of a fuzzy metric space $(X, M, *)$ into itself are* **weakly commuting** *if*
$M(ASx, SAx, t) \geq M(Ax, Sx, t)$ *for each* $x \in X$ *and* $t > 0$.

Definition 1.2.24. *[193] Two mappings A and S of a fuzzy metric space $(X, M, *)$ into itself are* **R-weakly commuting** *provided there exists some real number R such that*
$M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R})$ *for each* $x \in X$ *and* $t > 0$.

Definition 1.2.25. *[2] Let f and g be two self mapping of a fuzzy metric space $(X, M, *)$ then, f and g are said to satisfy the* **property (E.A.)** *if there exists a sequence $\{x_n\}$ such that $lim_{n\to\infty} f x_n = lim_{n\to\infty} g x_n = t$ for some $t \in X$.*

Definition 1.2.26. *[29] Let $(X, M, *)$ be a fuzzy metric space and $\epsilon > 0$. A finite sequence $x = x_0, x_1, ..., x_n = y$ is called* ϵ**-chainable** *from x to y if $M(x_i, x_{i-1}, t) > 1 - \epsilon$ for all $t > 0$ and i = 1, 2, 3, . . ., n.*

Definition 1.2.27. *[29] A fuzzy metric space (X, M, *) is called* ϵ **- chainable** *if for $x, y \in X$ there exists an ϵ -chain from x to y.*

Definition 1.2.28. *[114] Two self mappings A and S of a fuzzy metric space (X, M, *) are called* **pointwise R -**

weakly commuting *if there exists* $R > 0$ *such that for all* x *in* X *and* $t > 0$,

$$M(ASx, SAx, t) \geq M(Ax, Sx, \frac{t}{R})$$

Definition 1.2.29. *[125] Two self mappings f and g on a fuzzy metric space (X, M, *) are called* **reciprocally continuous** *on X if*

$$lim_{n\to\infty} fgx_n = x \quad and \quad lim_{n\to\infty} gfx_n = gx$$

whenever $\{x_n\}$ *is a sequence in X such that*

$$lim_{n\to\infty} fx_n = lim_{n\to\infty} gx_n = x$$

for some x in X.

If f and g are both continuous then they are obviously reciprocally continuous but the converse need not be true.[125]

Definition 1.2.30. *A function* $\psi : [0, \infty) \to [0, \infty)$ *is an* **altering distance function** *if* $\psi(t)$ *is monotone non-decreasing, continuous and* $\psi(t) = 0$ *if and only if t = 0.*

Definition 1.2.31. *[102] A sequence* $\{x_n\}$ *in a fuzzy metric space* $(X, M, *)$ *is called* **G-Cauchy sequence** *if,*

$$lim_{n\to\infty}(x_{n+p}, x_n, t) = 1$$

for every $t > 0$ *and for each* $p > 0$. *A fuzzy metric space* $(X, M, *)$ *is complete (respectively G-complete) if, every Cauchy sequence (respectively G-sequence) in X converges in X.*

1.2.1 Types of compatible mappings

In 1986, G. Jungck [86] introduced the notion of compatible mappings, which are more general than commuting and weakly commuting mappings. In 1993, G. Jungck, P.P. Murthy and Y. J. Cho [88] gave a generalization of compatible mappings called compatible mappings of type (A) which is equivalent to the concept of compatible mappings under some conditions. In 1995, H. K. Pathak and M. S. Khan [152] introduced the concept of compatible mappings of type (B) with some examples to show that compatible mappings of type (B) need not be compatible of type (A). In 1996, H. K. Pathak, Y. J. Cho, S.S Chang and S. M. Kang [150] introduced the concept of compatible mappings of type (P) and compared with compatible mappings of type (A) and compatible mappings. In 1998, H. K. Pathak, Y. J. Cho, S. M. Kang, B. Madharia [151] introduced another extension of compatible mappings of type (A) in normed spaces called compatible mappings of type (C). In 2007, Singh and Singh [188] introduced the concept compatible mappings of type (E) in metric space and we have extended to fuzzy metric space[111].

We have introduced a new notion of compatible mappings of type (K) in metric space[82] which was extended to fuzzy metric space and established some common fixed point theorems for the pairs of compatible mappings of type (K) with examples [111].

Definition 1.2.32. *[86] Two self mappings S and T of a metric space (X, d) are called* **compatible** *if,*

$$\lim_{n\to\infty} d(STx_n, TSx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad for \quad some \quad t \in X.$$

Definition 1.2.33. *[86] Two self maps S and T of a metric space X are called* **non compatible** *if they are not compatible.*

Definition 1.2.34. *[174] Let S and T be mappings from a metric space (X, d) into itself. Then, S and T are said to be* **weakly compatible** *if they commute at their coincident point;*
that is, $Sx = Tx$ for some $x \in X$ implies $STx = TSx$.

Definition 1.2.35. *[192] Two self mappings f and g of a set X are* **occasionally weakly compatible** *(owc) if and only if there is a point x in X which is a coincident point of f and g at which f and g commute..*

Definition 1.2.36. *[88] The Self mappings A and S of a metric space (X, d) are said to be* **compatible of type** (A) *if*

$$lim_{n\to\infty} d(ASx_n, SSx_n) = 0 \quad and \quad lim_{n\to\infty} d(SAx_n, AAx_n) = 0$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = t$ *for some* $t \in X$.

Definition 1.2.37. *[150] The Self mappings* A *and* S *of a metric space* (X, d) *are said to be* **compatible of type** (P) *if*
$lim_{n\to\infty}d(SSx_n, AAx_n) = 0$ *whenever* $\{x_n\}$ *is a sequence in* X *such that* $lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = t$ *for some* $t \in X$.

Definition 1.2.38. *[152] The Self mappings* A *and* S *of a metric space* (X, d) *are said to be* **compatible of type** (B) *if*

$$lim_{n\to\infty}d(ASx_n, SSx_n) \leq \frac{1}{2}(lim_{n\to\infty}d(ASx_n, At) + lim_{n\to\infty}d(At, AAx_n)), \text{and}$$

$$lim_{n\to\infty}d(SAx_n, AAx_n) \leq \frac{1}{2}(lim_{n\to\infty}d(SAx_n, St) + lim_{n\to\infty}d(St, SSx_n)),$$

whenever $\{x_n\}$ *is a sequence in* X *such that*
$lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = t$ *for some* $t \in X$.

Definition 1.2.39. *[151] The Self mappings* A *and* S *of a metric space* (X, d) *are said to be* **compatible of type**

(C), *if*

$$\begin{aligned} lim_{n\to\infty} d(ASx_n, SSx_n) &\le \frac{1}{3}[lim_{n\to\infty} d(ASx_n, At) + lim_{n\to\infty} d(At, AAx_n) \\ &\quad + lim_{n\to\infty} d(At, SSx_n)] and, \\ lim_{n\to\infty} d(SAx_n, AAx_n) &\le \frac{1}{3}[lim_{n\to\infty} d(SAx_n, St) + lim_{n\to\infty} d(St, SSx_n) \\ &\quad + lim_{n\to\infty} d(St, AAx_n)], \end{aligned}$$

whenever $\{x_n\}$ *is a sequence in* X *such that* $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$ *for some* $t \in X$.

Clearly, compatible of type (A) implies compatible of type (B) and compatible of type (C) but converse may not be true. However compatible, compatible of type (A), compatible of type (B) and compatible of type (C) are equivalent under the continuity of A and S.

Definition 1.2.40. *[188] The Self mappings* A *and* S *of a metric space* (X, d) *are said to be* **compatible of type (E)**, *if*

$$lim_{n\to\infty} AAx_n = lim_{n\to\infty} ASx_n = S(t) and,$$

$$lim_{n\to\infty} SSx_n = lim_{n\to\infty} SAx_n = A(t),$$

whenever $\{x_n\}$ *is a sequence in* X *such that* $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$ *for some* $t \in X$.

It is noted that, if A, S is compatible of type (E), then it is compatible of type (B) but may not be compatible of type

(C) however, the converse is not true.
We have introduced the following new compatible mappings of type (K):

Definition 1.2.41. *[82] The self mappings A and S of a metric space (X, d) are said to be* **compatible of type (K)** *if*

$$lim_{n\to\infty} AAx_n = St \quad and \quad lim_{n\to\infty} SSx_n = At,$$

whenever $\{x_n\}$ is a sequence in X such that
$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$ for some t in X.

The following example shows that the compatible of type (K) is independent with compatible, compatible of type (A) compatible of type (C) and compatible of type (P).

Example 1.2.42. *Let $X = [0, 2]$ with the usual metric $d(x, y) = |x - y|$. We define self-mappings A and S as $Ax = 2$, $Sx = 0$ for $x \in [0, 1] - 1/2, Ax = 0$, $Sx = 2$ for $x = \frac{1}{2}$ and $Ax = \frac{2-x}{2}$, $Sx = \frac{x}{2}$ for $x \in (1, 2]$. Then, A and S are not continuous at $x = 1$, $1/2$.*

Clearly, $\{A, S\}$ is compatible of type (K) but the pair $\{A, S\}$ is neither compatible nor compatible of type (A)(compatible of type (C), compatible of type (P)).

Example 1.2.43. *Let$X = [0, 2]$ with the usual metric $d(x, y) = |x - y|$. Define self-mappings A and S as*

$Ax = Sx = 1$ *for* $x \in [0, 1)$, $Ax = Sx = \frac{4}{3}$ *for* $x =$ *1 and* $Ax = 2 - x$, $Sx = x$ *for* $x \in (1, 2]$.

Clearly, $\{A, S\}$ are not compatible of type (K) but it is compatible, compatible of type (A), compatible of type (B), compatible of type (C), and compatible of type (P).

Definition 1.2.44. *Let* $(X, M, *)$ *be a fuzzy metric space. Then a sequence* $\{x_n\}$ *in* X *is said to be convergent to* x *in* X *if for each* $\epsilon > 0$ *and each* $t > 0$, *there exist* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - \epsilon$ *for all* $n \geq n_0$.

Definition 1.2.45. *A sequence* $\{x_n\}$ *in* X *is said to be* **Cauchy** *if for each* $\epsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x_m, t) > 1 - \epsilon$ *for all* $n, m \geq n_0$.
A fuzzy metric space in which every **Cauchy sequence** *is convergent is said to be complete.*

Definition 1.2.46. *[30] Let f and g be self mappings on a fuzzy metric space* $(X, M, *)$. *Then, f and g are said to be* **compatible** *or asymptotically commuting if, for all* $t > 0$, $limn \to \infty M(fgx_n, gfx_n, t) = 1$ *whenever* $\{x_n\}$ *is a sequence in* X *such that,*
$limn \to \infty fx_n = limn \to \infty gx_n = z$, *for some* $z \in X$.

Definition 1.2.47. *[89] The mappings f and g from a fuzzy metric space* $(X, M, *)$ *into itself are said to be* **weakly compatible** *if they commute at their coincidence point,* *that is,*

$fx = gx$ *implies that* $fgx = gfx$.

Definition 1.2.48. *[185] Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then, the mappings A and S are said to be* **semi-compatible** *if*
$lim_{n\to\infty} M(ASx_n, Sx, t) = 1$, *for all* $t > 0$,
whenever $\{x_n\}$ *is a sequence in* X *such that*
$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x \in X$.

Definition 1.2.49. *[159] A fuzzy metric space $(X, M, *)$ is said to be* **sequentially compact** *if every sequence in X has a convergent sub-sequence in it.*

Definition 1.2.50. *[88] The self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be* **compatible of type (A)** *if*
$\lim_{n\to\infty} d(ASx_n, SSx_n, t) = 1$ *and*
$\lim_{n\to\infty} d(SAx_n, AAx_n, t) = 1$
whenever $\{x_n\}$ *is a sequence in* X*such that*
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ *for some* x *in* X *and* $t > 0$.

Definition 1.2.51. *[150] The self mappings A and S of a fuzzy metric space $(X, M, *)$ are said to be* **compatible of type (P)** *if*
$\lim_{n\to\infty} d(SSx_n, AAx_n, t) = 1$ *whenever* $\{x_n\}$ *is a sequence in* X *such that*
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ *for some* x *in* X *and* $t > 0$.

Definition 1.2.52. *[10] The self mappings A and S of a fuzzy metric space* $(X, M, *)$ *are called* **reciprocally continuous** *on X if* $\lim_{n\to\infty} ASx_n = Ax$ *and* $\lim_{n\to\infty} SAx_n = Sx$
whenever $\{x_n\}$ *is a sequence in X such that*
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ *for some x in X.*

Definition 1.2.53. *[111] The self mappings A and S of a fuzzy metric space (X, M, *) are said to be* **compatible of type (E)** *iff*
$lim_{n\to\infty}M(AAx_n, ASx_n, t)$ $=lim_{n\to\infty}M(AAx_n, Sx, t)$
$=lim_{n\to\infty}M(ASx_n, Sx, t) = 1,$ *and*
$lim_{n\to\infty}M(SSx_n, SAx_n, t) = lim_{n\to\infty}M(SSx_n, Ax, t) =$
$lim_{n\to\infty}M(SAx_n, Ax, t) = 1,$
whenever $\{x_n\}$ *is a sequence in X such that*
$lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = x$ *for some x in X and* $t > 0$.

Definition 1.2.54. *[113] The self mappings A and S of a fuzzy metric space* $(X, M, *)$ *are said to be* **compatible of type (K)** *iff*
$\lim_{n\to\infty} M(AAx_n, Sx, t) = 1,$ *and*
$\lim_{n\to\infty} M(SSx_n, Ax, t) = 1,$
whenever $\{x_n\}$ *is a sequence in X such that*
$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ *for some x in X and* $t > 0$.

The following examples show that the compatible of type (K) in fuzzy metric space is independent with compatible,

compatible of type (A), compatible of type (P) and reciprocal continuous.

Example 1.2.55. *Let* $X = [0, 2]$ *with the usual metric* $d(x, y) = |x - y|$, *define* $M(x, y, t) = \frac{t}{t+d(x,y)}$ *for all* $x, y \in X, t > 0$ *and* $a * b = ab$ *for all* $a, b \in [0, 1]$ *then* $(X, M, *)$ *is a fuzzy metric space.*
We define self-mappings A *and* S *as*
$Ax = 2, Sx = 0$ *for* $x \in [0, 1] - \{\frac{1}{2}\}$,
$Ax = 0, Sx = 2$ *for* $x = \frac{1}{2}$ *and* $Ax = \frac{2-x}{2}, Sx = \frac{x}{2}$ *for* $x \in (1, 2]$.
Then, A *and* S *are not continuous at* $x = 1, \frac{1}{2}$.
Consider a sequence $\{x_n\}$ *in* X *such that* $x_n = 1 + \frac{1}{n}$ *for all* $n \in N$. *Then, we have* $Ax_n = \frac{(2-x_n)}{2} \to \frac{1}{2} = x$ *and* $Sx_n = \frac{x_n}{2} \to \frac{1}{2} = x$. *Also, we have* $AAx_n = A(\frac{(2-x_n)}{2}) = 2 \to 2, ASx_n = A(\frac{x_n}{2}) = 2 \to 2, S(x) = 2$ *and* $SSx_n = S(\frac{x_n}{2}) = 0 \to 0, SAx_n = S(\frac{2-x_n}{2}) = 0 \to 0, A(x) = 0.$
Therefore, (A, S) *is compatible of type (K) but the pair* (A, S) *is neither compatible nor compatible of type (A) (compatible of type (P), reciprocal continuous).*

Example 1.2.56. *Let* $X = [0, 2]$ *with the usual metric* $d(x, y) = |x - y|$, *define* $M(x, y, t) = \frac{t}{t+d(x,y)}$ *for all* $x, y \in X, t > 0$ *and* $a * b = ab$ *for all* $a, b \in [0, 1]$ *then* $(X, M, *)$ *is a fuzzy metric space.*
We define self-mappings A *and* S *as*

$Ax = Sx = 1$ *for* $x \in [0,1), Ax = Sx = \frac{4}{3}$ *for* $x = 1$ *and* $Ax = 2 - x, Sx = x$ *for* $x \in (1,2]$.

Consider a sequence $\{x_n\}$ *in* X *such that* $x_n = 1 + \frac{1}{n}$ *for all* $n \in N$. *Then,we have*

$Ax_n = (2 - x_n) \to 1 = x,$ *and* $Sx_n = x_n \to 1 = x$.

Since, $2 - x_n < 1$ *for all* $n \in N$,

we have $AAx_n = A(2 - x_n) = 1 \to 1,\ ASx_n = A(x_n) = 2 - x_n \to 1$ *and*

$SSx_n = S(x_n) = x_n \to 1, SAx_n = S(2 - x_n) = 1 \to 1$.

Also,we have

$A(x) = \frac{4}{3} = S(t)$ *but*

$AS(x) = AS(1) = A(\frac{4}{3}) = \frac{2}{3}, SA(x) = SA(1) = S(\frac{4}{3}) = \frac{4}{3}$.

However, we have $\frac{2}{3} = AS(x) \neq SA(x) = \frac{4}{3}$, *at* $x = 1$.

Therefore, (A, S) *is not compatible of type (K) but it is compatible, compatible of type (A) and compatible of type (P).*

1.3 Generalized forms of fuzzy metric space

In 1975, Fuzzy metric spaces have been introduced by Kramosil and Michalek[102] as a generalization of metric space. George and Veeramani [49] modified the notion of fuzzy metric spaces with the help of continuous t-norms and then many others introduced the different generalized forms like fuzzy 2- metric, fuzzy 3- metric and intuitionistic fuzzy metric spaces.

Definition 1.3.1. *[177] The 3-tuple* $(X, M, *)$ *is called a* **fuzzy-2 metric space** *if* X *is an arbitrary set, is a continuous t-norm and* M *is a fuzzy set in* $X^3 \times [0, \infty)$ *satisfying the following conditions: for all* $x, y, z \in X$ *and* $t_1, t_2, t_3 > 0$*,*

(a) $M(x, y, z, 0) = 0$*,*

(b) $M(x, y, z, t) = 1$ *for all* $t > 0$ *and when at least two of the three points are equal,*

(c) $M(x, y, z, t) = M(x, z, y, t) = M(y, z, x, t)$ *and*

(d) $M(x, y, u, t_1) * M(x, u, z, t_2) * M(u, y, z, t_3) \leq M(x, y, z, t_1 + t_2 + t_3)$*,*

(e) $M(x, y, z, .) : [0, \infty) \to [0, 1]$ *is left continuous.*

Definition 1.3.2. *[177] The 3-tuple* $(X, M, *)$ *is called a* **fuzzy-3 metric space** *if* X *is an arbitrary set, is a con-*

tinuous t-norm and M is a fuzzy set in $X^4 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t_1, t_2, t_3 > 0$,

(a) $M(x, y, z, w, 0) = 0$,

(b) $M(x, y, z, w, t) = 1$ *for all* $t > 0$ *(only when simplex* $< x, y, z, w >$ *degenerate)*

(c) $M(x, y, z, w, t) = M(x, y, w, z, t) = M(x, w, z, y, t) = M(w, y, z, x, t) = ...$

(d) $M(x, y, z, u, t_1) * M(x, y, u, w, t_2) * M(x, u, z, w, t_3) * M\ (u, y, z, w, t_4) M(x, y, z, w, t_1 + t_2 + t_3 + t_4)$, *and*

(e) $M(x, y, z, w, .) : [0, \infty) \to [0, 1]$*is left continuous.*

Definition 1.3.3. *[147] A 5-tuple $(X, M, N, *, \diamond)$ is said to be an* **intuitionistic fuzzy metric space** *if X is an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions;*

(a) $M(x, y, t) + N(x, y, t) \leq 1$,

(b) $M(x, y, 0) = 0$,

(c) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,

(d) $M(x, y, t) = M(y, x, t)$,

(e) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$, *for all* $x, y, z \in X$, *and* $s, t > 0$,

(f) $M(x, y,) : [0, \infty) \to [0, 1]$ *is left continuous,*

(g) $lim_{t\to\infty}M(x,y,t) = 1$ *for all* $x, y \in X$,

(h) $N(x,y,0) = 1$,

(i) $N(x,y,t) = 0$ *for all* $t > 0$ *if and only if* $x = y$,

(j) $N(x,y,t) = N(y,x,t)$,

(k) $N(x,y,t)\diamond N(y,z,s) \geq N(x,z,t+s)$ *for all* $x, y, z \in X$, *and* $s, t > 0$,

(l) $N(x,y,) : [0,\infty) \to [0,1]$ *is left continuous, and*

(m) $lim_{t\to\infty}N(x,y,t) = 0$ *for all* $x, y \in X$.

Definition 1.3.4. *[173] The triple* $(X, M, *)$ *is a* **M-fuzzy metic space** *if* X *is an arbitrary set,* $*$ *is a continuous t-norms and* M *is a fuzzy set in* $X^3 \times (0,\infty)$ *satisfying the following conditions:*
for each $x, y, z, a \in X$ *and* $t, s > 0$.

(a) $M(x,y,z,t) > 0$, *for all* $x, y, z \in X$

(b) $M(x,y,z,t) = 1$ *if and only if* $x = y = z$, *for all* $t > 0$,

(c) $M(x,y,z,t) = M(px,y,z,t)$, *where* p *is a permutation function,*

(d) $M(x,y,a,t) * M(a,z,z,s) \leq M(x,y,z,t+s)$,

(e) $M(x,y,z,.) : [0,\infty) \to [0,1]$ *is continuous*

Definition 1.3.5. *[166] The 3-tuple* (X, M, T) *is said to be an* L**-fuzzy metric space** *if* X *is an arbitrary (non-empty) set,* T *is a continuous t-norm on* L *and* M *is an* L*-fuzzy*

set on $X^2 \times (0,\infty)$ *satisfying the following conditions:*
for every x, y, z *in* X *and* t, s *in* $(0,1)$

(a) $M(x,y,t) >_L 0_L$;

(b) $M(x,y,t) = 1_L$ *for all* $t > 0$ *if and only if* $x = y$;

(c) $M(x,y,t) = M(y,x,t)$;

(d) $T(M(x,y,t), M(y,z,s)) \leq_L M(x,z,t+s)$ *and*

(e) $M(x,y,) :]0,\infty[\to L$ *is continuous and* $lim_{t\to\infty} M(x,y,t) = 1_L$.

Definition 1.3.6. *[6] The 3-tuple* $(X, M, *)$ *is called a* **non-Archimedean fuzzy metric space** *(shortly, N.A. FM-space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set in* $X^2 \times [0,\infty)$ *satisfying the following conditions:*
For all $x, y, z \in X$ *and* $s, t > 0$,

(a) $M(x,y,0) = 0$,

(b) $M(x,y,t) = 1$, *for all* $t > 0$ *if and only if* $x = y$,

(c) $M(x,y,t) = M(y,x,t)$,

(d) $M(x,y,t) * M(y,z,s) \leq M(x,z,maxt,s)$ *Or equivalently* $M(x,y,t) * M(y,z,t) \leq M(x,z,t)$

(e) $M(x,y,.) : [0,\infty) \to [0,1]$ *is left continuous.*

Definition 1.3.7. *[54] The 4-tuple* $(X, M, W, *)$ *is called a* **Random fuzzy metric space**, *if* X *and Ware arbitrary*

set,$*$ is a continuous t-norm, M is a fuzzy set in $X^2 \times [0, \infty)$ and is function from W to X, satisfying the following conditions:
for all $x, y, z \in X$ and $s, t > 0$,

(a) $M(x, y, 0) = 0$,

(b) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,

(c) $M(x, y, t) = M(y, x, t)$,

(d) $M(x, y, t) * M(y, z, s)M(x, z, t + s)$ *and*

(e) $M(x, y, a) : [0, 1) \to [0, 1]$ *is left continuous.*

Definition 1.3.8. *[170] The 3-tuple $(X, M, *)$ is called a* **semi fuzzy metric space** *(shortly FM- space) if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in $X^2 \times [0, \infty)$ satisfying the following conditions:*
for all $x, y, z \in X$ and $s, t > 0$,

(a) $M(x, y, 0) = 0$,

(b) $M(x, y, t) = 1$ *for all* $t > 0$ *if and only if* $x = y$,

(c) $M(x, y, t) = M(y, x, t)$,

By definition, it is clear that fuzzy metric space is a three dimensional extended form of metric space. Fuzzy 2-metric space, Fuzzy 3-metric space and M- fuzzy metric space are extended forms of fuzzy metric space into higher dimensions. Also, the intuitionistic fuzzy metric space is four dimensional notion in which two fuzzy sets and continuous t-conorm are consider. In case of Intuitionistic fuzzy metric space, if two fuzzy sets are equal and continuous t-conorm is assumed to be an identity mapping then intuitionistic fuzzy metric space reduces to unseal fuzzy metric space. Again, if function in random fuzzy metric space is an identity function, then the random fuzzy metric space changes into unseal fuzzy metric space. Moreover, non-Archimedean fuzzy metric space is different from fuzzy metric space only in triangle inequality property where as in semi fuzzy metric space, the triangle inequality is removed. Finally, L-fuzzy metric space is a fuzzy metric space defined on L- fuzzy set and ϵ- chain property is added in ϵ- chainable fuzzy metric space.

1.3.1 Historical Development of Fixed Point Theorems in metric space

Historically, the most important result in the fixed point theorem is due to L. E. J. Brouwer which asserts that every self continuous mapping of a closed unit ball in $\mathbb{R}^n$, the n - dimensional Euclidean Space, possess a fixed point. A particular case of Brouwer's theorem can be stated as follows.

Theorem 1.3.9. *[146] The closed unit interval* $[0,1]$ *on the real line posses a fixed point property, i.e each continuous mapping of* $[0,1]$ *into itself has a fixed point.*

Most application of topological theorems to analysis, involves infinite dimensional space of functions or sequences. The usual procedure is to extend a theorem from finite dimensional space to an infinite dimensional space. The infinite dimensional analogue of Brouwer's result was given by Schauder in 1930.

Theorem 1.3.10. *[146] Any compact convex nonempty subset of a normed linear space has the fixed point property for continuous mapping.*

Brouwer's and Shauder's fixed point theorems are fundamental theorems in the area of fixed point theory and its applications. Shauder's theorem is of great importance in the numerical treatment of equations in analysis. In 1935, Tychoff extended Brouwer's result to a compact convex subset of a locally convex linear topological space.

Theorem 1.3.11. *[146] Any compact convex nonempty subset of a locally convex Housdorff real topological vector space has the fixed point property for continuous mapping.*

Perhaps the most frequently cited and most widely applied fixed point theorem is due to S. Banach which appeared in his Ph.D. thesis(1920, published in 1922)

Theorem 1.3.12. *[146] Let (X, d) be complete metric space and $S : X \to X$ be a map such that $d(S(x), S(y)) \leq kd(x, y)$ for some $0 \leq k < 1$ and all $x, y \in X$. Then S has a unique fixed point in X. More over, for any $x_0 \in X$ the sequence of iterates x_0, $S(x_0)$, $S(S(x_0)), \ldots$ converges to the fixed point of S.*

if $d(S(x), S(y)) \leq k\ d(x, y)$ for some $0 \leq k < 1$ and all $x, y \in X$,then S is called a *contraction.* A contraction shrinks distances by a uniform factor k less then 1 for all pairs of points. The above theorem is called the contraction mapping theorem or Banach's fixed point theorem.

Banach contraction principle is simple in nature and its proof does not involve much of topological machinery. The proof is constructive, that is the existence of the fixed point is established by constructing the point as the limit of the sequence of the iterates tending to the fixed point. The construction of the sequence $\{x_n\}$ and the study of its convergence are known as the method of successive approximation.

Banach contraction mapping theorem has long been used as one of the most important tools in the study of nonlinear problems. It provides an impressive illustration of the unifying power of functional analysis in an analytic method and of the usefulness of fixed point theorems in analysis. Therefore numerous generalizations of this theorem have been obtained during the last five decades by weakening its hypothesis while retaining the convergence property of successive iterates to the unique fixed point of the mapping. The importance of these generalizations are notions of non expansive and contractive mappings.One of the most interesting generalizations of the Banach Contraction Principle consists of replacing the Lipschitz constant k by some real valued function whose values are less than 1.

One of the first extension of Banach's contraction principle to become widely known is the following theorem due to Rakoth [156].

Theorem 1.3.13. *[156] Let (X, d) be a non empty com-*

plete metric space and suppose $T : X \to X$ *satisfies*

$$d(Tx, Ty) \leq \alpha d(x, y) \forall x, y \in X$$

where $\alpha : [0, \infty] \to [0, \infty]$ *is monotonically decreasing. Then* T *has a unique fixed point* z*, and for all* $x_0 \in X$ *we have,*

$$T^n(x_0) \to z \quad as \quad n \to \infty.$$

Rakoth's theorem is related to the following theorem by Bailey [9].

Theorem 1.3.14. *[9] Let (X, d) be a compact metric space and* $T : X \to X$ *be continuous.If there exists* $n = n(x,y)$ *with*

$$d(T^n(x), T^n(y)) < d(x, y)$$

for $x \neq y$*, then* T *has a unique fixed point.*

In 1969, Meir and Keeler[115] introduced following theorem in complete metric space as a generalization of Banach contraction Principle.

Theorem 1.3.15. *[115] Let (X, d) be a non empty, complete metric space and suppose T:* $X \to X$ *satisfies the condition,*

given $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that for all* $x, y \in X$ *with* $x \neq y$*,*

$$\epsilon \leq d(x, y) < \epsilon + \delta \Rightarrow d(Tx, Ty) < \epsilon \tag{1.3}$$

Then T has a unique fixed point z, and for all $x_0 \in X$ we have $T^n(x_0) \to z$ as $n \to \infty$.

A mapping T: X $\to$ X on a metric space (X, d) which satisfies the condition (1.3) is called a Meir - keeler contraction.

1.3.2 Common Fixed Point Theorems in Metric Space

Definition 1.3.16. *[146] Let S and T be self mappings of a set X, then a point z in X is called a* **common fixed point** *of S and T if $Sz = z = Tz$. Also the point z is called a coincidence point of S and T provided $Sz = Tz$.*

G. Jungck [85] obtained a well known generalization of Banach contraction principle to obtain common fixed points of commuting mappings. Jungck introduced the following condition (called *Jungck Contraction*)

$$d(Sx, Sy) \leq k$$

$d(Tx, Ty), 0 \leq k < 1$.for a pair of self maps S and T of a complete metric space and established the following theorem.

Theorem 1.3.17. *[85] Let $S, T : X \to X$ be a pair of commuting continuous self maps satisfying the condition,*

$$d(Sx, Sy) \leq k\, d(Tx, Ty), \quad 0 \leq k < 1.$$

then S and T have a unique common fixed point whenever $S(X) \subset T(X)$.

In 1983, B. Fisher [47] established a common fixed point theorem for four mappings, A, B, S and T satisfying,

$$A(X) \subset T(X), B(X) \subset S(X)$$

and the condition,

$$d(Ax, By) \leq k\, max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\}, \quad 0 \leq k < 1.$$

In 1986, G. Jungck [86] obtained the following common fixed point theorem for four continuous mappings on a compact metric space.

Theorem 1.3.18. *[86] Let A, B, S and T be continuous self mappings of a compact metric space (X, d) with* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$. *If A, B, S and T be compatible pairs and*

$$d(Ax, By) < max(m(x, y)) > 0 \quad where,$$

$$m(x, y) = \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(By, Sx) + d(Ax, Ty)]\}$$

Then A, B, S and T have a unique common fixed point.

In 1986, R.P. Pant [120] simultaneously and independently established following common fixed point theorem satisfying Meir - Keeler type contractive condition. As Meir - Keeeler type contractive condition does ensure a common fixed point theorem unless δ satisfies some additional condition or some additional inequalities used, taking δ to be non decreasing.

Theorem 1.3.19. *[120] Let A, S and B, T be commuting self mapping of a complete metric space (X, d) satisfying* $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$ *and the condition given* $\epsilon > 0$*, there exists a* $\delta(\epsilon) > 0$, $\delta(\epsilon)$ *being non decreasing such that*

$$\epsilon \leq max(d(Sx, Ty), d(Ax, Sx), d(By, Ty)) < \epsilon + \delta$$

$\Rightarrow d(Ax, By) < \epsilon$.*If one of the mappings A, B, S and T is continuous, then A, B, S and T have a unique common fixed point.*

Also, the common fixed points for four mappings satisfying contractive condition were extended for sequences of mappings by Jungck et.al [87], J. Jachymski [62], R. P. Pant [123].

In 2007, K. Jha [71] established the following fixed point theorem for sequence of mappings involving two pairs of weakly compatible mappings under a Lipschitz type contractive condition.
Let $\{A_i\}, \quad i = 1, 2, 3, ..., \quad S \quad and \quad T$ be self mappings of

a metric space (X, d). In the theorem let us denote,

$$M_{1i}(x,y) = max[d(Sx,Ty), d(A_1x,Sx), d(A_iy,Ty), \frac{1}{2}\{d(Sx,A_iy)+d(A_1x,Ty)\}]$$

Theorem 1.3.20. *[71] Let* $\{A_i\}$, $i = 1,2,3,...,$ *S and T be self mappings of a metric space (X, d) such that,*

(a) $A_1X \subset TX, A_iX \subset SX$ *for* $i > 0$

(b) Given $\epsilon > 0$*, there exists a* $\delta > 0$ *such that for all x, y in X,* $\epsilon < M_{12}(x,y) < \epsilon + \delta \Rightarrow d(A_1x, A_2y) \leq \epsilon$*, and*

(c) $d(A_1x, A_iy) < \alpha[d(SxTy) + d(A_1x,Sx) + d(A_iy,Ty) + d(Sx,A_iy) + d(A_ix,Ty)]$ *for* $0 \leq \alpha \leq \frac{1}{3}$.

If one of A_iX*, SX, or TX is complete subspace of X and if the pairs* (A_1,S) (A_k,T) *for some* $k > 1$*, are weakly compatible, then all the* A_i*, S and T have a unique common fixed point.*

Fixed point theorems are statements containing sufficient conditions that ensure existence of a fixed point, so one of the central concerns in fixed point theory is to find a minimal set of sufficient conditions which ensures the existence of a common fixed point. The vigorous activity of last three decades has manifested in the form of significant contributions relating to fixed point and coincidence points of contractive mappings.

1.3.3 Historical Development of Fixed Point Theorems in fuzzy metric space

In 1965, the concept of fuzzy set was introduced by L.A Zade[200]. Then, fuzzy metric spaces have been introduced by O. Kramosil and J. Michalek [102]. A. George and P. Veeramani [49] modified the notion of fuzzy metric spaces with the help of continuous t-norms and also many others have been introduce and generalized in different way. Recently, many authors have studied the fixed point theory in the fuzzy metric space and number of fixed point theorems have been obtained in fuzzy metric space by using the contractive condition of self mappings.

In 1983, M. Grabiec [52] extended well known fixed point theorems of Banach and Edelstein contraction principle in in fuzzy metric space in the sense of Kramosil and Michelek.

Theorem 1.3.21. *[52] (Fuzzy Banach contraction Theorem)*
Let $(X, M, *)$ *be a complete fuzzy metric space such that*

(a) $lim_{t\to\infty} M(x, y, t) = 1$ *for all* $x, y \in X$*,*

(b) $M(Tx, Ty, kt) \geq M(x, y, t)$

for all $x, y \in X$ *where* $0 < k < 1$*. Then* T *has unique fixed point.*

Lemma 1.3.22. *[52] If* $lim_{t\to\infty} x_n = x$ *and* $lim_{t\to\infty} y_n =$

y, then $M(x, y, t-\epsilon) \leq lim_{t\to\infty} M(x_n, y_n, t) \leq M(x, y, t+\epsilon)$ *for all* $t > 0$ *and* $0 < \epsilon < \frac{t}{2}$.

Theorem 1.3.23. *[52] (fuzzy Edelstein contraction theorem) Let* $(X, M, *)$ *be a compact fuzzy metric space with* $(x, y, .)$ *continuous for all* $x, y \in X$. *Let* $T : X \to X$ *be a mapping satisfying*

$$M(Tx, Ty, t) > M(x, y, t) forall x \neq y \text{ and } t > 0.$$

Then T has unique fixed point.

In 1983, proved the contraction principle in the setting of fuzzy metric spaces introduced by J. Kramosil and J. Michalek. In 1993, P.V. Subramanyam generalized Grabiec's results for a pair mappings.

Theorem 1.3.24. *[190] Let* $(X, M, *)$ *be a complete fuzzy metric space and let* $f, g : X \to X$ *be mappings that satisfy the following conditions:*

(a) $g(X) \subseteq f(X)$,

(b) f *is continuous, and*

(c) $M(g(x), g(y), \alpha t) \geq M(f(x), f(y), t)$ *for all* x, y *in* X *and* $0 < \alpha < 1$.

Then, f *and* g *have unique common fixed point provided* f *and* g *commute.*

In 2009, V. Pant obtained the following fixed point theorems for pointwise R-weakly commuting self mappings in fuzzy metric space.

Theorem 1.3.25. *[137] Let f and g be pointwise R -weakly commuting self mappings of type (A_g) of a fuzzy metric space $(X, M, *)$ such that*

(a) $fX \subset gX$, *and*

(b) $M(fx, fy, t) > min\ \{M(gx, gy, th), M(fx, gx, th),$
$M(fy, gy, th), M(fy, gx, th), M(fx, gy, th), 0 \leq h < 1,$
for $t > 0$.

If f and g satisfy the property (E.A.) and the range of either of f or g is a complete subspace of X, then f and g have a unique common fixed point.

Theorem 1.3.26. *[137] Let f and g be non-compatible pointwise R-weakly commuting self mappings of type (A_g) of a fuzzy metric space $(X, M, *)$ such that*

(a) $fX \subset gX$, *and*

(b) $M(fx, fy, t) > min\ \{M(gx, gy, th), M(fx, gx, th),$
$M(fy, gy, th), M(fy, gx, th), M(fx, gy, th),\ 0 \leq h < 1,$
for $t > 0$

If the range of f or g is a complete subspace of X, then f and g have a unique common fixed point and the fixed point is the point of discontinuity.

1.3.4 Common fixed point theorems for Three Mappings in fuzzy metric space

In 2002, S. Sharma obtained the following common fixed point results for three self maps in fuzzy metric space.

Theorem 1.3.27. *[177] Let (X, M, *) be a complete fuzzy metric space with the condition* $(FM-6)$ *and let* S *and* T *be continuous mappings of* X*, then* S *and* T *have a common point in* X *if there exists continuous mapping* A *of* X *into* $S(X) \cap T(X)$ *which commute with* S *and* T *and*
$M(Ax, Ay, qt) \geq min\,\{M(Ty, Ay, t), M(Sx, Ax, t), M(Sx, Ty, t)\}$,
for all $x, y \in X, t > 0$ *and* $0 < q < 1$.
Then, S, T and A have a unique common fixed point.

Theorem 1.3.28. *[177] Let (X, M, *) be a complete fuzzy-2 metric space and let* S *and* T *be continuous mappings in* X*, then* S *and* T *have a common point in* X
if there exists continuous mapping A *of* X *into* $S(X) \cap T(X)$
which commute with S *and* T *and*

$$\begin{aligned} M(Ax, Ay, a, qt) \;\geq\; & min[M(Ty, Ay, a, t), M(Sx, Ax, a, t), \\ & M(Sx, Ty, a, t)], \end{aligned}$$

for all $x, y, a, b \in X, t > 0$ *and* $0 < q < 1$,
$\lim_{t\to\infty} M(x, y, z) = 1$ *for all* x, y, z *in* X.
Then S, T and A have a unique common fixed point.

Theorem 1.3.29. *[177] Let* $(X, M, *)$ *be a complete fuzzy-3 metric space and let* S *and* T *be continuous mappings in*

X, then S and T have a common point in X if there exists continuous mappings A of X into $S(X) \cap T(X)$ which commute with S and T and

M(Ax, Ay, a, b, qt) $\geq min[M(Ty, Ay, a, b, t), M(Sx, Ax, a, b, t),$ $M(Sx, Ty, a, b, t)$,for all $x, y, a, b \in X, t > 0$ and $0 < q < 1$, $\lim_{t\to\infty} M(x, y, z, w, t) = 1$ for all $x, y, z, w \in X$.
Then S, T and A have a unique common fixed point.

In 2006, J. H. Jung obtained the following common fixed point theorems for weakly pair of compatible self maps in fuzzy metric space.

Theorem 1.3.30. *[28] Let $(X, M, *)$ be a complete ϵ - chainable fuzzy metric space and let S be continuous self mapping of X and T be self mapping of X. Then, S and T have a common fixed point in X if and only if there exists a continuous self mapping A of X such that the following conditions are satisfied:*

(a) $AX \subset TX \cap SX$,

(b) the pairs (A, S) and (A, T) are weakly compatible,

(c) there exists $q \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$, $M(Ax, Ay, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Ay, Ty, t) * M(Ax, Ty, t)$.

In fact, A, S and T have a unique common fixed point in X.

1.3.5 Common fixed point theorem for four mappings in fuzzy metric space

In 2005, B. Singh and S. Jain obtained the following fixed point theorems in fuzzy metric space using implicit relation under semi-compatibility.

Theorem 1.3.31. *[183] Let A, B, S, and T be self-mappings of a complete fuzzy metric space (X, M, *) satisfying that*

(a) $A(X) \subseteq T(X), B(X) \subseteq S(X)$;

(b) The pair (A, S) *is semi-compatible and* (B, T) *is weak compatible; one of A or S is continuous; and*

(c) For some $\phi \in \phi$, *there exists* $k \in (0, 1)$ *such that for all* $x, y \in X$ *and* $t > 0$,
$\phi(M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, kt)) \geq 0$,
$\phi(M(Ax, By, kt), M(Sx, Ty, t), M(Ax, Sx, kt), M(By, Ty, t)) \geq 0$.

Then, A, B, S and T have unique common fixed point in X.

In 2005, S. H. Cho and J. H. Jung obtained the following results of common fixed point theorems in ϵ-chainable fuzzy metric space.

Theorem 1.3.32. *[28] Let* $(X, M, *)$ *be a complete* ϵ *- chainable fuzzy metric space and let* A*,* B*,* S *and* T *be self mappings of* X *satisfying the following conditions:*

(a) $AX \subset TX$ *and* $BX \subset SX$*,*

(b) A *and* S *are continuous,*

(c) the pairs (A, S) *and* (B, T) *are weakly compatible, and*

(d) there exists $q \in (0, 1)$ *such that* $M(Ax, By, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t)$ *for every* $x, y \in X$ *and* $t > 0$*.*

Then, A*,* B*,* S *and* T *have a unique common fixed point in* X*.*

In 2007, S. Kutukcu, S. Sharma and H. Tokgoz obtained the following results of fixed point theorem in fuzzy metric spaces for R-weakly commuting pairs of self maps.

Theorem 1.3.33. *[106] Let* (A, S) *and* (B, T) *be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space* $(X, M, *)$ *such that*

(a) $AX \subset TX, BX \subset SX$*, and*

(b) $M(Ax, By, t) \geq M(x, y, ht)$*,* $0 < h < 1$*,* $x, y \in X$ *and* $t > 0$*.*

Suppose that (A, S) and (B, T) are compatible pairs of reciprocally continuous mappings. Then, A, B, S and T have a unique common fixed point.

Theorem 1.3.34. *[106] Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space $(X, M, *)$ such that*

(a) $AX \subset TX$, $BX \subset SX$, and

(b) $M(Ax, By, t) \geq M(x, y, ht)$, $0 < h < 1$, $x, y \in X$ and $t > 0$.

Let (A, S) and (B, T) be compatible mappings. If any of the mappings in compatible pairs (A, S) and (B, T) is continuous then A, B, S and T have a unique common fixed point.

Theorem 1.3.35. *[106] Let $(X, M, *)$ be a complete fuzzy metric space with $a * a \geq a$ for all $a \in [0, 1]$ and the condition $(FM.6)$. Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of X such that*

(a) $AX \subset TX, BX \subset SX$; and

(b) there exists $k \in (0, 1)$ such that $M(Ax, By, kt) \geq M(x, y, t)$ for all $x, y \in X$, $\alpha \in (0, 2)$ and $t > 0$.

If one of the mappings in compatible pair (A, S) or (B, T) is continuous, then A, B, S and T have a unique common fixed point.

Theorem 1.3.36. *[106] Let A, B, S and T be self mappings on a complete metric space (X, d) satisfying*

(a) $AX \subset TX, BX \subset SX$;

(b) if there exists $k \in (0, 1)$ *such that* $d(Ax, By) \leq kmax\,\{d(Sx, Ax),$ $d(Ty, By), d(Sx, Ty), [d(Ty, Ax) + d(Sx, By)]/2$ *for all* $x, y \in X$.

Then A, B, S and T have a unique common fixed point in X.

In 2008, K. P. R. Rao, G. R. Babu and B. Fisher obtained the following results of common fixed point theorems in fuzzy metric spaces under implicit relations.

Theorem 1.3.37. *[160] Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$, for all $t \in [0, 1]$, and let f, g, S and T be self maps on X such that*

(a) $f(X) \subset T(X), g(X) \subset S(X)$,

(b) S and $T($ are continuous,

(c) The pairs (f, S) and (g, T) are compatible,

(d) There exists $k \in (0, 1)$ such that for every $x, y \in X$ and $t > 0$, $M(fx, gy, kt) \geq M(Sx, Ty, t) * M(fx, Sx, t) * M(gy, Ty, t) * M(fx, Ty, t)$, and

(e) $lim_{n \leftarrow \infty} M(x, y, t) = 1$, *for all* $x, y \in X$.

Then, f, g, S and T have a unique common fixed point in X.

Theorem 1.3.38. *[160] Let A, B, S and T be self-mappings of a complete L-fuzzy metric space (X, M, T), which has property (C), satisfying:*

(a) $A(X) \subseteq T(X), B(X) \subseteq S(X)$ *and* $T(X), S(X)$ *are two closed subsets of* X*;*

(b) the pairs (A, S) *and* (B, T) *are weakly compatible; and*

(c) $M(Ax, By, t) \geq_L M(Sx, Ty, kt)$*, for every* x, y *in* X *and some* $k > 1$*.*

Then, A, B, S and T have a unique common fixed point in X*.*

In 2009, M. Abbas, I. Altun and D. Gopal obtained the following results of common fixed point theorems for non compatible mappings in fuzzy metric spaces.

Theorem 1.3.39. *[2] Let* $(X, M, *)$ *be a fuzzy metric s-pace. Let* A, B, S *and* T *be mappings from* X *into itself with*
$A(X) \subseteq T(X)$ *and* $B(X) \subseteq S(X)$ *and there exists a constant* $k \in (0, 1/2)$ *such that*

$$\begin{aligned} M(Ax, By, kt) \geq\ & \phi(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ & M(Ax, Ty,), M(By, Sx, (2-\alpha)t)), \end{aligned}$$

for all $x, y \in X, \alpha \in (0, 2), t > 0$ *and* $\psi \in \phi$ *.*
Then A, B, S *and* T *have a unique common fixed point in* X

provided the pair (A, S) or (B, T) satisfies (E. A.) property, one of $A(X), T(X), B(X), S(X)$ is a closed subset of X and the pairs (B, T) and (A, S) are weakly compatible.

Theorem 1.3.40. *[2] Let $(X, M, *)$ be a fuzzy metric space. Let A, B, S and T be mappings from X into itself such that*

$$\begin{aligned} M(Ax, By, kt) &\geq \phi(M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ &\quad M(Ax, Ty,), M(By, Sx, (2-\alpha)t)), \end{aligned}$$

for all $x, y \in X, k \in (0, 1/2), \alpha \in (0, 2), t > 0$ and $\psi \in \phi$. Then A, B, S and T have a unique common fixed point in X provided the pair (A, S) and (B, T) satisfy common (E.A.) property, $T(X)$ and $S(X)$ are closed subsets of X and the pairs (B, T) and (A, S) are weakly compatible.

In 2009, R.K. Saini and M. Kumar proved the following fixed point theorem in fuzzy metric space using implicit relation.

Theorem 1.3.41. *[169] Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space $(X, M, *)$ such that*

(a) $AX \subset TX, BX \subset SX,$

(b) $M(Ax, By, t) \geq M(x, y, ht); 0 < h < 1, x, y \in X$ *and* $t > 0.$

Suppose that (A, S) and (B, T) is compatible pairs of reciprocally continuous mappings. Then, A, B, S and T have a unique common fixed point.

Theorem 1.3.42. *[169] Let (A, S) and (B, T) be pointwise R-weakly commuting pairs of self mappings of complete fuzzy metric space $(X, M, *)$ such that*

(a) $AX \subset TX, BX \subset SX$,

(b) $M(Ax, By, t) \geq M(x, y, ht); 0 < h < 1, x, y \in X$ *and* $t > 0$.

Let (A, S) and (B, T) be compatible mappings. If any of the mappings in compatible pairs (A, S) and (B, T) is continuous, then A, B, S and T have a unique common fixed point.

Theorem 1.3.43. *[169] Let $(X, M, *)$ be a complete fuzzy metric space with $t * t \geq t$, for all $t \in [0, 1]$ and the condition (FM 6). Let (A, S) and (B, T) be point wise R-weakly commuting pairs of self maps on X satisfying*

(a) $AX \subset TX$ *and* $BX \subset SX$*;*

(b) (A, S) *and* (B, T) *are compatible pairs and one of the mapping in each pair is continuous;*

(c) there exists $k \in (0, 1)$ *such that*
$F(M(A^2x, B^2y, kt), M(S^2x, A^2x, t), M(T^2y, S^2x, t),$
$M(T^2y, B^2y, kt), M(A^2x, T^2y, t)) \geq 0,$
for all $x, y \in X$ *and* $t > 0$*, where* $F \in F*$,

Then A, B, S and T have a unique common fixed point in X.

In 2010, A. S. Ranadive and A. P. Chouhan obtained the following results on absorbing mappings and fixed point theorem in fuzzy metric spaces.

Theorem 1.3.44. *[157] Let $(X, M, *)$ be a complete ϵ - chainable fuzzy metric space and let A, B, S and T be self mappings of X satisfying the following conditions:*

(a) $AX \subset TX$ and $BX \subset SX$;

(b) A and S are continuous ;

(c) the pairs (A, S) and (B, T) are weakly compatible; and there exists $q \in (0, 1)$ such that $M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) * M(Ax, Ty, t)$, for every $x, y \in X$ and $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

In 2010, C. T. Aage and J. N. Salunke obtained the following results of fixed point theorems in fuzzy metric spaces.

Theorem 1.3.45. *[1] Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly* *compatible. If there exists* $q \in (0, 1)$ *such that*

$$\begin{aligned} M(Ax, By, qt) \;\geq\; & min(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ & M(Ax, Ty, t), M(By, Sx, t)) \end{aligned}$$

for all $x, y \in X$ *and for all* $t > 0$*, then there exists a unique fixed point* w *in* X *such that* $Aw = Sw = w$ *and a unique point* $z \in X$ *such that* $Bz = Tz = z$*. Moreover,* $z = w$*, so that there is a unique common fixed point of* A*,* B*,* S *and* T*.*

Theorem 1.3.46. *[1] Let (X, M, *) be a complete fuzzy metric space and let* A*,* B*,* S *and* T *be self-mappings of* X*. Let the pairs* $\{A, S\}$ *and* $\{B, T\}$ *be occasionally weakly compatible. If there exists* $q \in (0, 1)$ *such that*

$$\begin{aligned} M(Ax, By, qt) \;\geq\; & \phi(min(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ & M(Ax, Ty, t), M(By, Sx, t))), \end{aligned}$$

for all $x, y \in X$ *and* $\phi : [0, 1] \to [0, 1]$ *such that* $\phi(t) > t$ *for all* $0 < t < 1$*, then there exists a unique common fixed point of* A*,* B*,* S *and* T*.*

Theorem 1.3.47. *[1] Let* $(X, M, *)$ *be a complete fuzzy metric space and let* A*,* B*,* S *and* T *be self-mappings of* X*. Let the pairs* $\{A, S\}$ *and* $\{B, T\}$ *be occasionally weakly* *compatible. If If there exists $q \in (0,1)$ such that*

$$\begin{aligned} M(Ax, By, qt) \geq & \ \phi(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ & M(Ax, Ty, t), M(By, Sx, t)) \end{aligned}$$

for all $x, y \in X$ and $\phi : [0,1] \to [0,1]$ such that $\phi(t, 1, 1, t, t) > t$ for all $0 < t < 1$, then there exists a unique common fixed point of A, B, S and T.

Theorem 1.3.48. *[1] Let $(X, M, *)$ be a complete fuzzy metric space and let A, B, S and T be self-mappings of X. Let the pairs $\{A, S\}$ and $\{B, T\}$ be occasionally weakly compatible. If there exists a point $q \in (0,1)$ such that for all $x, y \in X$ and $t > 0$,*

$$\begin{aligned} M(Ax, By, qt) \geq & \ M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ & * M(Ax, Ty, t) \end{aligned}$$

then there exists a unique common fixed point of A, B, S and T.

Theorem 1.3.49. *[1] Let $(X, M, *)$ be a complete fuzzy metric space. Then continuous self mappings S and T of X have a common fixed point in X if and only if there exists a self mapping A of X such that the following conditions are satisfied*

(a) $AX \subset TX \cup SX$

(b) the pairs $\{A, S\}$ and $\{A, T\}$ are weakly compatible, and

(c) there exists a point $q \in (0,1)$ *such that for every* $x, y \in X$ *and* $t > 0$, $M(Ax, Ay, qt) \geq M(Sx, Ty, t)M(Ax, Sx, t) * M(Ay, Ty, t) * M(Ax, Ty, t)$.

In fact A, S and T have a unique common fixed point.

Theorem 1.3.50. *[1] Let* $(X, M, *)$ *be a complete fuzzy metric space and let* A *and* S *be self-mappings of* X*. Let the* A *and* B *are occasionally weakly compatible. If there exists a point* $q \in (0,1)$ *such that for all* $x, y \in X$ *and* $t > 0$,

$$\begin{aligned} M(Sx, Sy, qt) \;\geq\; & \alpha M(Ax, Ay, t) + \beta min(M(Ax, Ay, t), \\ & M(Sx, Ax, t), M(Sy, Ay, t)) \end{aligned}$$

for all $x, y \in X$*, where* $\alpha, \beta > 0, \alpha + \beta > 1$*. Then,* A *and* S *have a unique common fixed point.*

In 2011, S. Kumar and B. Fisher obtained the following results of a common fixed point theorem in fuzzy metric space using property (E.A.).

Theorem 1.3.51. *[104] Let* $(X, M, *)$ *be a complete fuzzy metric space and let* A, B, S *and* T *be self -mappings of* X*. Let the pairs* A, S *and* B, T *be occasionally weakly compatible. If there exists* $q \in (0,1)$ *such that*

(a) $M(Ax, By, qt) \geq \phi(min\{M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), M(Ax, Ty, t), M(By, Sx, t)\})$ *, for all* $x, y \in X$ *and*

(b) $\phi : [0,1] \to [0,1]$ *such that* $\phi(t) > t$ *for all* $0 < t < 1$.

Then there exists a unique common fixed point of A, B, S *and* T*.*

Theorem 1.3.52. *[104] Let* $(X, M, *)$ *be a complete fuzzy metric space and let* A, B, S *and* T *be self -mappings of* X*. Let the pairs* A, S *and* B, T *be occasionally weakly compatible. If there exists* $q \in (0, 1)$ *such that*

$$\begin{aligned} M(Ax, By, qt) \;\geq\; & \phi(M(Sx, Ty, t), M(Sx, Ax, t), M(By, Ty, t), \\ & M(Ax, Ty, t), M(By, Sx, t)) \end{aligned}$$

for all $x, y \in X$ *and* $\phi : [0, 1]^5 \to [0, 1]$ *such that* $\phi(t, 1, 1, t, t) > t$ *for all* $0 < t < 1$*, then there exists a unique common fixed point of* A, B, S *and* T*.*

Theorem 1.3.53. *[104] Let* $(X, M, *)$ *be a complete fuzzy metric space and let* A, B, S *and* T *be self-mappings of* X*. Let the pairs* A, S *and* B, T *be occasionally weakly compatible. If there exists* $q \in (0, 1)$ *for all* $x, y \in X$ *and* $t > 0$

$$\begin{aligned} M(Ax, By, qt) \;\geq\; & M(Sx, Ty, t) * M(Ax, Sx, t) * M(By, Ty, t) \\ & * M(Ax, Ty, t) \end{aligned}$$

then there exists a unique common fixed point of A, B, S *and* T*.*

Theorem 1.3.54. *[104] Let* $(X, M, *)$ *be a complete fuzzy metric space. Then, continuous self mapping* S *and* T *of* X *have a common fixed point in* X *if and only if there exists a* *self mapping A of X such that the following conditions are satisfied:*

(a) $AX \subset TX \cap SX$

(b) *the pairs* $\{A, S\}$ *and* $\{A, T\}$ *are weakly compatible, and*

(c) *there exists a point* $q \in (0, 1)$ *such that for every* $x, y \in X$ *and* $t > 0$
$M(Ax, By, qt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Ay, Ty, t)$
$* M(Ax, Ty, t)$

In fact there exists a unique common fixed point for A, S *and* T.

Theorem 1.3.55. *[104] Let* $(X, M, *)$ *be a complete fuzzy metric space and Let* A *and* B *be self- mapping of* X. *Let* A *and* B *are occasionally weakly compatible. If there exists a point* $q \in (0, 1)$ *for all* $x, y \in X$ *and* $t > 0$,

$$\begin{aligned} M(Sx, Sy, qt) \;\geq\; & (Ax, Ay, t) + \beta min(M(Ax, Ay, t), \\ & M(Sx, Ax, t), M(Sy, Ay, t)) \end{aligned}$$

for all $x, y \in X$, *for* $\alpha, \beta > 0$, $\alpha + \beta > 1$. *Then,* A *and* S *have a unique common fixed point.*

In 2013,K. Jha obtained the following results of common fixed point theorem for four mapping in fuzzy metric space.

Theorem 1.3.56. *[75] Let (X, M, *) be a complete fuzzy metric space with additional condition (vi) and with* $a * a \geq$

a for all $a \in [0, 1]$. Let A, B, S and T be mappings from X into itself such that

(a) $AX \subseteq TX, BX \subseteq SX$, *and*

(b) $M(Ax, By, t) \geq r(N(x, y, t))$,

where $r : [0, 1] \to [0, 1]$ *is a continuous function such that* $r(t) > t$ *for some* $0 < t < 1$ *and for all* $x, y \in X$ *and* $t > 0$. *If (A, S) or (B, T) is semi-compatible pair of reciprocally continuous maps with respectively (B, T) or (A, S) as weakly compatible maps, then A, B, S and T have a unique common fixed point in X.*

1.3.6 Common fixed point theorems for sequence of mappings in fuzzy metric space.

In 2005, A. Razani proved the following results of a contraction theorem in fuzzy metric spaces.

Theorem 1.3.57. *[162] Let (X, M,*) be a fuzzy metric space, and A is a fuzzy contractive mapping of X into itself such that there exists a point $x \in X$ whose sequence of iterates $(A^n(x))$ contains a convergent subsequence $(A^{n_i}(x))$, then $\xi = lim_{i\to\infty} A^{n_i}(x) \in X$ is a unique fixed point.*

Theorem 1.3.58. *[162] Let $(X, M, *)$ be a fuzzy metric space, where the continuous t-norm * is defined as $a * b =$*

min $\{a, b\}$ *for* $a, b \in [0, 1]$. *Suppose* f *is a fuzzy* ϵ *- contractive self-mapping of* X *such that, there exists a point* $x \in X$ *whose sequence of iterates* $(f^n(x))$ *contains a convergent subsequence* $(f^{n_i}(x))$, *then* $\xi = lim_{i\to\infty} f^{n_i}(x) \in X$ *is a periodic point.*

In 2010, K. Jha obtained the following results of a common fixed point theorem for sequence of mapping in fuzzy metric space.

Theorem 1.3.59. *[73] Let* $(X, M, *)$ *be a fuzzy metric space with additional condition (FM6.) and with* $a * a \geq a$ *for all* $a \in [0, 1]$. *Let* $\{A_i\}$, *I = 1, 2, 3, 4, , S and T be self mappings of a fuzzy metric space from (X, M, *) such that*

(a) $A_1X \subset TX, A_iX \subseteq SX$ *for* $i = 1, 2, 3...1,$

(b) there exists $r \in (0, 1)$ *such that* $M(A_1x, A_ix, rt) \geq M_{1i}(x, y, t)$ *for all* $x, y \in X$, $\alpha \in (0, 2)$ *and* $t > 0$.

If one of A_iX, *SX or TX is complete subspace of X and if the pairs* (A_1, S) *and* (A_k, T), *for some* $k > 1$ *are weakly compatible then all the mapping* $\{A_i\}$, *S and T have a unique common point.*

In 2012, K. Jha obtained the following results of generalized common fixed point theorem for sequence of mapping in fuzzy metric space.

Theorem 1.3.60. *[74] Let $(X, M, *)$ be a fuzzy metric space. Let $\{A_i\}$, $i = 1, 2, 3, \ldots$, S and T be mappings of a fuzzy metric space from X into itself such that*

(a) $A_1X \subseteq TX, A_iX \subseteq SX$, *for $i > 1$, and*

(b) for a function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(r) > 0$ for $r > 0, \psi(0) = 0$
and an altering distance function ϕ such that for $i > 1$, the relation $\phi(\frac{1}{M(A_1x,A_iy,t)} - 1) \leq \phi\ (\frac{1}{M_{1i}(x,y,t)} - 1) - \psi(\frac{1}{M_{1i}(x,y,t)} - 1)$ holds for every $x, y \in X$ and each $t > 0$.

If one of A_iX, SX and TX is a G-complete subspace of X; if the pair (A_1, S) and (A_i, T), for $i > 1$, are weakly compatible,then all the mappings A_i, S and T have a unique common fixed point in X.

1.4 Some applications

Fixed point theory is a highly applicable branch of mathematics. It has numerous applications within and outside mathematics. These have been constantly used for the existence of the solution and to check the stability of the system. Recently, fixed point theorems have been used in new areas of mathematical economies, game theory, dynamical system, fluid flow and even equation derived from models in the life science. Since fixed point theory is recently developed field with wide applications, so it is a very active field of research. Fixed Point Theorems are one of the major tools economics use for proving existence. In 1996, J. Casti has used the Brouwer's fixed point theory in the theory in human affairs, including the identification of an optimal Earth-to-Moon Trajectory for space travel to analyze the intergenerational occupational modeling. Tarski's fixed point theorem has applications in Graph Theory. Lefschetz's fixed point theory has applications to Non convex differential inclusions on manifolds. H.K. Pathak and B. Fisher have shown some applications of fixed points in dynamical programming. Banach contraction principle is very useful in the existence and uniqueness theories. Out of all classical fixed point theorems, the contraction principle has many applications which are scattered throughout almost all branches of mathematics. The paper of Jha [32] deals with the survey work on some applications of Banach's fixed point theorem to different fields. We find other applications of fixed point theory in Ferreira to the problems such as signal and image reconstruction, tomography, telecommunications, interpolation, extrapolation, quantize design, signal enhancement, signal synthesis, filter synthesis (Artificial Neural Network).

The theory is a mixture of analysis (pure and applied), topology, and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics.

Fuzzy sets and fuzzy logic are powerful mathematical tools for modeling and controlling uncertain systems in industry, humanity, and nature; they are facilitators for approximate reasoning in decision making in the absence of complete and precise information. Their role is significant when applied to complex phenomena not easily described by traditional mathematics.

Chapter 2

Common fixed point theorems in metric space

In this chapter, we present the common fixed point theorems in metric space of mappings satisfying contractive type conditions. We have introduced a new notion of compatible mappings of type (K) which is independent of known conditions and established a common fixed point result for pair of compatible mappings of type (K). The established theorems have been published in peer reviewed international journals.

2.0.1 Introduction

The study of common fixed point theorem in metric space of mappings satisfying certain contractive type conditions has been a very active field of research. In 1986, G. Jungck

[86] introduced the notion of compatible mappings, which is more general than commuting and weakly commuting mappings.In 1999, Pant [126] introduced the concept of reciprocally continuous mappings and obtained common fixed point theorem. Recently, Singh and Singh [188] introduced the concept Compatible mappings of type (E) in metric space and established some common fixed point. Also, we have established a common fixed point theorem using compatible mapping of type (K).
We have introduced the new notion of compatible mappings of type (K) and established a common fixed point theorem for the pairs of compatible mappings of type (K) with example[82].

2.0.2 Basic definitions

Definition 2.0.1. *Two self mappings S and T of a metric space (X, d) are called compatible if,* $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, *whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad for \quad some \quad t \in X.$$

Definition 2.0.2. *The self mappings A and S of a metric space (X, d) are called reciprocally continuous on X if*

$$\lim_{n\to\infty} ASx_n = Ax \quad and \quad \lim_{n\to\infty} SAx_n = Sx$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x$ *for some x in X.*

Definition 2.0.3. *The Self mappings A and S of a metric space (X, d) are said to be compatible of type (E), if*

$$lim_{n\to\infty} AAx_n = lim_{n\to\infty} ASx_n = S(t) \quad and$$

$$lim_{n\to\infty} SSx_n = lim_{n\to\infty} SAx_n = A(t),$$

whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Definition 2.0.4. *The self mappings A and S of a metric space (X, d) are said to be compatible of type (K)if*

$$lim_{n\to\infty} AAx_n = St \quad and \quad lim_{n\to\infty} SSx_n = At,$$

whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$ for some t in X.

2.0.3 Common Fixed Point Theorems in metric space

In 2004, R.P.Pant, V. Pant and V.P. Pandey [145] established following theorems,

Theorem 2.0.5. *Let (A, S) and (B, T) be compatible pairs of self-mappings of a complete metric space (X,d) such that*

(a) $AX \subset TX$ and $BX \subset SX$,

(b) Given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$,

$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon$

where $M(x,\ y) = max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Ax,Ty)+d(Sx,By)}{2}\right\}$

(c) $d(Ax, By) < max\left\{d(Sx, Ty), k\frac{d(Ax,Sx)+d(By,Ty)}{2}, \frac{d(Ax,Ty)+d(Sx,By)}{2}\right\}$ *for* $0 < k \leq 2$

Suppose that the mappings in one of the pair (A,S) or (B,T) are reciprocal continuous. Then A, B, S and T have unique common fixed point.

In 2005, K. Jha and R.P. Pant [78] established following theorems for compatible pairs of mappings in complete metric space.

Theorem 2.0.6. *[78]Let (A, S) and (B, T) be compatible pairs of self-mappings of a complete metric space (X,d) such that*

(a) $AX \subset TX$ *and* $BX \subset SX$,

(b) Given $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all* $x, y \in X$,

$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon$

(c) $d(Ax, By) < max[k_1[d(Sx, Ty)+d(Ax, Sx)+d(By, Ty)], k_2\frac{d(Sx,By)+d(Ax,Ty)}{2}]$ *for* $0 \leq k_1 < 1, 1 \leq k_2 < 2$

If one of the mappings A,B,S and T is continuous then A,B,S and T have unique common fixed point.

We prove a common fixed point theorem for compatible and reciprocally continuous pairs of self-mappings in a complete metric space (X,d) with example. This theorem has been published in

Annals of Pure and Applied Mathematics Vol. 5, No. 2, 2014, 120-124 ISSN: 2279-087X (P), 2279-0888.

We start with the following proposition.

Proposition 2.0.7. *If A and S are compatible and reciprocal continuous mappings on a metric space (X, d). Then, we have A(t) = S(t) where* $\{x_n\}$ *is a sequence in X such that*

$$lim_{n\to\infty}Ax_n = lim_{n\to\infty}Sx_n = t$$

for some $t \in X$ *If there exist* $u \in X$ *such that* $Au = Su = t$*, then* $ASu = SAu$*.*

Proof (i) By definition of compatible mappings, we have

$$lim_{n\to\infty}SAx_n = lim_{n\to\infty}ASx_n.$$

Also, by definition of reciprocal continuous, we have

$$lim_{n\to\infty}ASx_n = A(t) \quad and \quad lim_{n\to\infty}SAx_n = S(t).$$

So, we get A(t) = S(t).

(ii)Suppose Au = Su = t for some $u \in X$. Then, we have ASu = At and SAu = St. But, from (i), we get At = St. So, we have ASu = SAu.

Lemma 2.0.8. *[62] Let A, B, S and T be self-mapping of metric space (X, d) such that*

$$AX \subset TX, BX \subset SX.$$

Also, assume further that given $\epsilon > 0$, there exists $\delta > 0$ such that for all x, y in X, we have

$$\epsilon < M(x,y) < \epsilon + \delta \Rightarrow d(Ax,By) \leq \epsilon, \quad \textit{and}$$

$$d(Ax,By) < M(x,y) \quad \textit{whenever} M(x,y) > 0$$

where M(x, y) = $max\Big\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{d(Sx,By)+d(Ax,Ty)}{2}\Big\}$. Then, for each $x_0 \in X$, the sequence $\{y_n\}$ in X defined by the rule $y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ is a Cauchy sequence.

Theorem 2.0.9. *[79] Let (A, S) and (B, T) be compatible and reciprocally continuous pairs of self-mappings in a complete metric space (X,d) such that*

(a) $AX \subset TX$ *and* $BX \subset SX$,

(b) Given $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, we have $\epsilon \leq M(x,y) < \epsilon + \delta \Rightarrow d(Ax,By) < \epsilon$, and $d(Ax,By) < M(x,y)$ where M(x, y) = max $\Big\{d(Sx,Ty), d(Ax,Sx), d(By,Ty), \frac{d(Sx,By)+d(Ax,Ty)}{2}\Big\}$, and

(c) $d(Ax,By) < max\,[k_1[d(Sx,Ty)+d(Ax,Sx)+d(By,Ty)], k_2\frac{d(Sx,By)+d(Ax,Ty)}{2}]$ *for* $0 \leq k_1, k_2 \leq 1$.

Then, the mappings A, B, S and T have unique common fixed point.

Proof: Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \qquad (2.1)$$

This can be down from (1).Then, by lemma (2.0.8), $\{y_n\}$ is a Cauchy Sequence. Also, since X is complete, so there exists a point z in X such that $y_n \to z$. Now, from (2.1), we get

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z, y_{n+1} = Bx_{2n+1} = Sx_{2n+2}. \qquad (2.2)$$

Since (A, S) is compatible and reciprocal continuous, using the proposition(2.0.7), we have

$$Az = Sz. \qquad (2.3)$$

We claim that Az = z. If $Az \neq z$, then, from (3), we get

$$d(Az, Bx_{2n+1}) < max[k_1 \{d(Sz, Tx_{2n+1}) + d(Az, Sz) + d(Bx_{2n+1}, Tx_{2n+1})\} k_2 \frac{d(Sz, Bx_{2n+1}) + d(Az, Tx_{2n+1})}{2}].$$

Letting $n \to \infty$,we get $d(Az, z) < d(Az, z)$, a contradiction. Hence, we get Az = z.
Therefore, we have Az = Sz = z.
Hence z be the common fixed point of A and S. Also, since

$AX \subset TX$ there exists a point w in X such that Az = Tw.
We claim that Bw = Tw. If $Bw \neq Tw$ from (3), we get

$$\begin{aligned} d(Tw, Bw) = d(Az, Bw) &< max[k_1 \{d(Sz, Tw) + d(Az, Sz) + d(Bw, Tw)\} \\ & k_2 \frac{d(Sz, Bw) + d(Az, Tw)}{2}] \end{aligned}$$

which implies $d(Az, Bw) < d(Az, Bw)$, a contradiction. So, we get Tw = Bw.
Hence,

$$Az = Sz = Tw = Bw = z. \tag{2.4}$$

Now, using the proposition(2.0.7), we get

$$BTw = TBw. \tag{2.5}$$

Moreover, we get BBw = BTw and TTw = TBw. Hence,

$$BBw = BTw = TTw = TBw. \tag{2.6}$$

Again, we claim BBw = Bw. If $BBw \neq Bw$. Then, from (2), we get

$$\begin{aligned} d(Bw, BBw) &= (Az, BBw) \\ &< max[d(Sz, TBw), d(Az, Sz), d(BBw, TBw), \\ & \frac{d(Sz, BBw) + (Az, TBw)}{2}] \\ &= max[d(Az, BBw), 0, 0, d(Az, BBw)] \end{aligned}$$

which implies $d(Az, BBw) < d(Az, BBw)$. This is a contradiction. Hence, we have Bw = BBw.
Now, from relation (2.6), we get BBw = TBw = Bw. So,

we have Bw = Az = z. Moreover, BBw = TBw implies that Bz = Tz = z which is the common fixed point of B and T. Hence z is the common fixed point of A, B, S and T.

For uniqueness of the common fixed point,
let $u \neq z$ is another fixed point. Then, we get Au = Su = Bu = Tu. Finally, using relation (3), we get

$$\begin{aligned} d(Az, Bu) &< max[k_1 \{d(Sz, Tu) + d(Az, Sz) + d(Bu, Tu)\}, \\ & k_2 \frac{d(Sz, Bu) + d(Az, Tu)}{2}]. \end{aligned}$$

This implies $d(Az, Bu) < d(Az, Bu)$ which is a contradiction. Hence, the common fixed point of A, B, S and T is unique. This establishes the theorem.

Example 2.0.10. *Let X = [2,10] and d the Euclidean metric on X. Define A, B, S and T : $X \to X$ as follows*
Ax = 2 for all x,
Bx = 2 if $x < 4$ and ≥ 5, Bx = 3+x if $4 \leq x < 5$,
Sx = x if $x \leq 8$, Sx = 8 if $x > 8$, and
Tx = 2 if $x < 4$ or ≥ 5, Tx = 5 +x if $4 \leq x < 5$. Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point x = 2.

If we take A = B and S = T. Then, the above theorem reduces to following corollary.

Corollary 2.0.11. *Let (A, S) be a reciprocal continuous* *and compatible self-mappings of a complete metric space(X, d) such that*

(a) $AX \subset SX$,

(b) *Given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x, y \in X$, *we have* $\epsilon \le M(x, y) < \epsilon + \delta \Rightarrow d(Ax, Ay) < \epsilon$ *with* $d(Ax, Ay) < M(x, y)$ *where*
$M(x,\ y) = max\left\{d(Sx, Sy), d(Ax, Sx), d(ay, Sy), \frac{d(Sx,Ay)+d(Ax,Sy)}{2}\right\}$

(c) $d(Ax, Ay) < max\left[k_1\left\{d(Sx, Sy) + d(Ax, Sx) + d(Ay, Sy)\right\}, k_2\frac{d(Sx,Ay)+d(Ax,y)}{2}\right]$ *for* $0 \le k_1, k_2 \le 1$.

Then, A and S have a unique common fixed point.

Remarks. The main theorem remains true for compatible of type (A), compatible of type (B) and compatible of type (C) in place of compatible if A, S, B and T are assumed to be continuous. Our result improves the result of Jha et al. [77, 78], Pant and Jha [140, 141], Pant [126] and other similar results in literature.

We introduce a new compatible mappings of type (K) and prove a Lemma and a fixed point theorem for pair of compatible mappings of type (K) on a metric space X with example. This theorem has been published in International J. of Math. Sci. and Engg. Appls. (IJMSEA) as follows:

International J. of Math. Sci. Engg. Appls. (I-JMSEA)ISSN 0973-9424, Vol. 8 No. I (January, 2014), pp. 383-391

We start with the following Lemma.

Lemma 2.0.12. *[82] Let A, B, S and T be a self mappings of a metric space (X, d) satisfying the following conditions:*

(a) $A(X) \subset T(X)$, $B(X) \subset S(X)$,

(b) Given $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all x, y in X.*
$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) \leq \epsilon$
and $d(Ax, By) < M(x, y)$,
where M(x, y) = Max [d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty)],

Then for each x_0 *in X, the sequence* $\{y_n\}$ *in X defined by the rule* $y_{2n-1} = Ax_{2n-2} = Tx_{2n-1}$*and*
$y_{2n} = Bx_{2n-1} = Sx_{2n}$ *, for all n = 1 ,2 ,... , is a Cauchy sequence.*

Proof: Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, so for any $x_0 \in X$, there exists $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this $x_1 \in X$, there exists $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we define a sequence $\{y_n\}$ in X such that
$y_{2n-1} = Ax_{2n-2} = Tx_{2n-1}$ and $y_{2n} = Bx_{2n-1} = Sx_{2n}$,

for all n = 1 ,2 ,...

From condition (2), we get

$$\begin{aligned} d(y_{2n+1}, y_{2n+2}) &= d(Ax_{2n}, Bx_{2n+1}) \\ &< Max[d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n}, Tx_{2n+1}), \\ &\quad d(Ax_{2n}, Tx_{2n+1})] \\ &= Max[d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\ &\quad d(y_{2n+1}, y_{2n+1})] \\ &< Max[d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2})], \end{aligned}$$

which implies

$$d(y_{2n+1}, y_{2n+2}) < d(y_{2n}, y_{2n+1}). \qquad (2.7)$$

Similarly, we have

$$d(y_{2n+2}, y_{2n+3}) < d(y_{2n+1}, y_{2n+2}). \qquad (2.8)$$

From (2.7) and (2.8) we have

$d(y_{n+1}, y_{n+2}) < (y_n, y_{n+1})$ and so on.

Thus, $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers and, therefore, tends to a limit $r \geq 0$. If possible suppose, $r > 0$. Then, for $\delta > 0$, there exists a positive number N such that for each $n \in N$ we have

$$r < d(y_{2n}, y_{2n+1}) = M(x, y) < r + \delta. \qquad (2.9)$$

Selecting δ in (2.9) in accordance with (2), we have

$d(y_{2n-1}, y_{2n}) < d(Ax_{2n-2}, Bx_{2n-1}) < r$, contradicting (2.9).

Hence, we get

$$lim_{n\to\infty}d(y_n, y_{n+1}) = 0. \tag{2.10}$$

We now show that $\{y_n\}$ is a Cauchy sequence. Since $d(y_{n+1}, y_{n+2}) \leq d(y_n, y_{n+1})$, it is sufficient to show that the sub-sequence $\{y_{2n}\}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exists $\epsilon > 0$ for which we can find subsequences $\left\{y_{2m(k)}\right\}$and $\left\{y_{2n(k)}\right\}$ of $\{y_{2n}\}$ such that n(k) is the least index for which $n(k) > m(k) > k$ and $d(y_{2m(k)}, y_{2n(k)}) \geq \epsilon$. This means that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \epsilon. \tag{2.11}$$

Using triangle inequality, we get

$$\begin{aligned}\epsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\ &\leq d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)1}, y_{2n(k)}) \\ &\leq \epsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)1}, y_{2n(k)}).\end{aligned}$$

Letting $k \to \infty$ and using (2.10) we conclude that

$$lim_{k\to\infty}d(y_{2m(k)}, y_{2n(k)}) = \epsilon. \tag{2.12}$$

Moreover, we have
$\left|d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)})\right| \leq d(y_{2n(k)}, y_{2n(k)+1})$.
Letting $k \to \infty$ and using (2.10) and (2.12),we get

$$lim_{k\to\infty}d(y_{2m(k)}, y_{2n(k)+1}) = \epsilon. \tag{2.13}$$

Similarly, we get

$$lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)+1}) = \epsilon \quad and \quad lim_{k\to\infty} d(y_{2m(k)+1}, y_{2n(k)+2}) = \epsilon. \tag{2.14}$$

Now, from (2), we have

$$\begin{aligned} d(y_{2m(k)+1}, y_{2n(k)+2}) &= d(Ax_{2m(k)}, Bx_{2n(k)+1}) \\ &< Max(d(Sx_{2m(k)}, Tx_{2n(k)+1}), d(Ax_{2m(k)}, Sx_{2m(k)}), \\ &\quad d(Bx_{2n(k)+1}, Tx_{2n(k)+1}), d(Ax_{2m(k)}, Tx_{2n(k)+1})) \\ &= Max((d(y_{2m(k)}, y_{2n(k)+1}), d(y_{2m(k)+1}, y_{2m(k)}), \\ &\quad d(y_{2n(k)+2}, y_{2n(k)+1}), d(y_{2m(k)+1}, y_{2n(k)+1})) \end{aligned}$$

If $k \to \infty$ then, we get $\epsilon < Max\{\epsilon, 0, 0, \}$, which is a contradiction. Hence, $\{y_n\}$ is a Cauchy sequence.

Theorem 2.0.13. *[82] Let (X, d) be a complete metric space and A, B, S and T be a self mappings of X satisfying the following conditions:*

(a) $A(X) \subset T(X), B(X) \subset S(X)$,

(b) Given $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all x, y in X,* $\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) \leq \epsilon$ *and* $d(Ax, By) < M(x, y)$
where $M(x, y) = Max\,[d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Ax, Ty)]$

(c) S and T are continuous.

If (A, S) and (B, T) compatible mappings of type (K), then A, B, S and T have a unique common fixed point.

Proof: Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, for any $x_0 \in X$, there exists $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this $x_1 \in X$, there exists $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we define a sequence $\{y_n\}$ in X such that
$y_{2n-1} = Ax_{2n-2} = Tx_{2n-1}$ and $y_{2n} = Bx_{2n-1} = Sx_{2n}$, for all $n = 1, 2, ...$
Then, from (2) and by Lemma(2.0.12) , we have $\{y_n\}$ is a Cauchy sequence in X. Since (X, d) is complete, $\{y_n\}$ converges to some point $z \in X$, and so that $\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z. Suppose (A, S) and (B T) are compatible of type (K), then we have

$$AAx_{2n-2} \to Sz, SSx_{2n} \to Az; BBx_{2n-1} \to Tz \quad and \quad TTx_{2n-1} \to Bz. \tag{2.15}$$

Also, from (2), we get
$d(AAx_{2n-2}, BBx_{2n-1}) < \text{Max}\,[d(SAx_{2n-2}, TBx_{2n-1}), d(AAx_{2n-2}, SAx_{2n-2}), d(BBx_{2n-1}, TBx_{2n-1}), d(AAx_{2n-2}, TBx_{2n-1})]$.
Taking limit as $n \to \infty$ and using (2.15), we have
$d(Sz, Tz) < \text{Max}\,\{d(Sz, Tz), d(Sz, Sz), d(Tz, Tz), d(Sz, Tz)\}$,
which implies $d(Sz, Tz) < d(Sz, Tz)$. It follows that

$$Sz = Tz. \tag{2.16}$$

Now, from (2), we get

$$d(Az, BBx_{2n-1}) \;<\; Max[d(Sz, TBx_{2n-1}), d(Az, Sz), d(Bz, TBx_{2n-1}) \\ d(Az, TBx_{2n-1})].$$

Again, taking limit as $n \to \infty$ and using (2.15) and (2.16), we have

$$\begin{aligned} d(Az, Tz) &< Max\{d(Sz, Sz), d(Az, Tz), d(Tz, Tz), d(Az, Tz)\} \\ &= d(Az, Tz). \end{aligned}$$

and hence

$$Az = Tz. \tag{2.17}$$

From (2), (2.16) and (2.17) we get,

$$\begin{aligned} d(Az, Bz) &< Max[d(Sz, Tz), d(Az, Sz), d(Bz, Tz), d(Az, Tz)] \\ &= Max[d(Az, Az), d(Az, Az), d(Bz, Az), d(Az, Az)] \\ &= d(Az, Bz) \end{aligned}$$

and hence

$$Az = Bz. \tag{2.18}$$

From (2.16), (2.17) and (2.18) we have

$$Az = Bz = Tz = Sz. \tag{2.19}$$

Now, we show that Az = z. From (2), we get

$$\begin{aligned} d(Az, Bx_{2n-1}) &< Max[d(Sz, Tx_{2n-1}), d(Az, Sz), d(Bx_{2n-1}, Tx_{2n-1}), \\ &\quad d(Az, Tx_{2n-1})]. \end{aligned}$$

And, taking limit as $n \to \infty$ and using (2.16) and (2.17), we have

$$\begin{aligned} d(Az, z) &< Max\{d(Sz, z), d(Az, Sz), d(z, z), d(Az, z)\} \\ &= Max\{d(Az, z), d(Az, Az), d(z, z), d(Az, z)\} \\ &= d(Az, z). \end{aligned}$$

Hence, we get Az = z. Thus, from (2.19), we have
z = Az = Bz = Tz = Sz and so z is a common fixed point of A, B, S and T.
In order to prove the uniqueness of fixed point, let w be another common fixed point of A, B, S and T. Then, we get Aw = Bw = Sw = Tw. Therefore, we have

$$\begin{aligned} d(z, w) &= d(Az, Bw) \\ &< Max\{d(Sz, Tw), d(Az, Sz), d(Bw, Tw), d(Az, Tw)\} \\ &= d(z, w). \end{aligned}$$

Hence, we get z = w. This completes the proof of the theorem.

Now, we have the following example.

Example 2.0.14. *Let X = [2, 10] with the metric d defined by* $d(x, y) = |x - y|$ *for all* $x, y \in X$. *Then, (X, d) is a complete metric space. Define A, B, S and* $T : X \to X$ *as follows:*
$Ax = 2$ *if* $x - 3, Ax = 3$ *if* $x > 3$;
$Bx = 2$ *if* $x - 5$, $Bx = 3$ *if* $x > 5$

$Sx = Tx = x$ for all $x \in X$, Then A, B, S and T satisfy all the conditions of the above Theorem and have a unique common fixed point $x = 2$.

If A = B and T = S in above theorem, then we get following corollary.

Corollary 2.0.15. *Let (X, d) be a complete metric space and A and S be a self mappings of X satisfying the following conditions:*

(a) $A(X) \subset S(X), Given \epsilon > 0$ *there exists* $\delta > 0$ *such that for all x, y in X,*
$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, Ay) \leq \epsilon$ *and*
$d(Ax, Ay) < M(x, y)$
where $M(x, y) = Max\{d(Sx, Sy), d(Ax, Sx), d(Ax, Sy), d(Ax, Sy)\}$,
for all $x, y \in X$, *and*

(b) S is continuous.

If (A, S) is compatible mappings of type (K), then A and S have a unique common fixed point.

Remarks: The main theorem remains true for compatible, compatible of type (A), compatible of type (B) and compatible of type (C) and compatible of type (P) in place of compatible of type (K) if A, S, B and T are assumed to be continuous. Also, our result improves other similar results in literature.

We prove a fixed point theorem for pair of compatible mappings of type (E) on a metric space X with example.This theorem has been accepted for the publication in **Applications and Applied Mathematics : An International Journal.**

We start with the following proposition.

Proposition 2.0.16. *[79] If A and S be compatible mappings of type (E) on a metric space X and if one of function is continuous .Then,*

(a) $A(t) = S(t)$ *and* $lim_{n\to\infty} AAx_n = lim_{n\to\infty} SSx_n = lim_{n\to\infty} ASx_n = lim_{n\to\infty} SAx_n$ *whenever* $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$, *for some point* $t \in X$

(b) If these exist $u \in X$ *such that* $Au = Su = t$ *then* $ASu = SAu$.

Proof:(a) Let $\{x_n\}$ be a sequence of X such that

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t,$$

for some point $t \in X$, by definition of compatible of type (E), we have

$$lim_{n\to\infty} AAx_n = ASx_n = S(t).$$

If A is a continuous mapping. Then, we get

$$lim_{n\to\infty}AAx_n = A(lim_{n\to\infty}Ax_n) = A(t).$$

So, we have A(t) = S(t).
Similarly, if S is continuous we get the same result. By definition of compatible of type (K), clearly

$$lim_{n\to\infty}AAx_n = lim_{n\to\infty}SSx_n = lim_{n\to\infty}ASx_n = lim_{n\to\infty}SAx_n.$$

(b)Suppose Au = Su =t for some $u \in X$. Then,

$$ASu = A(Su) = At \quad and \quad SAu = S(Au) = St. But At = St.$$

So, ASu = SAu, Hence proved.

We use the following lemma to prove our main result.

Lemma 2.0.17. *[78]Let A, B, S and T be self-mapping of metric space (X, d) such that*

$$AX \subset TX, BX \subset SX.$$

Assume further that given $\epsilon > 0$ there exists $\delta > 0$ such that for all x, y in X.

$$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, By) \le \epsilon \quad and \quad d(Ax, By) < M(x, y)$$

whenever $M(x, y) > 0$,where

$$M(x,\ y) = max\left\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), [\frac{d(Sx,By)+d(Ax,Ty)]}{2}\right\}$$

Then, for each $x_0 \in X$, the sequence $\{y_n\}$in X define by the rule $y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ is a Cauchy sequence.

Theorem 2.0.18. *[79] Let (A, S) and (B, T) be pairs of self-mappings compatible of type (E) of a complete metric space (X,d) such that*

(a) $AX \subset TX$ *and* $BX \subset SX$,

(b) Given $\epsilon > 0$ *there exists* $\delta > 0$ *such that for all*$x, y \in X$, $\epsilon < M(x, y), < \epsilon + \delta \Rightarrow d(Ax, By) < \epsilon$ *with* $d(Ax, By) < M(x, y)$
where M(x, y) = max $\{d(Sx, Ty), d(Ax, Sx), d(By, Ty),$ *If one of the mappings A, B, S and T is continuous then A, B, S and T have unique common fixed point.*

Proof: Let x_0 be any point in X. Define sequence $\{x_n\}$ and $\{y_n\}$in X given by the rule

$$y_{2n} = Ax_{2n} = Tx_{2n+1}; y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \quad (2.20)$$

This can be down from condition (1).
By lemma (2.0.17), $\{y_n\}$ is a Cauchy sequence. Also, since X is complete, so there exists a point z in X such that $y_n \to z$ and from relation (2.20), we get

$$y_{2n} = Ax_{2n} = Tx_{2n+1} \to z \quad and \quad y_{n+1} = Bx_{2n+1} = Sx_{2n+2}. \quad (2.21)$$

Suppose S is continuous. Since, (A, S) is compatible of type (E), then, by proposition (2.0.16), we get

$$Az = Sz. \quad (2.22)$$

We claim that Az = z. If $Az \neq z$ then using condition (3), we get

$$d(Az, Bx_{2n+1}) < max[k_1 \{d(Sz, Tx_{2n+1}) + d(Az, Sz) + d(Bx_{2n+1}, Tx_{2n+1})\} \\ k_2 \frac{d(Sz, x_{2n+1}) + d(Az, Tx_{2n+1})}{2}].$$

Letting $n \to \infty$ we get $d(Az, z) < d(Az, z)$, which is a contradiction. Hence Az = z.So, we have Az = Sz = z.

Hence z is the common fixed point of A and S.

Also, since $AX \subset TX$ there exist a point w in X such that Az = Tw. We claim that Bw = Tw. If $Bw \neq Tw$. Then, using condition (3), we get

$$d(Tw, Bw) = d(Az, Bw) < max[k_1[d(Sz, Tw) + d(Az, Sz) + d(Bw, Tw)], \\ k_2 \frac{d(Sz, Bw) + d(Az, Tw)}{2}],$$

which implies $d(Az, Bw) < d(Az, Bw)$, a contradiction. Hence, we get Tw = Bw. Therefore, we get

$$Az = Sz = Tw = Bw = z. \tag{2.23}$$

Now, using the above proposition(2.0.16), we get

$$BTw = TBw. \tag{2.24}$$

Moreover, we get

$$BBw = TTw = BTw = TBw. \tag{2.25}$$

Again, we claim BBw = Bw. If $BBw \neq Bw$, from using condition (2), we get

$$\begin{aligned} d(Bw, BBw) &= d(Az, BBw) \\ &< max[d(Sz, TBw), d(Az, Sz), d(BBw, TBw), \\ &\quad \frac{d(Sz, BBw) + (Az, TBw)}{2}] \\ &= max[d(Az, BBw), 0, 0, d(Az, BBw)], \end{aligned}$$

which implies $(Az, BBw) < d(Az, BBw)$ which is a contradiction. Hence, we get Bw = BBw.
Now, from relation (2.25), we get BBw = TBw = Bw.
Hence, we get Bw = Az = z. So, z is the common fixed point of B and T. Hence z is the common fixed point of A, B, S and T.

For uniqueness

Let $u \neq z$ is another fixed point. Then, we get
Au = Su = Bu = Tu. Now from relation (3), we get

$$\begin{aligned} d(Az, Bu) &< max[k_1 \{d(Sz, Tu) + d(Az, Sz) + d(Bu, Tu)\}, \\ &\quad k_2 \frac{d(Sz, Bu) + d(Az, Tu)}{2}]. \end{aligned}$$

This implies $d(Az, Bu) < d(Az, Bu)$ which is a contradiction. Hence fixed point is unique. Also, in the proof of theorem we can choose function A is continuous in place of S. Moreover, the proof is similar when T or B is assumed to be continuous in place of S.This establishes the theorem.

Example 2.0.19. *Let X = [2, 10] be a set and d the*

Euclidean metric on X. Define A, B, S and $T : X \to X$ *as follows;*

$Ax = 2$ *for all* x ,

$Bx = 2$ *if* $x < 4$ *and* ≥ 5 $Bx = 3 + x$ *if* $4 \leq x < 5$

$Sx = x$ *if* $x \leq 8$, $Sx = 8$ *if* $x > 8$;

$Tx = 2$ *if* $x < 4$ *or* $\geq 5, Tx = 5 + x$ *if* $4 \leq x < 5$ *Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point* $x = 2$.

If A = B and S = T.Then, we have the following corollary:

Corollary 2.0.20. *Let (A, S) be a pairs of the self-mappings of compatible of type (E) of a complete metric space(X, d) such that*

(a) $AX \subset SX$,

(b) Given $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x, y \in X$,
$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(Ax, Ay) < \epsilon$,
with $d(Ax, Ay) < M(x, y)$, *where*
$M(x, y) = \max\left\{d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), \frac{d(Sx,Ay)+d(Ax,Sy)}{2}\right\}$,

(c) $d(Ax, Ay) < \max[k_1[\{d(Sx, Sy) + d(Ax, Sx) + d(Ay, Sy)\}$,

$k_2 \frac{d(Sx,Ay)+d(Ax,Sy)}{2}]$ *for* $0 \leq k_1, k_2 \leq 1$

If one of the mapping A and S is continuous then, A and S have a unique common fixed point.

Remarks The main theorem remain true for compatible maps and also true for compatible of type (A), compatible of type (B) and compatible of type (C) if A, S, B and T continuous. Our result improves the result of [77, 78], Pant and Jha [141] and other similar results in literature.

Chapter 3

Common fixed point theorems in fuzzy metric space

In this chapter, we study the some common fixed point theorems in fuzzy metric space, extended the notion of compatible of type (K) and the compatible of type (E) in fuzzy metric space and obtain a common fixed point theorems on complete fuzzy metric space with example. Result generalizes and improves other similar results in literature. The theorems which has been established and published peer reviewed international journals.

3.0.4 Introduction

In 1965, the concept of fuzzy set was introduced by Zadeh[200]. Then, fuzzy metric spaces have been introduced by Kramosil and Michalek [102]. George and Veeramani [49] modified the notion of fuzzy metric spaces with the help of continuous t-norms. In 1986, G. Jungck [86] introduced notion of compatible mappings in metric space. In 1998, Y.J. Cho, H.K. Pathak, S.M. Kang and J.S. Jung [30] introduced the concept of compatible mappings in fuzzy metric space.We have extended compatible mappings of type (E) in fuzzy metric space and established a common fixed point theorem with example.Recently, We have [82] introduced the concept of compatible mappings of type (K) in metric space and extended it to fuzzy metric space and obtained a common fixed point theorem.

3.0.5 Basic definition

Definition 3.0.21. *[49] A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a* **continuous t-norm** *if* $*$ *is satisfying the following conditions:*
(a) $*$ *is commutative and associative;*
(b) $*$ *is continuous;*
(c) $a * 1 = a$ *for all* $a \in [0,1]$*;*
(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$*, and* $a, b, c, d \in [0,1]$.

Definition 3.0.22. *[49] A 3-tuple* $(X, M, *)$ *is said to be a* **fuzzy metric space** *if* X *is an arbitrary set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0, \infty)$ *satisfying* *the following conditions: for all* $x, y, z \in X$ *and* $s, t > 0$,

(FM1) $M(x, y, t) > 0$;

(FM 2) $M(x, y, t) = 1$ *if and only if* $x = y$;

(FM 3) $M(x, y, t) = M(y, x, t)$;

(FM 4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(FM 5) $M(x, y, .) : (0, \infty) \to (0, 1]$ *is continuous.*

Then M is called a fuzzy metric on X. *The function* $M(x, y, t)$ *denote the degree of nearness between* x *and* y *with respect to* t. *Also, we consider the following condition in the fuzzy metric spaces* $(X, M, *)$.

(FM6) $t \overset{lim}{\to} \infty M(x, y, t) = 1$, *for all* $x, y \in X$.

Example 3.0.23. *[49] Let* (X, d) *be a metric space. Denote* $a * b = ab$ *for all* $a, b \in [0, 1]$ *and let* M *be fuzzy set on* $X^2 \times (0, \infty)$ *defined as follows:*
$M(x, y, t) = \frac{t}{t+d(x,y)}$. *Then* $(X, M, *)$ *is a fuzzy metric space. We call this fuzzy metric induced by a metric d is the standard fuzzy metric.*

Definition 3.0.24. *[49] Let* $(X, M, *)$ *be a fuzzy metric space. Then a sequence* $\{x_n\}$ *in* X *is said to be convergent to* x *in* X *if for each* $\epsilon > 0$ *and each* $t > 0$, *there exist* $n_0 \in N$ *such that* $M(x_n, x, t) > 1 - \epsilon$ *for all* $n \geq n_0$.
a sequence $\{x_n\}$ *in* X *is said to be Cauchy if for each* $\epsilon > 0$ *and each* $t > 0$, *there exists* $n_0 \in N$ *such that* $M(x_n, x_m, t) > 1 - \epsilon$ *for all* $n, m \geq n_0$. *A fuzzy metric space in which every Cauchy sequence is convergent is said*

to be complete.

Definition 3.0.25. *[30] The self mappings A and S of a fuzzy metric space* $(X, M, *)$ *are said to be* **compatible** *iff* $\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1$ *whenever* $\{x_n\}$ *is a sequence in X such that*

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x.$$

for some $xinX$ *and* $t > 0$.

Definition 3.0.26. *The self mappings A and S of a fuzzy metric space (X, M, *) are said to be* **compatible of type (E)** *iff*
$lim_{n\to\infty} M(AAx_n, ASx_n, t) = lim_{n\to\infty} M(AAx_n, Sx, t) =$
$lim_{n\to\infty} M(ASx_n, Sx, t) = 1$ *and* $lim_{n\to\infty} M(SSx_n, SAx_n, t) =$

$lim_{n\to\infty} M(SSx_n, Ax, t) = lim_{n\to\infty} M(SAx_n, Ax, t) = 1,$
whenever $\{x_n\}$ *is a sequence in X such that*

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x$$

for some x in X and $t > 0$.

Definition 3.0.27. *[82] The self mappings A and S of a fuzzy metric space (X, M, *) are said to be* **compatible of type (K)** *iff*

$$lim_{n\to\infty} M(AAx_n, Sx, t) = 1 \quad and \quad lim_{n\to\infty} M(SSx_n, Ax, t) = 1,$$

whenever $\{x_n\}$ *is a sequence in X such that*
$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x$ *for some x in X and* $t > 0$.

Lemma 3.0.28. *[106] In a fuzzy metric space (X, M, *), if* $a * a \geq a$ *for all* $a \in [0,1]$ *then* $a * b = min\ \{a, b\}$ *for all* $a, b \in [0,1]$.

Lemma 3.0.29. *[177] Let (X, M, *) be a fuzzy metric space with the condition: (FM6)* $lim_{n\to\infty} M(x, y, t) = 1$ *for all* $x, y \in X$. *If there exists* $k \in (0,1)$ *such that* $M(x, y, kt) \geq M(x, y, t)$ *then* $x = y$.

Lemma 3.0.30. *[30] Let* $\{y_n\}$ *be a sequence in a fuzzy metric space (X, M, *) with the condition (FM6). If there exists* $k \in (0,1)$ *such that* $M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ *for all* $t > 0$ *and* $n \in N$, *then* $\{y_n\}$ *is a Cauchy sequence in X.*

Lemma 3.0.31. *[52] Let* $(X, M, *)$ *be a fuzzy metric s-pace. Then for all* x, y *in* X, $M(x, y, .)$ *is non-decreasing.*

Lemma 3.0.32. *[28] Let* $(X, M, *)$ *be a fuzzy metric space. If there exists* $q \in (0,1)$ *such that* $M(x, y, qt) \geq M(x, y, t)$ *for all* x, y *and* $t > 0$ *then* $x = y$.

Lemma 3.0.33. *[100] The only t-norm* $*$ *satisfying* $r * r \geq r$ *for all* $r \in [0,1]$ *is the minimum t-norm, that is,* $a * b = min\ \{a, b\}$ *for all* $a, b \in [0,1]$.

Proposition 3.0.34. *If A and S be compatible mappings of type (E) on a fuzzy metric space (X, M, *) and if one of function is continuous .Then,*

(a) $A\ (x) = S(x)$ *and* $lim_{n\to\infty} AAx_n = lim_{n\to\infty} SSx_n = lim_{n\to\infty} ASx_n = lim_{n\to\infty} SAx_n$ *where* $lim_{n\to\infty} Ax_n = x$,$lim_{n\to\infty} Sx_n = x$, *for some point* $x \in X$ *and sequence* $\{x_n\}$.

(b) *If there exist* $u \in X$ *such that* $Au = Su = x$ *then* $ASu = SAu$.

Proof : *Let* $\{x_n\}$ *be a sequence of X such that*

$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x$ *for some x in X.*

Then by definition of compatible of type (E), we have

$$lim_{n\to\infty} AAx_n = ASx_n = S(x).$$

If A is a continuous mapping, then we get

$$lim_{n\to\infty} AAx_n = A lim_{n\to\infty} Ax_n = A(x).$$

This implies $A(x) = S(x)$ *and*
$lim_{n\to\infty} AAx_n = lim_{n\to\infty} SSx_n = lim_{n\to\infty} ASx_n = lim_{n\to\infty} SAx_n$
.
Similarly, if S is continuous, then we get the same result. This is the proof of part (a)
Again, suppose $Au = Su = x$ *for some* $u \in X$.
Then, $ASu = A(Su) = At$ *and* $SAu = S(Au) = St$.
From (a), we have $At = St$. *Hence,* $ASu = SAu$.
This is the proof of part (b).

3.0.6 Common Fixed Point Theorems for pair of self mappings in complete fuzzy metric space

In 2002, P. Balasubramaniam et al.[10] established following common fixed point theorem in complete fuzzy metric space.

Theorem 3.0.35. *[10] Let (A, S) and (B, T) be pointwise R- weakly commuting pairs of self-mappings of complete fuzzy metric space (X, M,*) such that*

i. $AX \subset TX$ *and* $BX \subset SX$,

ii. $M(Ax, By, t) \geq M(x, y, ht), 0 < h < 1, x, y \in X$ *and* $t > 0$

Suppose that (A,S) or (B,T) is compatible pair of reciprocal continuous mappings. Then A,B,S and T have unique common fixed point.

In 2004, R.P.Pant and K. Jha [142] established following theorem in complete fuzzy metric space using pointwise R- weakly commuting pairs of self-mappings.

Theorem 3.0.36. *[142] Let (A, S) and (B, T) be pointwise R- weakly commuting pairs of self-mappings of complete fuzzy metric space (X, M,*) such that*

i. $AX \subset TX$ *and* $BX \subset SX$,

ii. $M(Ax, By, t) \geq M(x, y, ht), 0 < h < 1, x, y \in X$ *and* $t > 0$

Let (A,S) and (B,T) be compatible mappings.If any of the mappings in compatible pair (A,S) and (B,T) is continuous. Then A,B,S and T have unique common fixed point.

In 2007, S. Kutukcu et al. [107] established following theorem in complete fuzzy metric space.

Theorem 3.0.37. *[107] Let (X, M, *) be a complete fuzzy metric space with $a * a \geq a$ for all $a \in [0,1]$ and with the condition (FM 6). Let (A, S) and (B, T) be pointwise R- weakly commuting pair self mapping of X such that*

i. $A(X) \subset T(X)$, $B(X) \subset S(X)$,

ii. there exists $k \in (0,1)$ *such that* $M(Ax, By, kt) \geq N(x, y, t)$ *for all* $x, y \in X$,$\alpha \in (0,2)$ *and* $t > 0$,*where* $N(x, y, t) = M(Sx, Ax, t) * M(Ty, By, t) * M(Sx, Ty, t) * M(Ty, Ax, \alpha t) * M(Sx, By, (2-\alpha)t)$,

If one of the mappings in compatible pair (A, S) or (B, T) continuous then A, B, S and T have a unique common fixed point.

In 2012, M. Koireng and Y. Rohen [101] established following theorem in complete fuzzy metric space using compatible mappings of type (P).

Theorem 3.0.38. *[101] Let $(X, M, *)$ be a complete fuzzy metric space and A, B, S and T be a self mappings of X satisfying the following conditions:*

i. $A(X) \subset T(X), B(X) \subset S(X)$,

ii. *S and T are continuous*

iii. *the pair (A, S) and (B, T) compatible mappings of type (P)*

iv. $M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Bx, Ty, t) * M(Ax, Ty, t)$,
for all $x, y \in X$, $k \in (0, 1)$ *and* $t > 0$,

Then A, B, S and T have a unique common fixed point.

We have established a common fixed point theorem for two pairs of self mappings using compatible mappings of type (E) in fuzzy metric space with example. This theorem has been published in

Applied Mathematical Sciences, Vol. 8 (2014), pp.2007 - 2014.

If A, B, S and T are self mappings in fuzzy metric space (X, M, *), we denote

$$\begin{aligned} M_\alpha(x, y, t) &= M(Sx, Ax, t) * M(Ty, By, t) * M(Sx, Ty, t) \\ &\quad * M(Ty, Ax, \alpha t) * M(Sx, By, (2 - \alpha)t), \end{aligned}$$

for all $x, y \in X$,$\alpha \in (0, 2)$ and $t > 0$.

Theorem 3.0.39. *Let (X, M, *) be a complete fuzzy metric space with $a * a \geq a$ for all $a \in [0, 1]$ and with the condition (FM 6). Let one of the mapping of self mappings (A, S) and (B, T) of X be continuous such that*

i. $A(X) \subset T(X)$*,* $B(X) \subset S(X)$*,*

ii. there exists $k \in (0, 1)$ *such that* $M(Ax, By, kt) \geq M_\alpha(x, y, t)$ *for all* $x, y \in X$*,*$\alpha \in (0, 2)$ *and* $t > 0$*.*

If (A, S) and (B, T) compatible of type of (E) then A, B, S and T have a unique common fixed point.

Proof Let x_0 be any point in X. From condition (1), there exists $x_1, x_2 \in X$ such that
$Ax_0 = Tx_1 = y_0$ and $Bx_1 = Sx_2 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $Ax_{2n} = Tx_{2n+1} = y_{2n}$ and $Bx_{2n+1} = Sx_{2n+2} = y_{2n+1}$ for n = 0, 1, 2, ... For $t > 0$ and $\alpha = 1 - q$ with $q \in (0, 1)$

in (2), Then, we have

$$\begin{aligned}
M(Ax_{2n}, Bx_{2n+1}, kt) &\geq M(Sx_{2n}, Ax_{2n}, t) * M(Tx_{2n+1}, Bx_{2n+1}, t) \\
&\quad * M(Sx_{2n}, Tx_{2n+1}, t) * M(Tx_{2n+1}, Ax_{2n}, \\
&\quad (1-q)t) * M(Sx_{2n}, Bx_{2n+1}, (1+q)t), \\
M(y_{2n}, y_{2n+1}, kt) &\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) \\
&\quad * M(y_{2n-1}, y_{2n}, t) * M(y_{2n-1}, y_{2n+1}, (1+q)t) \\
&\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) \\
&\quad * M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, qt) \\
&\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) \\
&\quad * M(y_{2n}, y_{2n+1}, qt).
\end{aligned}$$

Since t-norm * is continuous, letting $q \to 1$, we have

$$\begin{aligned}
M(y_{2n}, y_{2n+1}, kt) &\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t) \\
&\quad M(y_{2n}, y_{2n+1}, t) \\
&\geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t).
\end{aligned}$$

It follows that $M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t) * M(y_{2n}, y_{2n+1}, t)$.
Similarly, $M(y_{2n+1}, y_{2n+2}, kt) * M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t)$.
Therefore, for all n even or odd, we have
$M(y_n, y_{n+1}, kt) * M(y_{n-1}, y_n, t) * M(y_n, y_{n+1}, t)$.
Consequently, $M(y_n, y_{n+1}, t) * M(y_{n-1}, y_n, k^{-1}t) * M(y_n, y_{n+1}, k^{-1}t)$
and hence $M(y_n, y_{n+1}, t) * M(y_{n-1}, y_n, t) * M(y_n, y_{n+1}, k^{-1}t)$.

Since $M(y_n, y_{n+1}, k^{-m}t) \to 1$ as $k \to 0$, it follows that

$M(y_n, y_{n+1}, kt) \geq M(y_{n-1}, y_n, t)$ for all $n \in N$ and $t > 0$. Therefore, by Lemma (3.0.30), $\{y_n\}$ is a Cauchy sequence. Since X is complete, then there exists a point z in X such that $y_n \to z$ as $n \to \infty$. Moreover, we have $y_{2n} = Ax_{2n} = Tx_{2n+1} \to z$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2} \to z$. If A and S are compatible of type (E) and one of mapping of the pair (A, S) is continuous then by Lemma (3.0.31) we have Az = Sz. Since $AX \subset TX$, there exists a point w in X such that Az = Tw. Using condition (2), with $\alpha = 1$, we have

$$\begin{aligned} M(Az, Bw, kt) &\geq M(Sz, Az, t) * M(Tw, Bw, t) * M(Sz, \\ &\quad Tw, t) * M(Tw, Az, t) * M(Sz, Bw, t) \\ &= M(Az, Az, t) * M(Az, Bw, t) * M(Az, Az, t) \\ &\quad * M(Az, Az, t) * M(Az, Bw, t) \\ &\geq M(Az, Bw, t). \end{aligned}$$

This implies Az = Bw. Thus, we have Az = Sz = Bw = Tw. Also, we get

$$\begin{aligned} M(Az, Bx_{2n+1}, kt) &\geq M(Sz, Az, t) * M(T_{2n+1}, Bx_{2n+1}, t) * \\ &\quad M(Sz, Tx_{2n+1}, t) * M(Tx_{2n+1}, Az, t) \\ &\quad * M(Sz, Bx_{2n+1}, t). \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} M(Az, z, kt) &\geq M(Az, Az, t) * M(Az, z, t) * M(Az, Az, t) * \\ &\quad M(Az, Az, t) * M(Az, z, t) \\ &\geq M(Az, z, t). \end{aligned}$$

Hence, we get Sz = Az =z. Therefore, z is common fixed point of A and S.

Again, if B and T are compatible of type (E) and one of mappings of (B, T) is continuous, so we get Bw =Tw = Az = z. By using proposition (3.0.34), we get

BBw = BTw = TBw = TTw. Thus, we get Bz = Tz.

Also, we get

$$\begin{aligned} M(Ax_{2n}, Bz, kt) &\geq M(Sx_{2n}, Ax_{2n}, t) * M(Tz, Bz, t) * M(Sx_{2n}, Tz, t) \\ &\quad * M(Tz, Ax_{2n}, t) * M(Sx_{2n}, Bz, t). \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} M(z, Bz, kt) &\geq M(z, z, t) * M(Tz, Bz, t) * M(z, Tz, t) \\ &\quad * M(Tz, z, t) * M(z, Bz, t) \\ &\geq M(z, Bz, t). \end{aligned}$$

Hence, we have Bz = Tz =z. Therefore z is common fixed point of B and T. Hence z is common fixed point of A, B, S and T.

For uniqueness,

suppose that $Aw(Az \neq z)$ is another common fixed point of A, B, S and T. Then, using condition (2) with $\alpha = 1$, we have

$$
\begin{aligned}
M(AAz, BAw, kt) &= M(Az, Aw, kt) \\
&\geq M(SAz, AAz, t) * M(TAw, BAw, t) * M(SAz, \\
&\quad TAw, t) * M(TAw, AAz, t) * M(SAz, BAw, t) \\
&= M(Az, Az, t) * M(Aw, Aw, t) * M(Az, Aw, t) \\
&\quad M(Aw, Az, t) * M(Az, Aw, t) \\
&\geq M(Az, Aw, t).
\end{aligned}
$$

That is, $Aw = Az = z$. Thus, z is a unique common fixed point of A, B, S and T.

We have the following example.

Example 3.0.40. *Let X = [2, 10] with the metric d defined by $d(x, y) = |x - y|$ and define $M(x, y, t) = \frac{t}{t+d(x,y)}$ for all $x, y \in X$, $t > 0$. Clearly (X, M, *) is a complete fuzzy metric space. Define A, B, S and T : X$\rightarrow$ $X as follows$;*

$Ax = 2 for all x$,

$Bx = 2 if x < 4 and \geq$ 5 *Bx = 3+x if 4*$\leq x < 5$

Sx = x if $x \leq$ 8, Sx = 8 if $x >$ 8;

Tx = 2 if $x <$ 4 or $\geq$ 5, Tx = 5 +x if 4 $\leq x <$ 5 Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point x = 2.

If we take $S = T = I_X$, an identity mapping of X in Theorem , we get the following result.

Corollary 3.0.41. *Let (X, M, *) be a complete fuzzy metric space with $a * a \geq a$ for all $a \in [0, 1]$ and with the condition (FM6).*
If one of the mapping of self mappings (A, B) of X is continuous such that for $k \in (0, 1)$ we have

$$M(Ax, By, kt) \geq N(x, y, t) \quad for \quad all x, y \in X, \alpha \in (0, 2)$$

and $t > 0$, and if (A, B) is compatible of type of (E) then A and B have a unique common fixed point.

Remarks: The main theorem remains true if (A, S) and (B, T) are pointwise R- weakly commuting pairs and one of the mappings (A, B) or (S, T) is continuous and true for compatible, compatible of type (A) and compatible of type (P) in place of compatible of type (E) if A, S, B and T are assumed to be continuous. Also, our result extend and generalize the results of Pant and Jha[16], Singh and Singh [10,11] and S. Kutukcu et al.[18] and improves other similar results in literature.

Now, we prove a common fixed point theorem for two pairs compatible mappings of type (K) in complete fuzzy metric space with example. This theorem has been published in

Electronic J. Math. Analysis and Appl,(2)(2014), 248-253.

Theorem 3.0.42. *[112] Let $(X, M, *)$ be a complete fuzzy metric space and A, B, S and T be a self mappings of X satisfying the following conditions:*

i. $A(X) \subset T(X), B(X) \subset S(X)$,

ii. $M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t) * M(Bx, Ty, t) * M(Ax, Ty, t)$,
for all $x, y \in X$, $k \in (0, 1)$ *and* $t > 0$, *and*

iii. S *and* T *are continuous.*

If (A, S) and (B, T) compatible of type of (K), then A, B, S and T have a unique common fixed point.

Proof: Since $A(X) \subset T(X)$ and $B(X) \subset S(X)$, so for any $x_0 \in X$, there exists $x_1 \in X$ such that $Ax_0 = Tx_1$ and for this x_1, there exists $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we define a sequence $\{y_n\}$ in X such that $y_{2n-1} = Ax_{2n-2} = Tx_{2n-1}$ and $y_{2n} = Bx_{2n-1} = Sx_{2n}$, for all $n = 1, 2, ...$

From (b), we get

$$\begin{aligned} M(y_{2n+1}, y_{2n+2}, kt) &= M(Ax_{2n}, Bx_{2n+1}, kt) \\ &\geq M(Sx_{2n}, Tx_{2n+1}, t) * M(Ax_{2n}, Sx_{2n}, t) \\ &\quad * M(Bx_{2n}, Tx_{2n+1}, t) * M(Ax_{2n}, Tx_{2n+1}, t) \\ &= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) \\ &\quad * M(y_{2n+2}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t) \\ &\geq M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+2}, t) \end{aligned}$$

From lemma (3.0.31) and (3.0.33) we have

$$M(y_{2n+1}, y_{2n+2}, kt) \geq M(y_{2n}, y_{2n+1}, t) \qquad (3.1)$$

Similarly, we have

$$M(y_{2n+2}, y_{2n+3}, kt) \geq M(y_{2n+1}, y_{2n+2}, t) \qquad (3.2)$$

From (3.1) and (3.2), we have

$$M(y_{n+1}, y_{n+2}, kt) \geq M(y_n, y_{n+1}, t) \qquad (3.3)$$

From (3.3), we have $M(y_{n+1}, y_{n+2}, t) \geq M(y_n, y_{n+1}, \frac{t}{k}) \geq M(y_{n-1}, y_n, \frac{t}{k^2}) \geq ... \geq M(y_1, y_2, \frac{t}{k^n}) \to 1 as n \to \infty$.
So, $M(y_n, y_{n+1}, t) \to 1 as n \to \infty$ for any $t > 0$. For each $\epsilon > 0$ and each $t > 0$, we can choose $n_0 \in N$ such that $M(y_n, y_{n+1}, t) > 1 - \epsilon for all n > n_0$. For $m, n \in N$, we suppose $m \geq n$. Then, we have that

$$\begin{aligned} M(y_n, y_m, t) &\geq M(y_n, y_{n+1}, \frac{t}{m-n}) * M(y_{n+1}, y_{n+2}, \frac{t}{m-n}) * ... \\ &\quad * M(y_{m-1}, y_m, \frac{t}{m-n}) \\ &\geq (1-\epsilon) * (1-\epsilon) * ...(m-n) \end{aligned}$$

times. This implies
$M(y_n, y_m, t) \geq (1-\epsilon)$ and hence $\{y_n\}$ is a Cauchy sequence in X.
Since $(X, M, *)$ is complete, $\{y_n\}$ converges to some point $z \in X$, and so that $\{Ax_{2n-2}\}, \{Sx_{2n}\}, \{Bx_{2n-1}\}$ and $\{Tx_{2n-1}\}$ also converges to z. Since (A, S) and (B, T) are compatible of type (K), we have

$$AAx_{2n-2} \to Sz, SSx_{2n} \to Az, BBx_{2n-1} \to Tz, TTx_{2n-1} \to Bz \tag{3.4}$$

From (b), we get
$M(AAx_{2n-2}, BBx_{2n-1}, kt) \geq M(SAx_{2n-2}, TBx_{2n-1}, t)*$

$M(AAx_{2n-2}, SAx_{2n-2}, t) * M(BBx_{2n-1}, TBx_{2n-1}, t)$
$* M(AAx_{2n-2}, TBx_{2n-1,t})$
Taking limit as $n \to \infty$ and using (3.4), we have
$M(Sz, Tz, kt) \geq M(Sz, Tz, t)*M(Sz, Sz, t)*M(Tz, Tz, t)*$
$M(Sz, Tz, t)$
$(Sz, Tz, t) * 1 * 1 * M(Sz, Tz, t) \geq M(Sz, Tz, t)$.
It follows that

$$Sz = Tz \tag{3.5}$$

Now, from (b), we get

$$M(Az, BBx_{2n-1}, kt) \;\geq\; M(Sz, TBx_{2n-1}, t) * M(Az, Sz, t)$$
$$*M(Bz, TBx_{2n-1}, t) * M(Az, TBx_{2n-1}, t)$$

Again, taking limit as $n \to \infty$ and using (3.4) and

(3.5), we have

$$\begin{aligned} M(Az,Tz,kt) &\geq M(Sz,Sz,t) * M(Az,Tz,t) \\ &\quad * M(Tz,Tz,t) * M(Az,Tz,t) \\ &\geq M(Az,Tz,t). \end{aligned}$$

and hence

$$Az = Tz \tag{3.6}$$

From (b), (3.5) and (3.6), we get

$$\begin{aligned} M(Az,Bz,kt) &\geq M(Sz,Tz,t) * M(Az,Sz,t) \\ &\quad * M(Bz,Tz,t) * M(Az,Tz,t) \\ &= M(Az,Az,t) * M(Az,Az,t) \\ &\quad * M(Bz,Az,t) * M(Az,Az,t) \\ &\geq M(Az,Bz,t). \end{aligned}$$

and hence

$$Az = Bz \tag{3.7}$$

From (3.5),(3.6) and (3.7), we have

$$Az = Bz = Tz = Sz \tag{3.8}$$

Now, we show that $Bz = z$. From (b), we get

$$\begin{aligned} M(Ax_{2n},Bz,kt) &\geq M(Sx_{2n},Tz,t) * M(Ax_{2n},Sx_{2n},t) \\ &\quad * M(Bz,Tz,t) * M(Ax_{2n},Tz,t). \end{aligned}$$

And, taking limit as $n \to \infty$ and using (3.5) and (3.4), we have

$$\begin{aligned} M(z, Bz, kt) &\geq M(z, Tz, t) * M(z, z, t) \\ &\quad *M(Bz, Tz, t) * M(z, Tz, t) \\ &= M(z, Bz, t) * 1 * M(Az, Az, t) * M(z, Bz, t) \\ &\geq M(z, Bz, t). \end{aligned}$$

And hence $Bz = z$. Thus from (3.8), we get $z = Az = Bz = Tz = Sz$ and so z is a common fixed point of A, B, S and T.
In order to prove the uniqueness of fixed point, let w be another common fixed point of A, B, S and T. Then, $Aw = Bw = Sw = Tw$, therefore, using (ii), we get

$$\begin{aligned} M(z, w, kt) &= M(Az, Bw, kt) \\ &\geq M(Sz, Tw, t) * M(Az, Sz, t) \\ &\quad *M(Bw, Tw, t) * M(Az, Tw, t) \\ &\geq M(z, w, t). \end{aligned}$$

From Lemma (3.0.33), we get $z = w$. This completes the proof of theorem.

We have the following example.

Example 3.0.43. *Let* $X = [2, 10]$ *with the metric* d *defined by* $d(x, y) = |x - y|$ *and define* $M(x, y, t) = \frac{t}{t+d(x,y)}$ *for all* $x, y \in X, t > 0$. *Clearly* $(X, M, *)$ *is a complete fuzzy metric space. Define* A, B, S *and* $T : X \to X$ *as follows:* $Ax = 2$ *if* $x \leq 3, Ax = 3$ *if* $x > 3$; $Bx = 2$ *if* $x \leq 5, Bx = 3$ *if* $x > 5$ *and* $Sx, Tx = x$ *for all* $x \in X$,
Then A, B, S *and* T *satisfy all the conditions of the above theorem and have a unique common fixed point* $x = 2$.

If $A = B$ and $T = S$ in above theorem, then we get following result.

Corollary 3.0.44. *Let* $(X, M, *)$ *be a complete fuzzy metric space and* A *and* S *be a self mappings of* X *satisfying the following conditions:*
(i) $A(X) \subset S(X)$,
(ii) $M(Ax, Ay, kt) \geq M(Sx, Sy, t)*M(Ax, Sx, t)*M(Ax, Sy, t)* M(Ax, Sy, t)$,
for all $x, y \in X, k \in (0, 1)$ *and* $t > 0$, *and*
(iii) S *is continuous. If* (A, S) *compatible of type of (K), then* A *and* S *have a unique common fixed point.*

Remarks. Our result extends and generalizes the results of Cho[28], Koireng and Rohon[101] and Jha et al. [82]. Also, our result improves other similar results in literature.

Chapter 4

Common fixed point theorems in intuitionistic fuzzy metric space

In this chapter, we introduce the notion of compatible of type (K) in intuitionistic fuzzy metric space and obtain some common fixed point theorems in complete intuitionistic fuzzy metric space with example. Our result in intuitionistic fuzzy metric space generalizes and improves other similar results in literature.Also, the established results has been published in peer reviewed international journals.

4.0.7 Introduction

In 1986, the concept of intuitionistic fuzzy set was introduced by K. Atanassov [5] as a generalization of

fuzzy set. In 2004, the intuitionistic fuzzy metric spaces have been introduced by J.H. Park [147] with the help of continuous t-norm and continuous t- conorm as a generalization of fuzzy metric space.
Recently, K. Jha et al. [82] introduced the concept of compatible mappings of type (K) in metric space and Manandhar et al. [112] extended compatible mappings of type (K) in fuzzy metric space.
we have following basic definitions.

4.0.8 Basic definition

Definition 4.0.45. *[172] A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a* **continuous t-norm** *if* $*$ *is satisfying the following conditions:*

(a) $*$ *is commutative and associative;*

(b) $*$ *is continuous;*

(c) $a * 1 = a$ *for all* $a \in [0,1]$;

(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$*, and* $a, b, c, d \in [0,1]$.

Definition 4.0.46. *[172] A binary operation* $\diamond : [0,1]\times[0,1] \to [0,1]$ *is a* **continuous t-conorm**,*if it satisfies the following conditions:*

i. $\diamond$ *is commutative and associative;*

ii. $\diamond$ *is continuous;*

iii. $a \diamond 0 = a$ *for all* $a \in [0,1]$;

iv. $a \diamond b \leq c \diamond d$ *whenever* $a \geq c$ *and* $b \geq d$, *for each* $a, b, c, d \in [0, 1]$.

Definition 4.0.47. *[4] A 5-tuple* $(X, M, N, *, \diamond)$ *is said to be an* **intuitionistic fuzzy metric space** *(shortly IFM-Space) if* X *is an arbitrary set,* $*$ *is a continuous t-norm,* $\diamond$ *is a continuous t-conorm and M, N are fuzzy sets on* $X^2 \times (0, \infty)$ *satisfying the following conditions: for all* $x, y, z \in X$ *and* $s, t > 0$;

(IFM-1) $M(x, y, t) + N(x, y, t) \leq 1$;

(IFM-2) $M(x, y, 0) = 0$;

(IFM-3) $M(x, y, t) = 1$ *if and only if* $x = y$;

(IFM-4) $M(x, y, t) = M(y, x, t)$;

(IFM-5) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(IFM-6) $M(x, y, .): [0, \infty) \to [0, 1]$ *is left continuous;*

(IFM-7) $lim_{t \to \infty} M(x, y, t) = 1$

(IFM-8) $N(x, y, 0) = 1$;

(IFM-9) $N(x, y, t) = 0$ *if and only if* $x = y$;

(IFM-10) $N(x, y, t) = N(y, x, t)$;

(IFM-11) $N(x, y, t) \diamond N(y, z, s) \geq N(x, z, t + s)$;

(IFM-12) $N(x, y, .): [0, \infty) \to [0, 1]$ *is right continuous;*

(IFM-13) $lim_{t \to \infty} N(x, y, t) = 0$.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions M(x, y, t) and N(x, y, t) denote the degree of nearness and degree of non-nearness between *x and y with respect to t, respectively.*

Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space if X of the form $(X, M, 1 - M, *, \diamond)$ such that t- norm $*$ and t-conorm $\diamond$ are associated, that is, x $\diamond$ y =1-((1 - x) $*$ (1 - y)) for any $x, y \in X$. But the converse is not true [196].

Example 4.0.48. *[102] Let (X, d) be a metric space. We define a$*$ b $=$ ab and a $\diamond$ b $=$ min$\{1, a + b\}$ for all a, b $\in$ [0, 1] and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:*
$M_d(x, y, t) = \frac{t}{t+d(x,y)}$, *and* $N_d(x, y, t) = \frac{d(x,y)}{t+d(x,y)}$.
Then (M_d, N_d) is an intuitionistic fuzzy metric on X. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

It is notated that the above example holds even with the t-norm a $*$ b =min $\{a, b\}$ and the t-conorm a$\diamond$b = max$\{a, b\}$ and hence (M_d, N_d) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t- conorm.

Definition 4.0.49. *[29] Let $(X, M, *)$ be a fuzzy metric space and $\epsilon > 0$. A finite sequence $x = x_0, x_1, ..., x_n = y$ is called*
$\epsilon -$ chainable from x to y if $M(x_i, x_{i-1}, t) > 1 - \epsilon$ for all $t > 0$ and i = 1, 2, 3, . . ., n.

Definition 4.0.50. *[29] A fuzzy metric space (X, M, *) is called* ϵ **- chainable** *if for* $x, y \in X$ *there exists an* ϵ *-chain from x to y.*

Definition 4.0.51. *[4] Let (X, M, N,* $*$ *,*$\diamond$*) be an intuitionistic fuzzy metric space.*

i. *A sequence* $\{x_n\}$ *in X is called Cauchy sequence if for each* $t > 0$ *and* $p > 0$,
   $lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ *and*
   $lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$.

ii. *A sequence* $\{x_n\}$ *in X is convergent to* $x \in X$ *if*
   $limn\to \infty M(x_n, x, t) = 1$ *and*
   $lim_{n\to\infty} N(x_n, x, t) = 0$ *for each* $t > 0$.

iii. *An intuitionistic fuzzy metric space is said to be complete if every Cauchy sequence is convergent.*

Definition 4.0.52. *[196] Let A and B be two self maps on a intuitionistic fuzzy metric space (X, M,N,* $*$*,*$\diamond$ *) then A is called* **B - absorbing** *if there exists a positive integer* $R > 0$ *such that*
M (B x,BAx, t) $\geq$ *M* $(Bx, Ax, \frac{t}{R})$ *and*
N(Bx,BAx, t) $\leq$ *N* $(Bx, Ax, \frac{t}{R})$ *for all* $x \in X$.
Similarly B is called A- absorbing if there exists a positive integer $R > 0$ *such that M(Ax, ABx, t)* $\geq$ *M* $(Ax, Bx, \frac{t}{R})$ *and*
N(Ax,ABx, t) $\leq$ *N* $(Ax, Bx, \frac{t}{R})$ *for all* $x \in X$

Lemma 4.0.53. *[178] Let (X, M, N,* $*$ *,*$\diamond$*) be an*

intuitionistic fuzzy metric space. If there exists a constant $k \in$ *(0, 1) such that,*
$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), and$
$N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t)$ *for every* $t > 0$ *and* $n =$ *1, 2,. . . then* $\{y_n\}$ *is a Cauchy sequence in* X *.*

Lemma 4.0.54. *[177] Let (X, M, *) be a fuzzy metric space with the condition: (FM6)* $lim_{n\to\infty} M(x, y, t) = 1$ *for all* $x, y \in X$*. If there exists* $k \in (0, 1)$ *such that* $M(x, y, kt) \geq M(x, y, t)$ *then* $x = y$*.*

Proposition 4.0.55. *If A and S be compatible mappings of type (K) on a intuitionistic fuzzy metric space (X, M, N, ∗ ,⋄) and if one of function is continuous. Then, we have*

i. A(x) = S(x) where $lim_{n\to\infty} Ax_n = x, lim_{n\to\infty} = x$*, for some point* $x \in X$ *and sequence* $\{x_n\}$*,*

ii. If these exist $u \in X$ *such that* $Au = Su = x$ *then* $ASu = SAu$*.*

Proof : Let $\{x_n\}$ be a sequence of X such that

$$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = x$$

for some x in X. Then, by the definition of compatible of type (K),
we have $lim_{n\to\infty} AAx_n = S(x)$.
If A is a continuous mapping, then we get
$lim_{n\to\infty} AAx_n = Alim_{n\to\infty} Ax_n =$ A(x).

This implies A(x) = S(x). Similarly, if S is continuous then, we get the same result. This is the proof of part (a).
Again, suppose Au = Su = x for some u $\in$ X. Then, ASu =A(Su) = Ax and SAu = S(Au) = Sx. From (a), we have Ax = Sx. Hence, we get ASu = SAu.
This is the proof of part (b).

4.0.9 Common Fixed Point Theorems for pair of mappings in intuitionistic fuzzy metric spaces

In 2006, S.H. Cho and Ji Hong Jung [29] established the following theorems for pair of compatible mappings in ϵ chanable fuzzy metric space.

Theorem 4.0.56. *[29] Let (X,M,*) be a ϵ - chanable fuzzy metric space and A, B, S and T be self maps of X satisfying the following condition:*

i. $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,
ii. *the pairs (A, S) and (B, T) weakly compatible,*
iii. *If A and S are continuous,*
iv. *there exist* $q \in (0,1)$ *such that*
   $M(Ax, By, t) \geq M(Sx, Ty, t) * M(Ax, Sx, t)$
   $*M(By, Ty, t) * M(Ax, Ty, t)$
   for every x, y in X and $t > 0$.

Then A, B, S and T have a unique common fixed point *in X.*

In 2010, S. Manro et al.[114] established the following theorem for pair of mapping in ϵ - chanable intuitionistic fuzzy metric spaces.

Theorem 4.0.57. *[114] Let A, B, S and T be self maps of a complete ϵ chanable intuitionistic fuzzy metric spaces (X, M, N, $*$,$\diamond$) with continuous t-norm * and continuous t-conorm $\diamond$ defined by $a * a \geq a$ and $(1-a) \diamond (1-a) \leq (1-a)$ for all $a \in [0,1]$ satisfying the following condition:*

i. $A(X) \subset T(X)$ *and* $B(X) \subset S(X)$,

ii. If A and S are continuous.

iii. the pairs (A, S) and (B, T) are weakly compatible,

iv. there exist $k \in (0,1)$ *such that*
$M(Ax, By, kt) \geq M(Sx, Ty, t) * M(Ax, Sx, t)$
$*M(By, Ty, t) * M(Ax, Ty, t)$ *and*
$N(Ax, By, t) \leq N(Sx, Ty, t) \diamond N(Ax, Sx, t)$
$\diamond N(By, Ty, t) \diamond N(Ax, Ty, t)$,
for every x, y in X and $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

In 2012, M. Verma and R.S. Chandel [196] established following theorem for absorbing mappings in complete Intuitionistic fuzzy metric space.

Theorem 4.0.58. *Let A be S- absorbing and B be T-absorbing self mapping on a complete Intuitionistic fuzzy metric space*
(X, M, N, * *,* $\diamond$*) with contineous t- norm defined by a*b = min*$\{a, b\}$ *where* $a, b \in [0, 1]$ *satisfying the conditions:*

i. $A(X) \subset TX), B(X) \subset S(X)$.

ii. *there exists* $k \in (0, 1)$ *such that for every* $x, y \in X$ *and* $t > 0$,

$$\begin{aligned} M(Ax, By, kt) &\geq min[M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(Ax, By, t), M(Sx, By, t)], and \\ N(Ax, By, kt) &\leq max[N(Sx, Ty, t), N(Ax, Sx, t), N(By, Ty, t), \\ &\quad N(Ax, Ty, t), N(Ax, By, t), N(Sx, By, t)]. \end{aligned}$$

iii. *for all* $x, y \in X$, $lim_t \to \infty$ *M(x, y, t)=1, and* $lim_t \to \infty$ *N(x,y,t)=1.*

If the pair of mappings (A, S) is reciprocal continuous mappings, then A, B, S and T have a unique common fixed point in X.

We have introduce compatible mappings of type (K) in Intuitionistic fuzzy metric space and proved the following common fixed point theorem with example. This theorem has been published in

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and Standards Vol. 3 No. 2 (2014), pp. 81-87.

Theorem 4.0.59. *[113] If A, B, S and T are self mapping on a complete Intuitionistic fuzzy metric space (X, M, N, * , ◇), satisfying the conditions:*

i. $A(X) \subset TX), B(X) \subset S(X)$.

ii. there exists $k \in (0, 1)$ *such that for every* $x, y \in X$ *and* $t > 0$,

$$\begin{aligned} M(Ax, By, kt) &\geq min[M(Sx, Ty, t), M(Ax, Sx, t), M(By, Ty, t), \\ &\quad M(Ax, Ty, t), M(Ax, By, t), M(Sx, By, t)], and \\ N(Ax, By, kt) &\leq max[N(Sx, Ty, t), N(Ax, Sx, t), N(By, Ty, t), \\ &\quad N(Ax, Ty, t), N(Ax, By, t), N(Sx, By, t)]. \end{aligned}$$

iii. B and T are weakly compatible mappings.

If the pair of mappings (A, S) is compatible of type (K) and one of the mapping is continuous then A, B, S and T have a unique common fixed point in X.

Proof: let x_0 be any arbitrary point in X, Construct a sequence $\{y_n\}$ in X such that $y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n+1}$, for n = 1, 2, 3 . . . This can be done by (1). By using contractive condition (2), we obtain

$$
\begin{aligned}
M(y_{2n+1}, y_{2n+2}, kt) &= M(Ax_{2n}, Bx_{2n+1}, kt) \\
&\geq min[M(Sx_{2n}, Tx_{2n+1}, t), M(Ax_{2n}, Sx_{2n}, t), \\
&\quad M(Bx_{2n+1}, Tx_{2n+1}, t), M(Ax_{2n}, Tx_{2n+1}, t), \\
&\quad M(Ax_{2n}, Bx_{2n+1}, t), M(Sx_{2n}, Bx_{2n+1}, t)] \\
&= min[M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), \\
&\quad M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n+1}, t), \\
&\quad M(y_{2n+1}, y_{2n}, t), M(y_{2n}, y_{2n}, t)] \\
&= min[M(y_{2n}, y_{2n+1}, t), M(y_{2n+1}, y_{2n}, t), \\
&\quad M(y_{2n}, y_{2n+1}, t), 1, M(y_{2n+1}, y_{2n}, t), 1] \\
&= M(y_{2n}, y_{2n+1}, t), That \quad is \\
M(y_{2n+1}, y_{2n+2}, kt) &\geq M(y_{2n}, y_{2n+1}, t),
\end{aligned}
$$

Similarly, we have $M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$,
So, we get

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), \qquad (4.1)$$

Also, we get

$$\begin{aligned}
N(y_{2n+1}, y_{2n+2}, kt) &= N(Ax_{2n}, Bx_{2n+1}, kt) \\
&\leq max[N(Sx_{2n}, Tx_{2n+1}, t), N(Ax_{2n}, Sx_{2n}, t), \\
&\quad N(Bx_{2n+1}, Tx_{2n+1}, t), N(Ax_{2n}, Tx_{2n+1}, t), \\
&\quad N(Ax_{2n}, Bx_{2n+1}, t), N(Sx_{2n}, Bx_{2n+1}, t)] \\
&\leq max[N(y_{2n}, y_{2n+1}, t), N(y_{2n+1}, y_{2n}, t), \\
&\quad N(y_{2n}, y_{2n+1}, t), N(y_{2n+1}, y_{2n+1}, t), \\
&\quad N(y_{2n+1}, y_{2n}, t), N(y_{2n}, y_{2n}, t)] \\
&\leq max[N(y_{2n}, y_{2n+1}, t), N(y_{2n+1}, y_{2n}, t), \\
&\quad N(y_{2n}, y_{2n+1}, t), 0, N(y_{2n+1}, y_{2n}, t), 0] \\
&= N(y_{2n}, y_{2n+1}, t), \\
so, we have & \\
N(y_{2n+1}, y_{2n+2}, kt) &\leq N(y_{2n}, y_{2n+1}, t).
\end{aligned}$$

Similarly, we have $N(y_{2n}, y_{2n+1}, kt) \leq (y_{2n-1}, y_{2n}, t)$.
So, we get

$$N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t). \qquad (4.2)$$

From (4.1), (4.2) and using Lemma (4.0.53), we get that $\{y_n\}$ is a Cauchy sequence in X.
But (X, M, N, * , ⋄) is complete so there exists a point z in X such that $\{y_n\} \to z$.
Also, we have $\{Ax_{2n-2}\}, \{Tx_{2n-1}\}, \{Sx_{2n}\}, \{Bx_{2n+1}\} \to z$.
Since, (A, S) is compatible of type (K) and one of the

mapping is continuous. So, using proposition (4.0.55), we get

$$Az = Sz. \tag{4.3}$$

Since $A(X) \subset T(X)$ then there exists a point u in X such that Az = Tu. Now, by contractive condition (2), we get

$$\begin{aligned} M(Az, Bu, kt) &\geq min[M(Sz, Tu, t), M(Az, Sz, t), M(Bu, Tu, t), \\ &\quad M(Az, Tu, t), M(Az, Bu, t), M(Sz, Bu, t)] \\ &= min[M(Az, Az, t), M(Az, Az, t), M(Bu, Az, t), \\ &\quad M(Az, Az, t), M(Az, Au, t), M(Az, Bu, t)] \end{aligned}$$

So, we have

$$M(Az, Bu, kt) \geq M(Az, Bu, t). \tag{4.4}$$

Also, we have

$$\begin{aligned} N(Az, Bu, kt) &\leq max[N(Sz, Tu, t), N(Az, Sz, t), N(Bu, Tu, t), \\ &\quad N(Az, Tu, t), N(Az, Bu, t), N(Sz, Bu, t] \\ &= max[N(Az, Az, t), N(Az, Az, t), N(Bu, Az, t), \\ &\quad N(Az, Az, t), N(Az, Au, t), N(Az, Bu, t)] \end{aligned}$$

So, we have

$$N(Az, Bu, kt) \leq N(Az, Bu, t) \tag{4.5}$$

Now, from (4.4), (4.5) and Lemma(4.0.53), we get Az = Bu. Thus, we get

$$Az = Sz = Bu = Tu. \tag{4.6}$$

To prove Az = z, we have

$$\begin{aligned} M(Az, Bx_{2n+1}, kt) \ &\geq \ min[M(Sz, Tx_{2n+1}, t), M(Az, Sz, t), \\ &\quad M(Bx_{2n+1}, Tx_{2n+1}, t), M(Az, Tx_{2n+1}, t), \\ &\quad M(Az, Bx_{2n+1}, t), M(Sz, Bx_{2n+1}, t)]. \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} M(Az, z, kt) \ &\geq \ min[M(Sz, z, t), M(Az, Sz, t), M(z, z, t), \\ &\quad M(Az, z, t), M(Az, z, t), M(Sz, z, t)] \\ &= \ min[M(Az, z, t), 1, 1, M(Az, z, t), \\ &\quad M(Az, z, t), M(Az, z, t)], \end{aligned}$$

that is,

$$M(Az, z, kt) \geq M(Az, z, t). \qquad (4.7)$$

Also, we have

$$\begin{aligned} N(Az, Bx_{2n+1}, kt) \ &\leq \ max[N(Sz, Tx_{2n+1}, t), N(Az, Sz, t), \\ &\quad N(Bx_{2n+1}, Tx_{2n+1}, t), \\ &\quad N(Az, Tx_{2n+1}, t), N(Az, Bx_{2n+1}, t), \\ &\quad N(Sz, Bx_{2n+1}, t)] \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} N(Az, z, kt) \ &\leq \ max[N(Sz, z, t), N(Az, Sz, t), N(z, z, t), \\ &\quad N(Az, z, t), N(Az, z, t), N(Sz, z, t)] \\ &= \ max[N(Az, z, t), 0, 0, N(Az, z, t), \\ &\quad N(Az, z, t), N(Az, z, t)] \end{aligned}$$

that is,

$$N(Az, z, kt) \leq N(Az, z, t). \tag{4.8}$$

Now, from (4.7), (4.7) and Lemma (4.0.53), we get, Az = z.

Hence, we have

$$Az = Sz = z. \tag{4.9}$$

So, z is a common fixed point of A and S. Also, we get

$$Bu = Tu = z. \tag{4.10}$$

Since B and T are weakly compatible, we have TBu =BTu. So, from (4.6), we get

$$Tz = Bz. \tag{4.11}$$

Again, we get

$$\begin{aligned} M(Ax_{2n-2}, Bz, kt) \geq \; & min[M(Sx_{2n-2}, Tz, t), M(Ax_{2n-2}, Sx_{2n-2}, t), \\ & M(Bz, Tz, t), M(Ax_{2n-2}, Tz, t) \\ & M(Ax_{2n-2}, Bz, t), M(Sx_{2n-2}, Bz, t)]. \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$\begin{aligned} M(z, Bz, kt) &\geq min[M(z, Tz, t), M(z, z, t), M(Bz, Tz, t), \\ & \quad M(z, Tz, t)M(z, Bz, t), M(z, Bz, t)] \\ &= min[M(z, Bz, t), 1, 1, M(z, Bz, t), \\ & \quad M(z, Bz, t), M(z, Bz, t)], \end{aligned}$$

that is,

$$M(z, Bz, kt) \geq M(z, Bz, t). \qquad (4.12)$$

Also, we have

$$\begin{aligned} N(Ax_{2n-2}, Bz, kt) &\leq max[N(Sx_{2n-2}, Tz, t), N(Ax_{2n-2}, Sx_{2n-2}, t), \\ &\quad N(Bz, Tz, t), N(Ax_{2n-2}, Tz, t), \\ &\quad N(Ax_{2n-2}, Bz, t), N(Sx_{2n-2}, Bz, t)] \end{aligned}$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} N(z, Bz, kt) &\leq max[N(z, Tz, t), N(z, z, t), N(Bz, Tz, t), \\ &\quad N(z, Tz, t), N(z, Bz, t), N(z, Bz, t)] \\ &= max[N(z, Bz, t), 1, 1, M(z, Bz, t), \end{aligned}$$

$N(z, Bz, t), N(z, Bz, t)]$,

that is,

$$N(z, Bz, kt) \leq N(z, Bz, t). \qquad (4.13)$$

Now, from (4.12), (4.13) and Lemma (4.0.53), we get Bz = z. Therefore, we have

$$Tz = Bz = z. \qquad (4.14)$$

Hence, we get z is a common fixed point of B and T. From (4.9) and (4.14), we get Az = Sz = Bz =Tz = z. So z is a common fixed point of A, B, S, and T.

For uniqueness, Let w be the another common fixed($w \neq$

z) point then Aw= Bw = Sw = Pw = w.

Using contractive condition (2), we get

$$\begin{aligned} M(Az, Bw, kt) &\geq min[M(Sz, Tw, t), M(Az, Sz, t), M(Bw, Tw, t), \\ &\quad M(Az, Tw, t), M(Az, Bw, t), M(Sz, Bw, t)] \\ &= min.[M(Az, Bw, t), 1, 1, M(Az, Aw, t), \\ &\quad M(Az, Bw, t), M(Az, Bw, t)], \end{aligned}$$

that is,

$$M(Az, Bw, kt) \geq M(Az, Bw, t). \tag{4.15}$$

Also, we have

$$\begin{aligned} N(Az, Bw, kt) &\leq max.[N(Sz, Tw, t), N(Az, Sz, t), N(Bw, Tw, t), \\ &\quad N(Az, Tw, t), N(Az, Bw, t), N(Sz, Bw, t)] \\ &= max.[N(Az, Bw, t), 0, 0, N(Az, Aw, t), \\ &\quad N(Az, Bw, t), N(Az, Bw, t)], \end{aligned}$$

that is,

$$N(Az, Bw, kt) \leq (Az, Bw, t). \tag{4.16}$$

From (4.15), (4.15) and Lemma (4.0.53), we get Az = Bw, this implies Az = Aw and hence z is a unique fixed point.

We have the following example.

Example 4.0.60. *Let (X, M, N, * ,◇) be a intuitionistic fuzzy metric, where X = [2, 20]*

M(x, y, t) $= \frac{t}{t+d(x,y)}$, *N(x, y , t)* $= \frac{d(x,y)}{t+d(x,y)}$ *and d is the Euclidean metric on X.*

Define A, B, S and T $: X \to X$ *as follows;*

Ax = 2 for all x;

Bx = 2 if $x < 4$ *and* $\geq$ *5 Bx = 3 + x if* $4 \leq x < 5$*;*

Sx = if $x \leq 8$, *Sx = 8 if* $x > 8$*;*

Tx = 2 if $x < 4$ *or* $\geq$ *5, Tx = 5 + x if* $4 \leq x < 5$.

Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point x = 2.

Remarks: Our result (4.0.59) is also true for the pair of mappings (A, S) is compatible of type (E) in place of compatible mapping of type (K). Our result extends and generalizes the results of M. Verma and R.S. Chandel [196],K. Jha et al.[82], K.B. Manandhar et al.[111] and improve the result of Manandhar et al.[112]. Also, our result improves other similar results in literature.

Now, we prove a common fixed point theorem for compatible mappings of type (K) in complete Intuitionistic Fuzzy Metric space.This theorem has been accepted for publication in

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Theorem 4.0.61. *Let A, B, S and T be self maps of a* *complete intuitionistic fuzzy metric spaces (X, M, N, $*$,$\diamond$) with continuous t-norm * and continuous t-conorm $\diamond$ defined by $a * a \geq a$ and $a \diamond a \leq a$ for all $a \in [0, 1]$ satisfying the following condition:*

i. *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

ii. *the pairs (A, S) and (B, T) are compatible mappings of type (K),*

iii. *If A, S and one of the mapping of pair (B, T)is continuous.*

iv. *there exist $k \in (0, 1)$ such that*
*M(Ax, By, kt) $\geq$ M(Sx, Ty, t) * M(Ax, Sx, t)*
**M(By, Ty, t) * M(Ax, Ty, t) and*
N(Ax, By, t) $\leq$ N(Sx, Ty, t) $\diamond$ N (Ax, Sx, t)
$\diamond$N(By, Ty, t) $\diamond$ N(Ax, Ty, t),
for every x, y in X and $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

Proof: As A(X) $\subset$ T(X), for any $x_0 \in X$, there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1, we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$. Inductively, we can find a sequence $\{y_n\}$ in X as follows:
$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for n = 1, 2,...This can be done by (1). By using contrac-

tive condition (4), we obtain

$$\begin{aligned}
M(y_{2n+1}, y_{2n+2}, kt) &= M(Ax_{2n}, Bx_{2n+1}, kt) \\
&\geq M(Sx_{2n}, Tx_{2n+1}, t) * M(Ax_{2n}, Sx_{2n}, t) * \\
&\quad M(Bx_{2n+1}, Tx_{2n+1}, t) * M(Ax_{2n}, Tx_{2n+1}, t) \\
&\quad *M(Ax_{2n}, Bx_{2n+1}, t) * M(Sx_{2n}, Bx_{2n+1}, t) \\
&= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * \\
&\quad M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n+1}, t) * \\
&\quad M(y_{2n+1}, y_{2n}, t) * M(y_{2n}, y_{2n}, t) \\
&= M(y_{2n}, y_{2n+1}, t) * M(y_{2n+1}, y_{2n}, t) * \\
&\quad M(y_{2n}, y_{2n+1}, t) * 1 * M(y_{2n+1}, y_{2n}, t) * 1 \\
&= M(y_{2n}, y_{2n+1}, t), \\
That \quad is' & \\
M(y_{2n+1}, y_{2n+2}, kt) &\geq M(y_{2n}, y_{2n+1}, t).
\end{aligned}$$

Similarly, we have $M(y_{2n}, y_{2n+1}, kt) \geq M(y_{2n-1}, y_{2n}, t)$.,
So, we get

$$M(y_{n+2}, y_{n+1}, kt) \geq M(y_{n+1}, y_n, t), \qquad (4.17)$$

Also, we get

$$\begin{aligned}
N(y_{2n+1}, y_{2n+2}, kt) &= N(Ax_{2n}, Bx_{2n+1}, kt) \\
&\leq N(Sx_{2n}, Tx_{2n+1}, t) \diamond N(Ax_{2n}, Sx_{2n}, t) \diamond \\
&\quad N(Bx_{2n+1}x_{2n+1}, t)(Ax_{2n}, Tx_{2n+1}, t) \diamond \\
&\quad N(Ax_{2n}, Bx_{2n+1}, t) \diamond N(Sx_{2n}, Bx_{2n+1}, t) \\
&\leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n}, t) \\
&\quad \diamond N(y_{2n}, y_{2n+1}, t)(y_{2n+1}, y_{2n+1}, t) \diamond \\
&\quad N(y_{2n+1}, y_{2n}, t) \diamond N(y_{2n}, y_{2n}, t) \\
&\leq N(y_{2n}, y_{2n+1}, t) \diamond N(y_{2n+1}, y_{2n}, t) \diamond \\
&\quad N(y_{2n}, y_{2n+1}, t) \diamond 0 \diamond N(y_{2n+1}, y_{2n}, t) \diamond 0 \\
&= N(y_{2n}, y_{2n+1}, t),
\end{aligned}$$

so we have,

$$N(y_{2n+1}, y_{2n+2}, kt) \leq N(y_{2n}, y_{2n+1}, t).$$

Similarly, we have $N(y_{2n}, y_{2n+1}, kt) \leq (y_{2n-1}, y_{2n}, t)$,
So, we get

$$N(y_{n+2}, y_{n+1}, kt) \leq N(y_{n+1}, y_n, t). \tag{4.18}$$

From (4.17), (4.18) and Lemma (4.0.53), we get that the $\{y_n\}$ is a Cauchy sequence in X.
Since X is complete, therefore sequence $\{y_n\}$ in X converges to z for some z in X and so the sequences $\{Tx_{2n-1}\}$, $\{Ax_{2n-2}\}$, $\{Sx_{2n}\}$ and $\{Bx_{2n-1}\}$ also converges to z. Since pair (A, S) is compatible mappings of type (K), we

get

$$Az = Sz. \tag{4.19}$$

From (2), we have $ASx_{2n} \to Az$ and from (4.19) $ASx_{2n} \to Sz$. Also, from continuity of S, we have, $SSx_{2n} \to Sz$.
From (4), we get

$$\begin{aligned} M(ASx_{2n}, Bx_{2n-1}, kt) &\geq M(SSx_{2n}, Tx_{2n-1}, t) * M(ASx_{2n}, SSx_{2n}, t) \\ &\quad * M(Bx_{2n-1}, Tx_{2n-1}, t) * M(ASx_{2n}, Tx_{2n-1}, t) \\ N(ASx_{2n}, Bx_{2n-1}, kt) &\leq N(SSx_{2n}, Tx_{2n-1}, t) \diamond N(ASx_{2n}, SSx_{2n}, t) \\ &\quad \diamond N(Bx_{2n-1}, Tx_{2n-1}, t) \diamond N(ASx_{2n}, Tx_{2n-1}, t) \end{aligned}$$

Proceeding limit as $n \to \infty$, we have
M(Sz, z, kt) $\geq$ M(Sz, z, t) * M(Sz, Sz, t) * M(z, z, t) * M(Sz, z, t) and
$N(Sz, z, kt) \leq$ N(Sz, z, t) $\diamond N(Sz, Sz, t) \diamond N(z, z, t) \diamond N(Sz, z, t)$. From Lemma (4.0.54), we get Sz = z, and hence from (4.19)

$$Az = Sz = z. \tag{4.20}$$

Since (B, T) is compatible mappings of type (K)and one of the mapping is continuous,we get,

$$Tz = Bz. \tag{4.21}$$

From (4),

$$\begin{aligned} M(Ax_{2n}, Bz, kt) &\geq M(Sx_{2n}, Tz, t) * M(Ax_{2n}, Sx_{2n}, t) \\ &* M(Bz, Tz, t) * M(Ax_{2n}, Tz, t) \quad and \\ N(Ax_{2n}, Bz, kt) &\leq N(Sx_{2n}, Tz, t) \diamond N(Ax_{2n}, Sx_{2n}, t) \diamond \\ & N(Bz, Tz, t) \diamond N(Ax_{2n}, Tz, t), \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} M(z, Bz, kt) &\geq M(z, Tz, t) * M(z, z, t) * M(Bz, Tz, t) * M(z, Tz, t) \\ &= M(z, Bz, t) * M(z, z, t) * M(Bz, Bz, t) * M(z, Bz, t) \\ &\geq M(z, Bz, t) and \\ N(z, Bv, kt) &\leq N(z, Tz, t) \diamond N(z, z, t) \diamond N(Bz, Tz, t) \diamond N(z, Tz, t) \\ &= N(z, Bz, t) \diamond N(z, z, t) \diamond N(Bz, Bz, t) \diamond N(z, Bz, t) \\ &\leq N(z, Bz, t), \end{aligned}$$

which implies that Bz = z .From (4.19), (4.20), (4.21) therefore, Az = Sz = Bz = Tz = z. Hence A, B, S and T have common fixed point z in X.

For uniqueness, let w be another common fixed point of A, B, S and T.Then

$$\begin{aligned}
M(z,w,kt) &= M(Az,Bw,qt)\\
&\geq M(Sz,Tw,t)*M(Az,Sz,t)\\
&\quad *M(Bw,Tw,t)*M(Az,Tw,t)\\
&\geq M(z,w,t) \quad and\\
N(z,w,kt) &= N(Az,Bw,qt)\\
&\leq N(Sz,Tw,t)\diamond N(Az,Sz,t)\diamond\\
&\quad N(Bw,Tw,t)\diamond N(Az,Tw,t)\\
&\leq N(z,w,t).
\end{aligned}$$

From Lemma (4.0.54), we conclude that z = w. Hence A, B,S and T have unique common fixed point z in X. We have the following example.

Example 4.0.62. *Let (X, M, N, * ,$\diamond$) be a intuitionistic fuzzy metric, where X = [2, 20]*
M(x, y, t) = $\frac{t}{t+d(x,y)}$, *N(x, y , t) =* $\frac{d(x,y)}{t+d(x,y)}$ *and d is the Euclidean metric on X.*
Define A, B, S and T : $X \to X$ *as follows;*
Ax = 2 for all x;
Bx = 2 if x < 4 *and* $\geq$ *5 Bx = 3 + x if 4* $\leq$ *x ¡ 5;*
Sx = if x $\leq$*8, Sx = 8 if x* > 8;
Tx = 2 if x < 4 *or* $\geq$ *5, Tx = 5 + x if* $4 \leq x < 5$.
Then A, B, S and T satisfy all the conditions of the above theorem and have a unique common fixed point x = 2.

Corollary 4.0.63. *Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces (X, M, N, $*$,$\diamond$) with continuous t-norm * and continuous t-conorm $\diamond$ defined by $a * a \geq a$ and $a \diamond a \leq a$ for all $a \in [0, 1]$ satisfying the following condition:*

i. *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,*

ii. *the pairs (A, S) and (B, T) are compatible mappings of type (K),*

iii. *If A, S and one of the mapping of pair (B, T)is continuous.*

iv. *there exist $k \in (0, 1)$ such that*
*M(Ax, By, kt) $\geq$ M(Sx, Ty, t) * M(Ax, Sx, t)*
**M(Sx, By, 2t)*M(By, Ty, t) * M(Ax, Ty, t) and*
N(Ax, By, t) $\leq$ N(Sx, Ty, t) $\diamond$ N (Ax, Sx, t)$\diamond$
N (Sx, By, 2t)$\diamond$N(By, Ty, t) $\diamond$ N(Ax, Ty, t),
for every x, y in X and $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

Corollary 4.0.64. *Let A, B, S and T be self maps of a complete intuitionistic fuzzy metric spaces (X, M, N, $*$,$\diamond$) with continuous t-norm * and continuous t-conorm $\diamond$ defined by $a * a \geq a$ and $a \diamond a \leq a$ for all $a \in [0, 1]$ satisfying the following condition:*

i. *$A(X) \subset T(X)$ and $B(X) \subset S(X)$,* *ii. the pairs (A, S) and (B, T) are compatible mappings of type (K),*

iii. If A, S and one of the mapping of pair (B, T)is continuous.

iv. there exist $k \in (0, 1)$ *such that*
M(Ax, By, kt) $\geq$ *M(Sx, Ty, t) and*
N(Ax, By, t) $\leq$ *N(Sx, Ty, t)*
for every x, y in X and $t > 0$.

Then A, B, S and T have a unique common fixed point in X.

Remarks: Our result (4.0.61) is also true fore the pair (A,S) and (B,T) are compatible mappings of type (E) in place of compatible mappings of type (K). Our result (4.0.61) extends and generalizes the results of S. Manro et al. [114], C. Alaca et al. [4], K.B. Manandhar et. al [113, 112]and similar other results of fixed point in the literature.

Conclusion

Fixed point theory is an important part of non- linear functional analysis and is one of the more dynamic areas of research since last sixty years with wide applications to other disciplines. Theorems concerning the existence properties of fixed point are fixed point theorems. Also, it provide a powerful tool when it is applied to concrete problem in mathematics or applied mathematics to established the existence of solution of given problems or solution with particular properties.

The fuzzy metric space is one of the important extension of metric space. On dealing with two or more self mappings defined on metric and fuzzy metric space for the establishment of common fixed point, we need to choose suitable contractive condition in terms of compatibility.Establishing common fixed point results under certain contractive condition has become an interesting and challenging task and it continues to be an active and very wide open area of research activities.

Among different type of compatible mappings one of the latest introduced compatible mapping is compatible mapping of type (K). Therefore, some common fixed point theorems have been established by using compatible mappings of type (K) in metric, fuzzy metric and intuitionistic fuzzy metric spaces with example and other corollaries. Our results generalizes and improves similar existing fixed point results.

Research Scope

The future research scope of fixed point theorems in various generalized fuzzy metric spaces is as follows:

i. To study generalized forms of fuzzy metric spaces and to extend fixed point results under weaker contractive definitions like compatible, semi-compatible, weakly compatible, occasionally weakly compatible and non compatible mappings.

ii. To establish common fixed point theorems for three pairs of mappings and even for sequence of mappings.

iii. To Study the interrelationship between various type of contractive mappings in generalized forms of fuzzy metric space.

iv. To obtain common fixed point theorems without completeness of the space and continuity or under reciprocal continuity.

v. To find applications of fixed point results for pair of self mappings in different fields.

Bibliography

[1] Aage, C.T. and Salunke, J.N., On fixed point theorems in fuzzy metric spaces, *Internat. J. Open Problems Compt. Math.*, **3**(2)(2010), 123-131.

[2] Abbas, M., Altun, I., and Gopal, D., Common fixed point theorems for non compatible mappings in fuzzy metric spaces, *Bull. Math. Anal. and Appl*, **1**(2) (2009), 47-56.

[3] Agrawal, R.P., Maria, M. and Donal, O. R., Fixed Point Theory and Applications, *Cambridge University Press*, India, 2001.

[4] Alaca, C., Turkoglu. D and Yildiz. C, Fixed points in intuitionistic fuzzy metric Spaces, *Chaos, Solitons and Fractals*, **29**(2006), 1073-1078.

[5] Atanassov, K. , Intuitionistic fuzzy sets, *Fuzzy Sets and Syst.*, **20**,(1986)87-96.

[6] Altun, I. , Some fxed point theorems for single and multi valued mappings on ordered non-

Archimedean fuzzy metric spaces, *Iranian J. Fuzzy Systems*, **7**(1), (2010), 91-96.

[7] Azam, A. and Beg, I., Common fixed points of fuzzy maps,*Math. Com. Modeling*, (2009), 1331-1336.

[8] Baez-Duarte, L., Brouwer's fixed point theorem and a generalization of the formula for change of variables in multiple integrals, *J. Math. Anal. Appl.*, **177** (1993), 412-414.

[9] Bailey, D. F., Some theorems on cotractive mappings, *J. Indian Math. Soc.* **41**, (1966), 101-106

[10] Balasubramaniam, P., Muralisankar, S. and Pant, R.P., Common fixed points of four mappings in a fuzzy metric space, *J. Fuzzy Math.*, **10**(2) (2002), 379-384.

[11] Banach, S., Sur les operations dans les ensembles abstraits et leur applications aux equations integrales, *fundamental Mathematicae*,**3**(7)(1922), 133-181.

[12] Banerjee, A.K. and Dey, A., Metric Spaces and Complex Analysis, New Age Publ., India, 2008.

[13] Beitollahi, A and Azhdari, P., Some fixed point theorems in fuzzy metric space, *Internat. J. Contemp. Math. Sciences*, **4**(21)(2009), 1013-1019.

[14] Bohnenblust, H. F. and Karlin, S., *On a theorem*

of Ville: Contributions to the Theory of Games, Princeton University Press, New Jersey, 1950.

[15] Bollenbacher, A. and Hicks, T. L., A fixed point theorem revisited, *Proc. Amer. Math. Soc.*, **102**(1988), 898-990.

[16] Border, K. C., *Fixed-point theorems with applications to economics and game theory*, Cambridge University Press, UK 1999.

[17] Brosowski, B., Fixpunkt satze in der approximations-theories, *Mathematica*, **11**(1969), 192-220.

[18] Browder, F. E., On the convergence of successive approximations for non linear funcctional equations, *Nederlandse Akademie van Wetenschappen. Proceedings series A*, **71**(1968), 27-35.

[19] Browder, F. E., *Nonexpansive nonlinear operators in Banach Space*, Procideeengs of the National Academy of Science of the United States of America, **54** (1965), 1041-1044.

[20] Brown, R. F. (Ed.),*Contemporary Mathematics: Fixed point theory and its applications*, Proceedings of a conference held at the International Congress of Mathematics, Aug 4 - 6, 1986, American Mathematical Society.

[21] Carbone, A., B. E Rhoades and S. P Singh., A

fixed point theorem for generalized contraction map, *Indian J. Pure Appl. Math.*, **20** (1989), 543-548.

[22] Castanda, N., A note on a fixed point theorem and Newton method, *Indian J. Pure Applied Math.*, **30**(2)(1999), 199 - 201.

[23] Chandra, K. and Rao, S., Functional Analysis, Narosa Publ. New Delhi, India, 2002.

[24] Chauhan, S. S., Common fixed point theorem for two pairs of weakly compatible mappings in M-fuzzy metric spaces, *Internat. J. Math. Anal.*, **3**(8)(2009), 393-398.

[25] Chenny, E. W., Applications of fixed point theorems to approximation theory, In Theory of Approximations with Applications, *Academic Press*, (Ed. Law and Sahney), (1976), 1-8.

[26] Cho, Y.J., P. P. Murthy and G. Jungck, A common fixed point theorem of Meir and Keleer type, *Internat. J. Math. Math. Sci.*, **16**(4) (1993), 669-674.

[27] Cho, S. H. and Jung, J. H., On common fixed point theorems in fuzzy metric spaces, *Internat. Math. Forum*, **1**(2006), 1441-1451.

[28] Cho, S. H., On common fixed point theorems in fuzzy metric spaces, *J. Appl. Math. Computing*,

20(1-2)(2006), 523-533.

[29] Cho, S.H., Jung, J. H., On common fixed point theorems in fuzzy metric spaces *Internat. Maths Forum* **29** (2006), 1441-1451.

[30] Cho, Y.J., Pathak, H.K., Kang, S.M., and Jung, J.S., Common fixed points of compatible maps of type (B) on fuzzy metric spaces, *Fuzzy Sets and Systm* **93** (1998), 99-111.

[31] Choudhary, B. and S. Nanda, Functional Analysis with Applications, *Wiley Eastern Limited*, New Delhi, 1989.

[32] Ciric, L. B., A new fixed point theorem for contractive mappings, *Inst. Mathematique Publ., Nouvelle Serie.*, **30**(44)(1981), 25-27.

[33] Ciric, L.B., A generalization of Banach's contraction principle, *Proc. Amer. Math. Soc.*, **45** (1974), 267-273.

[34] Collaco, P. and J. Carvalho, A complete comparision of 25 contraction conditions, *Nonlinear Analysis. Theory, Methods and Applications*, **30**(1997), 471-476.

[35] Daffer, Z. P., and Kaneko, H., On expansive mappings, *Math. Japonica*, **37** (1992), 733-735.

[36] Danes, J., Some fixed point theorems, *Comment. Math. Univ. Carolin*, **9**(1968), 223- 235.

[37] Darbo, G., Punti uniti in transformazioni a condominio non compatto, *Rend Sem.Mat. Univ. Padova,* **24** (1955), 84 - 92.

[38] Deng, Z., Fuzzy pseudo-metric space, *J. Math. Anal. Appl.*, **86**(1982), 74-95.

[39] Dhage,B. C., Applications of some fixed point theorems in approximation theory, *Math. Sci. Res. Hotline*, **4**(2000), 63-69.

[40] Dieudonne, J., History of Functional Analysis, *Mathematics Studies 49, North -Holland Publ. Comp., Amsterdam*, 1981.

[41] Dugundgi, J., Topology, Universal Book Stall, New Delhi, 1986.

[42] Edward, R.E., Functional Analysis and Applications, Holt. Rinehart and Winston, New York, 1965.

[43] Erceg, M. A., Metric space in fuzzy set theory, *J. Math. Anal. Appl.*, **69**(1979), 205-230.

[44] Fan, K., Fixed point and minimax theorems in locally convex topological linear spaces, *Proc. Nat. Acad. Sci.,* **38** (1952), 121-126.

[45] Fisher, B. and Sessa S., Common fixed points for a pair of mappings, *J. Univ. Kuwait (Sci.)*, **15** (1988), 9-11.

[46] Fisher, B., Mappings with a common fixed point, *Math. Sem. Notes*, **7**(1979), 81-84.

[47] Fisher, B., Common fixed points of four mappings, *Bull. Inst. Math. Acad. Sinica.*, **11**(1983), 101-113.

[48] Franklin, J., *Mathematical Methods of Economics*, Springer-Verlag, New York, America, 1983.

[49] George, A. and Veeramani, P., On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, **64** (1994), 395-399.

[50] Goebel, K. and Kirk, W.A., Topics in Metric Fixed Point Theory, University Press, New York, 1990.

[51] Goffman, C., and Fedrick, G., First Course in Functional Analysis, Prentice-Hall of India Pvt. Ltd., New Delhi, 1996.

[52] Grabiec, M., Fixed points in fuzzy metric spaces, *Fuzzy Sets and Syst*, 27(1983), 385-389.

[53] Groger , K., A simple proof of the Brouwer fixed point theorem, *Math. Nachr.,* **102** (1980), 293-295.

[54] Gupta, R., Dhagat, V. and Shrivastava, R. , Fixed Point Theorem in Fuzzy Random Spaces, *Internat. J. Contemp. Math. Sci*, **5**, 2010, 1943 - 1949.

[55] Hegedius, M., New generalization of Banach's contraction principle, *Acta. Sci. Math., (Szeged)*, **42** (1980), 87-89.

[56] Hicks, T.L. and Rhoades, B.E., A Banach type fixed point theorem, *Math. Japon*, **24** (1979), 327-330.

[57] Hicks, M. D., A note on fixed point theorem, *J. Approx. Theory*, **34**(1982), 221-225.

[58] Hicks, T. L. and Sharma, P. L., Contractive type mappings and fixed points, *Mathematics Colloquium University of Cape Town*, (1984), 29-34.

[59] Hosseiniun, S. A. R. and Mona Nabiei, Some applications of fixed point theorem in economics and nonlinear functional analysis, *Internat. Math. Forum*, **5**(9)(2010), 2407 - 2414.

[60] Hussain, S. A. and Seghal, V. M. , On common fixed points for a family of mappings, *Bull. Austral. Math. Soc.*, **63**(1974), 261-264.

[61] Istratescu, V.I., Fixed point theory: an introduction to mathematics and its application, D.Reidal Publ. Company, Holland, **7**(1981), 15-466.

[62] Jachymski J., Common fixed point theorems for some families of mappings, *Indian J. Pure Appl. Math.*, **25**(1994) 925-937.

[63] Jain, S., Mundra, B. and Aske, S., Common fixed

point theorem in fuzzy metric space using implicit relation, *Internat. Math. Forum*, **4**(3) (2009), 135-141.

[64] Jain, A. and Singh, B., A fixed point theorem for compatible mappings of type (A) in fuzzy metric space, *Acta Ciencia Indica, XXXIII M*, (2007), 339-346.

[65] Jha K., Pant, R.P. and Singh, S.L., On the existence of common fixed point for compatible mappings,*Punjab Univ. J. Maths*, **37** (2005) 39-48.

[66] Jha, K., *Banach contraction principle and some generalizations*, M. Phil. Thesis submitted to Kathmandu University, Nepal, 1999.

[67] Jha K., Some applications of Banach's contractin principle, *Nepal J. Sci. Tech*, **4**(2002), 135-140.

[68] Jha, K., Pant, R. P., and Singh, S. L., Common fixed points for compatible mappings in metric spaces, *Radovi Math.*, **12** (2003), 107-114.

[69] Jha, K. and Pant, R.P., A generalization of a Meir- Keeler type common fixed point theorem for compatible maps, *Varahmihir J. Math. Sci.,* **3** (2003), 27-34.

[70] Jha, K., A common fixed-point theorem for compatible mappings in metric space, Proceedings of IV National Conference on Sci. and Tech., March

23-26, 2004, pp. 1565 - 1570.

[71] Jha, K., common fixed point for weakly compatible maps in metric space, *Kathmandu Univ. J. Sci. Eng. and Tech.,* **I** (IV)(2007), 1 - 6.

[72] Jha, K., Common fixed point theorem for weakly compatible non-continuous mappings, *Thai J. Math.,* **5** (2) (2007), 191-197.

[73] Jha, K., A common fixed point theorem for sequence of mapping in fuzzy metric space, *J. Fuzzy Math.*, **(3)** (2010), 1-7.

[74] Jha, K., Generalized common fixed point theorem for (ϕ, ψ) weak contraction in fuzzy metric space.*Bul. Allahabad Math. Society* **27** (2012), 9-26.

[75] Jha, K., A common fixed point theorem for semi compatible maps in fuzzy metric space,*Kathmandu University sci, engg,Technology* **9**, (2013), 83-89.

[76] Jha, K., Pant, R. P., and Porru, G., On Meir-Keeler type common fixed points for compatible maps,*The Nepali Math. Sc. Report.,* **25** (2) (2005), 29 - 34.

[77] Jha K., Pant, R.P. and Singh, S.L., Common fixed points for compatible mappings in metric spaces,*Radovi Math.*, **12** (2003) 107-114.

[78] Jha, K., Pant, R. P. and Singh, S. L., On the Existence of Common Fixed Point for Compatible Mappings, *Punjab Univ. J. Math.*, **37**(2005), 39-48.

[79] Jha, K., Pant, R. P., and Manandhar, K. B., A common fixed point theorem for reciprocal continuous compatible mapping in metric space, *Annals of Pure and Appl. Math.* **5** (2014), 120 - 124.

[80] Jha, K. and Panthi, D., A note on Brouwer's fixed point theorem, *Research Journal*, 5(2) (2011), 1-12.

[81] Jha, K., Pant, R. P. and Singh, S. L., On the existence of common fixed point for compatible mappings, *Punjab Univ. J. Math.,* **37** (2005), 39-48.

[82] Jha, K., Popa, V. and Manandhar, K.B., A common fixed point theorem for compatible mapping of type(K) in metric space, *Internat. J. Math. Sci. Engg. Appl. (IJMSEA)*, *8* (I) (2014), 383-391.

[83] Jha, K. and Pant, V., A generalization of unified common fixed-point theorem, *The Nepali Math. Sci. Report*, **21** (1,2)(2003), 62- 72.

[84] Joshi, M.C. and Bose, R.K., Some Topics in Nonlinear Functional Analysis, *Wiley Eastern Ltd., New Delhi*, 1985.

[85] Jungck, G., Commuting mappings and fixed points, *Amer. Math. Monthly,* **83** (1976), 261-263.

[86] Jungck, G., Compatible mappings and common fixed points, *Int. J. Math. Sci.,* **9** (1986), 771-779.

[87] Jungck, G., K. B. Moon, S. Park and B. E. Rhoades, On generalizations of the Meir-Keeler type contractive maps: Corrections, *J. Math Anal. Appl.,* **180** (1993), 221-222.

[88] Jungck G., Murthy, P.P. and Cho, Y. J., Compatible mappings of type (A) and common fixed points,*Math. Japonica.*, **38** (1993) 381-390.

[89] Jungck, G. and Rhoades, B. E. , Fixed point for set-valued functions without continuity,*Indian J. Pure and Appl. Math.*, **29**(3)(1998), 227238.

[90] Kakutani, S., A generalization of Brouwer's fixed point theorem, *Duke Math. J.*, **8** (1941), 457 - 459.

[91] Kalev, O. and Seikkala, A., On fuzzy metric spaces, *Fuzzy Sets and Systems*, **12**(1984), 215-229.

[92] Kannan, R., Some results on fixed points, *Bull. Calcatta. Math. Soc.*, **60**(1968), 71-76.

[93] Kannan, R., Some results on fixed points, *Amer. Math. Monthly*, **76**(1969), 405-408.

[94] Kantorovich, L.V., and Aktlov, G.P., Functional Analysis, *Pergamum Press*, 1982.

[95] Kesavan, S., Topics in Functional Analysis and Application, *Wiley Eastern, Ltd. New Delhi*, 1989.

[96] Khan, M. S. and Fisher,B. , Some fixed point theorems for commuting mappings, *Math. Nachr.,* **106** (1982), 323-326.

[97] Kim C, B., Fixed Point Theorems with Applications to Economics and Game Theory, *Cambridge University Press, Cambridge* 1999.

[98] Kirk, W. A., A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly*, **72**(1965), 1004-1006.

[99] Kirk, W.A. and Sims, B., Handbook of Metric Fixed Point Theory, Academic Publ., Dordrecht, The Netherlands, 2001.

[100] Klement, E.P., Mesiar, R. and E. Pap, Triangular Norms, *Kluwer Academic Publishers, Dordrecht*, 2000.

[101] Koireng, M. and Rohen, Y., Common fixed point theorems of compatible mappings of type (P) in fuzzy metric space, *Internat. J. Math. Anal*, **6**(2012), 181-188.

[102] Kramosil, O. and Michalek, J., Fuzzy metric and

statistical metric spaces, *Kybernetika* **11** (1975), 326-334.

[103] Kreyszig, E., *Introductory functional analysis with applications,* John Wiley and Sons, New York, 1978.

[104] Kumar, S. and Fisher, B., A common fixed point theorem in fuzzy metric space using property (E.A.) and implicit relation,*Thai J. Math.*, **9**(2011), 21-28.

[105] Kutukcu, S., Sharma, S. and Tokgoz, H., A fixed point theorem in fuzzy metric spaces, *Internat. J. Math. Anal.*, **1**(2007), 861-872.

[106] Kutukcu, S., A fixed point theorem for contraction type mappings in Mengar spaces, *Amer. J. Appl. Sci.*, **4**(6) (2007),371 - 373.

[107] Kutukcu, S., Sharma, S. and Tokgoz, H., A fixed point theorem in fuzzy metric spaces, *Internat. J. Math. Anal.*, **1** (2007), 861-872.

[108] Ky Fan, Extension of two fixed point theorems of F. E. Brouder, *Math. Z.*, **112**(1969), 234 - 240.

[109] Limaye, B.V., Functional Analysis, *New Age Internat. Pvt. Ltd Publishers, New Delhi*, 1989.

[110] Maddox, J. J., Elements of Functional Analysis, *Universal Book Stall, New Delhi*, 1972.

[111] Manandhar, K. B., Jha, K. and Pathak, H. K., A common fixed point theorem for compatible mappings of type (E) in fuzzy metric space, *App. Math. Sci*, **8** (2014), 2007 - 2014.

[112] Manandhar, K. B., Jha, K. and Porru, G., common fixed point theorem of compatible mappings of type (K) in fuzzy metric pace, *Electronic J. Math. Anal. and App.*, **2**(2)(2014), 248-253.

[113] Manandhar, K. B., Jha, K. and Cho,Y.J., common fixed point theorem in intuitionistic fuzzy metric spaces using compatible mappings of type (K),*Bull. Society for Math. Services and Standards* **3**(2014), 81-87.

[114] Manro, S., Kumar,S. and Singh,S., common fixed point theorem in intuitionistic fuzzy metric spaces. *App. Maths*, **1**(2010), 510-514.

[115] Meir, A. and Keeler, E. , A theorem on contraction mapping, *J. Math. Anal. Appl.*, **28** (1969), 326-329.

[116] Menger, K., Statistical metrics, *Proc. Nat. Acad. Sci.*, **28**(1942), 535.

[117] Mishra, U., Ranadive, A.S and Gopal. D., Some fixed point theorems in fuzzy metric spaces, *Tamkang J. Math.*,**39**(4)(2008), 309-316.

[118] Palais, R.S., A simple proof of Banach contrac-

tion principle,*J. Fixed Point Theory and Appl.*, **2**(2007), 221-223.

[119] Pant, R. P. and Pant, V. , A Unified fixed-point theorem, *Bull. Calcutta Math. Soc.*, **91** (1999), 227-232.

[120] Pant, R. P., Common fixed point of two pairs of commuting mappings, *Indian J. Pure Appl. Math.,* **17** (1986), 187-192.

[121] Pant, R. P., Common fixed point of weakly commuting mappings,*Math. Student,* bf 62 (1993), 97-102.

[122] Pant, R. P., Common fixed points of non-commuting mappings, *J. Math. Anal. Appl.*, **188** (1994), 436-440.

[123] Pant, R. P., Common fixed point of sequence of mappings, *Ganita,* **47** (1996), 43-49.

[124] Pant, R. P., Common fixed-point theorem for contractive mappings, *J. Math. Anal. Appl.,* **226** (1998), 251-258.

[125] Pant, R. P., A common fixed points of four maps,V Bull. Calcutta Math. Soc., **90** (1998) 281-286.

[126] Pant, R. P., A common fixed point theorem under a new condition, *Indian. J. Pure Appl. Math.*, **30** (2) (1999) 147-152.

[127] Pant, R. P. and Pant, V. , Common fixed point under strict contractive condition, *J. Math. Anal. Appl.*, **248** (2000), 327-332.

[128] Pant, R. P, A generalization of contraction principle, *J. Indian Math. Soc.,* **68** (1-4) (2001), 25-32.

[129] Pant, R. P., A new common fixed-point principle, *Soochow J. Math.*, **27** (3) (2001), 287-297.

[130] Pant, R. P. and Jha, K. , A generalization of Meir-Keeler type common fixed-point theorem for four mappings, *J. Nat. Physical Sci.*, **16** (1-2) (2002), 77-84.

[131] Pant, R. P.,Lohani, A. B. and Jha, K. , Meir-Keeler type fixed-point theorem and reciprocal continuity, *Bull. Calcatta. Math. Soc.,* **94** (6) (2002), 459-466.

[132] Pant, R. P., Fixed-point theorems and dynamics of functions, *J. Indian Math. Soc.*, **70** (1-4) (2003), 135 - 143.

[133] Pant, R. P. and Jha, K. , A generalization of Meir Keeler type common fixed point theorem for four mappings, *J. Ultra Sci. Phys. Sci.*, **15** (2003), 97-102.

[134] Pant, R. P., Jha, K. , Pant, V. and Pande, V. P. , A generalization of Meir-Keeler type common fixed-point theorem, *Ganita,* **55** (2004), 87-93.

[135] Pant, R.P., Lohani, A. B. and Jha, K., A history of fixed point theorems, GANITA BHARATI : *Bull. Soc. Hist. Math. India*, bf 24(1-4)(2004), 147-159.

[136] Pant, V. and Padaliya, S.,Fixed point in fuzzy metric spaces, In Mathematical Sciences and Application, *Proceeding of the NCMSA 2006*, Edit by P.R. Adhikary and K. Jha, 2006, pp.1-8.

[137] Pant, V., Some fixed point theorems in fuzzy metric space, *Tamkang J. Math.* **40**(2009), 59-66.

[138] Pant, R. P., Meir-Keeler type fixed point theorems and dynamics of functions, *Demonostratio Math.*, **36** (1) (2003), 199-206.

[139] Pant, R. P., A study on fixed points of noncommuting mappings, D.Sc. Thesis, Kumaon University, Nainital, 1998.

[140] Pant, R. P. and Jha, K., A generalization of Meir Keeler type common fixed point theorem for four mappings, . Nat. Phys. Sci., **16**(1-2) (2002) 77-84.

[141] Pant R.P. and Jha, K., A generalization of Meir Keeler type common fixed point theorem for four mappings,*J. Ultra Sci. Phys. Sci.*, **15** (2003) 97-102.

[142] Pant, R.P. and Jha, K., A remark on common fixed points of four mappings in a fuzzy metric

space, *J. Fuzzy Math.*, **12**(2) (2004), 433-437.

[143] Pant, R. P., K. Jha and A. B. Lohani, Generalization of a Meir-Keleer type fixed point theorem, *J. Indian Math. Soc.*, **70** (1-4)(2003), 57-65.

[144] Pant, R. P., Joshi P. C. and Gupta, V. , A Meir-Keleer type fixed point theorem, *Indian J. Pure Appl. Math.,* **32** (6)(2001), 779-787.

[145] Pant R.P., Pant,V. and Pandey,V.P, Generalization of Metr-Keeler type fixed point theorems,*Tamkang J. Math.*, **35**(2004), 179-187.

[146] Panthi, D., *Some fixed point results in dislocated and dislocated quasi metric space*, Ph. D. Thesis submitted to Kathmandu University, Nepal,2013.

[147] Park, J.H., Intuitionistic fuzzy metric spaces, *Chaos, Solitons and Fractals* **2**(2004), 1039-1046.

[148] Park, S., Generalizations of Ky Fan's matching theorems and their applications,

[149] Pathak, H. K, Cho, Y. J. and Kang, S. M. , Remarks on R-weakly commuting mappings and common fixed-point theorems, *Bull. Korean Math. Soc.*, **34** (2) (1997), 247- 257.

[150] Pathak, H. K., Cho, Y. J., Chang, S. S. and Kang, S. M., Compatible mappings of type (P) and fixed point theorem in metric spaces and Probabilistic

metric spaces, *Novi Sad J. Math.*, **26**(2)(1996), 87-109.

[151] Pathak H.K., Y.J. Cho, S. M. Kang and B. Madharia, Compatible mappings of type(C) and common fixed point theorems of Gregu's type,*Demonstr. Math.*, **31** (3) (1998) 499-517.

[152] Pathak, H.K. and Khan, M.S., Compatible mappings of type (B) and common fixed point theorems of Gregu's type,*Czechoslovak Math. J.*, **45** (120) (1995) 685-698.

[153] Popa, V. (2000). A general common fixed point theorem of Meir-Keeler type for non continuous weak compatible mappings,*FILOMAT*, **18**, pp. 433-40.

[154] Popa V., A general fixed point theorem for weakly compatible mappings in compact metric spaces, *Turkish. J. Math.*, **25**(2001), 465-474.

[155] Rajendra, A., Nonlinear Functional Analysis, *Narosa Publishing House*, India, 1990.

[156] Rakoth, E., A note on contractive mappings, Proceedings of the Amer. Math. Society, **13**(1962), 459-465.

[157] Ranadive, A. S. and Chouhan, A. P., Absorbing maps and fixed point theorem in fuzzy metric spaces, *Internat. Math. Forum*, **5**(10)(2010), 261-

268.

[158] Rani, A., Chugh, R. and Mehra, S., fixed points of absorbing maps in fuzzy metric space, *Internat. Math. Forum*, **4**(32)(2009), 1591-1601.

[159] Rao, K.P.R.,K.R.K,Rao and Rao, T.R., Common fixed point theorems in Sequentially Compact fuzzy metric spaces under implicit relations, *Internat. Math. Forum*, **2** (2007), 2543 - 2549.

[160] Rao, K.P.R., Aliouche, A. and Babu, G. R., Related fixed point theorems in fuzzy metric spaces, *J. Nonlinear Sci. Appl.*, **1**(2008), 194- 202.

[161] Rao, K.P.R., Babu, G.V. R. and Fisher, B., Common fixed point theorems in fuzzy metric spaces under implicit relations, *Hacettepe Math. Stat.*, **37**(2)(2008), 97-106.

[162] Razani, A.R., A Contraction Theorem in Fuzzy Metric Spaces, *Hindawi Publishing Corporation*, **3**(2005), 257-265

[163] Rezapour, S., Common fixed point of self-maps in intuitionistic fuzzy metric spaces, *MATEMAT BECHK*, **60**(2008), 261-268.

[164] Rhoades, B. E., A Meir-Keeler type fixed point theorem for three mappings, *Jnanabha,* **14** (1984), 17-21.

[165] Rudin, W., Functional Analysis, *Tata McGraw-Hill Publ. Comp. Ltd.*, India, 1974.

[166] Saadati, R., Razani, A., Adibi, H., A common fixed point theorem in L-fuzzy metric spaces, *Chaos,Solitons Fractals* **33** (2007), 358-363

[167] Saadati, R., Sedghi, S. S., Vaespour, S. M., Some common fixed point theorems in complete L-fuzzy metric spaces, *Bull. Malaysian Math. Sci. Soc.*, **31**(1)(2008), 77-84.

[168] Saadati, R. and Vaezpour, S. M., A fixed point theorems in L-fuzzy quasi-metric spaces, *Amer. J. Applied Sci.*, **6**(2)(2009), 273-275,

[169] Saini, R.K. and Kumar, M., Common fixed point theorems in fuzzy metric space using implicit relations, *Advances in Fuzzy Math.*, **4** (2)(2009),181-192.

[170] Samantha, T. K. and Mohinta, S., Common fixed point theorems under contractive condition in fuzzy symmetric spaces *Annals of Fuzzy Math. and Informatics*, **5**(2013), 337-347.

[171] Samelson, H., On the Brouwer fixed point theorem, *Portugal. Math.,* **22** (1963), 189-191.

[172] Schweizer, B. and Sklar, A. Statistical metric spaces, *Pacific J. Math.*, **10** (1960) 314-334.

[173] Sedghi, S. and Shobe, N., Fixed point theorem

in M-fuzzy metric spaces with property (E), *Advances in Fuzzy Math.*, **1**(1) (2006), 55- 65.

[174] Sessa, S., On a weakly commutativity condition in a fixed point considerations,*Publ. Inst. Math.*, **32** (46) (1986) 149-153.

[175] Shaken, S., Ciri, L.J.B. and Saadati, R., Common fixed point theorem in partially ordered L-fuzzy metric spaces, *Fixed Point Theory and Appl.*, 2010 (2010), 1-7.

[176] Sharma, S., On fuzzy metric space, *Southeast Asian Bull. Math.*, **26**(2002), 133-145.

[177] Sharma, S., Common fixed point theorems in fuzzy metric spaces, *Fuzzy Sets and Systems*, **127** (2002), 345-352.

[178] Sharma, S., Kutukcu, S. and Rathore, R.S., Common fixed point for Multivalued mappings in intuitionistic fuzzy metric space, *Communication of Korean Math. Society*, **22** (3),(2007), 391-399.

[179] Shashkin, A.Y., Fixed point, University Press, India, 1991.

[180] Singh, S. L. and V. Chhada, Remarks on recent fixed point theorems compatible maps, *Internat. J. Math. Math. Sci.,* **19** (4) (1996), 801-804.

[181] Singh, S. L. and Mishra, S. N. , Remarks on recent fixed point theorems and applications to in-

tegral equations,*Demonstratio Math.,* **24**(2001), 847-857.

[182] Singh, S.L. and Tomor A., Weaker forms of commuting mappings and existence of fixed point, *J. Korean Soc. Math. Edu. Seriec(B); Pure Appl. Math.*, **10**(3)(2003),145-161.

[183] Singh, B. and Jain, S., Semi-compatibility and fixed point theorems in fuzzy metric space using implicit relation, *Internat. J. Math. Math Sci*, **16**(2005), 2617-2629.

[184] Singh, B., and Chouhan, M.S., Common fixed points of compatible maps in Fuzzy metric spaces, *Fuzzy sets and Syst*, **115** (2000), 471-475.

[185] Singh, B. and Jain, S., Semi-compatibility, compatibility and fixed point theorems in fuzzy metric space, *J.Chungecheong Math. Soc.* **18**(1), 2005, 1-22.

[186] Singh, S. L. and Kasahara, S. , On some recent results on common fixed points, *Indian J. Pure Appl. Math.,* **13** (1982), 757-761.

[187] Singh, S. P and B. A. Meade, Fixed point theorems in complete complete metrc space, *Bull. Austral. Math. Soc.*, **16**(1977), 49-53.

[188] Singh, M.R. and Singh, Y. M., Compatible mappings of type (E) and common fixed point theo-

rems of Meir-Keeler type, *International J. Math. Sci. and Engg. Appl.* **1** (2), pp. 299- 315.

[189] Singh, M.R. and Singh, Y. M., On various types of compatible maps and common fixed point theorems for non- continuous maps,*Hacettepe J. Math. and Stat.* **40**(4)(2011), 503 - 513.

[190] Subarahmanyam, P.V., A common fixed point theorem in fuzzy metric space, *Inf. Sci*, **83**(1995), 109-112.

[191] Suzuki, T., Several fixed-point theorems concerning -distance, *Fixed-Point Theory and Appl.*, **2004** (2004), 195-209.

[192] Thagafi, A. A. and Shahzad, N., Generalized I-Non expansive Self maps and Invariant Approximations, *Acta Mathematica Sinica, English Series* **24** (2008), 867-876.

[193] Vasuki, R., Common fixed points for R-weakly commuting maps in fuzzy metric spaces, *Indian J. Pure Appl. Math.*, **30** (1999), 419-423.

[194] Vats, S. K., Weakly compatible maps in metric spaces, *J. Indian Math. Soc.*, **69** (1-4), (2002), 139-143

[195] Veerapandi, T., Jeyaraman, M. and Joseph, J. P., Some fixed point and coincident point theorem in generalized M-fuzzy metric space, *Internat. J.*

Math. Anal., **3**(13)(2009), 627-635.

[196] Verma, M. and Chandel, R. S., Common fixed point theorem for four mappings in intuitionistic fuzzy metric space using absorbing maps, *IJRRAS*, **10** (2) (2012), 286 - 291.

[197] Vijayaraju, P. and Sajath, Z. M. I., Some common fixed point theorems in fuzzy metric spaces, *Internat. J. Math. Anal.*, **3**(15)(2009), 701-710.

[198] Wilansky, A., Functional Analysis, *Blaisdell Publ Comp, New York*, 1964.

[199] Yosida, K., Functional Analysis, *Springer Verlag, Berlin-Heidelberg, New York*, 1966.

[200] Zadeh, L.A, Fuzzy Sets, *Inform. and Control*, **8**(1965), 338-353.