TRIBHUVAN UNIVERSITY
INSTITUTE OF ENGINEERING
CENTRAL CAMPUS, PULCHOWK

MATRIX TRANSFORMATION BETWEEN SEQUENCE SPACES AND THEIR PRACTICAL APPLICATIONS

By

Vinod Parajuli

A THESIS SUBMITTED TO
THE DEPARTMENT OF ENGINEERING SCIENCE AND HUMANITIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF DOCTOR OF PHILOSOPHY
IN MATHEMATICS

DEPARTMENT OF ENGINEERING SCIENCE AND HUMANITIES
LALITPUR, NEPAL

June, 2014

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Abstract

The study of sequence spaces was motivated by the classical results of Summability theory in Functional Analysis. The results obtained by Cesaro, Borel, Nörlund and others at the turn of 20th century stimulated interest in general matrix transformation theory which deals with characterization of matrix mappings between sequence spaces by giving necessary and sufficient conditions on the entries of the infinite matrices. The first application of analysis to the theory of Summability was done by Mazur in 1927 when he proved now his famous Mazur's consistency theorem. An outstanding contribution and plenty of work have been done in the field of sequence spaces in last 50+ years.

Kizmaz [41] introduced the concept of difference sequence spaces. The work of Kizmaz was further generalized by Et and Cloak [66], Tripathy and Esi [19], Tripathi, Esi and Tripathi [20], Esi, Tripathy and Sarma [3] and others. In the meantime in constructing new sequence spaces the role of the infinite matrices

$$G(u,v) = (g_{nk}) = \begin{cases} u_n v_k, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

called generalized weighted mean;

$$\Delta = (\delta_{nk}) = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n \\ 0, & 0 \le k < n \text{ or } k > n \end{cases}$$

called the difference operator matrix;

$$S = (s_{nk}) = \begin{cases} 1, & 0 \le k \le n \\ 0, & k > n \end{cases};$$

the operator matrix Δ_j which can be expressed as a sequential double band matrix given by

$$\Delta_j = \begin{pmatrix} 1 & -2 & 0 & 0 & \cdots \\ 0 & 2 & -3 & 0 & \cdots \\ 0 & 0 & 3 & -4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and combination of them has been considered to represent difference operator. In this connection we have constructed new matrices

$$S^n = \lambda = (\lambda_{nk}) = \begin{cases} n-k+1, & n \geq k \\ 0, & \text{otherwise} \end{cases}$$

which is a lower unitriangular matrix and an operator sparse band matrix λ_j which can be expressed as a sequential double band matrix given by

$$\lambda_j = \begin{pmatrix} \frac{1}{t_1} & -\frac{1}{t_1} & 0 & 0 & \cdots \\ 0 & \frac{1}{t_2} & -\frac{1}{t_2} & 0 & \cdots \\ 0 & 0 & \frac{1}{t_3} & -\frac{1}{t_3} & \cdots \\ 0 & 0 & 0 & \frac{1}{t_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

to introduce the new sequence spaces.

This thesis consists of six chapters.

Chapter one contains introduction with preliminaries and reviews.

Chapter two has been divided into two parts. The sequence spaces $w(p)$, $w_0(p)$ and $w_\infty(p)$ were introduced and studied by Maddox [45]. In [12], the authors have introduced the sequence spaces $c_0(u,v;p)$, $c(u,v;p)$, $l_\infty(u,v;p)$ and in [29] $l(u,v;p)$ and established some properties. Following this in the first part of chapter two, we introduce a set of sequence spaces $w(u,v;p)$, $w_0(u,v;p)$, $w_\infty(u,v;p)$ by the application of the generalized weighted mean matrix $G(u,v)$ as the operator, study some properties and find β- dual of $w(u,v;p)$. We also characterize the matrix classes $(w(u,v;p), l_\infty)$, $(w(u,v;p), c)$ and $(w(u,v;p), c_0)$. Recently in [78] , the sequence spaces $c_0(u,v;p,\Delta)$, $c(u,v;p,\Delta)$, $l_\infty(u,v;p,\Delta)$ and $l(u,v;p,\Delta)$ have been introduced. Following this in the second part of chapter two, we introduce the sequence spaces $w(u,v;p,\Delta)$, $w_0(u,v;p,\Delta)$ and $w_\infty(u,v;p,\Delta)$ by using the combination of the matrix $G(u,v)$ and the difference operator matrix Δ, study some properties and find β-dual of $w(u,v;p,\Delta)$. We also characterize the matrix classes $(w(u,v;p,\Delta),c)$, $(w(u,v;p,\Delta),c_0)$ and $(w(u,v;p,\Delta),\Omega(t))$.

Chapter three has also been divided into two parts. In [15] Choudhary and Mishra have introduced and studied the sequence space $\overline{l(p)}$ which is the set of all sequences whose S- transforms are in the space $l(p)$. Following this in the first part we introduce a new sequence space $l(p,\lambda)$ which is the set of all sequences whose $S^n = \lambda$ transforms are in $l(p)$. We compute β- dual of $l(p,\lambda)$ and characterize the matrix classes $(l(p,\lambda),\mathrm{c}),(l(p,\lambda),c_0)$ and $(l(p,\lambda),l_\infty)$. Similarly in the second part we introduce a set of new paranormed sequence spaces $l_\infty(p,\lambda)$, $c(p,\lambda)$ and $c_0(p,\lambda)$ which are generated by the infinite matrix λ . We also compute the basis for the spaces $c(p,\lambda)$ and $c_0(p,\lambda)$, obtain β- dual of them and characterize the matrix classes $(l_\infty(p,\lambda),l_\infty),(l_\infty(p,\lambda),c)$ and $(l_\infty(p,\lambda),c_0)$.

In Chapter four, we introduce a set of new paranormed sequence spaces $l_\infty(u,v;p,\lambda_j)$, $c(u,v;p,\lambda_j)$ and $c_0(u,v;p,\lambda_j)$ generated by the combination sparse band matrix λ_j and the generalized weighted mean matrix $G(u,v)$. We establish some topological properties, obtain the basis for $c(u,v;p,\lambda_j)$ and $c_0(u,v;p,\lambda_j)$ and find β- duals. We characterize the matrix classes $(l_\infty(u,v;p,\lambda_j),l_\infty)$, $(l_\infty(u,v;p,\lambda_j),c)$ and $(l_\infty(u,v;p,\lambda_j),c_0)$. Besides we give characterization theorem for the case of mapping that guarantees the given rate of convergence from the sequence space $l_\infty(p)$ to the new sequence space $l_\infty(u,v;p,\lambda_j)$.

In chapter five, we present a practical application of sequence space. In [26], the sequence spaces and function spaces on interval [0, 1] for DNA sequencing have been introduced and studied. The authors have introduced new sequence spaces by using generalized p- summation method and proved that these spaces of sequences and functions are Banach space. Based on the sequence spaces and function spaces on [0,1], we examine the behaviors of sequences generated by DNA nucleotides. We extend the results of authors [26] by introducing new basis function $\sum_{k=1}^{\nu}\frac{x^k}{k!}$, $(\nu = 1,2,3,\dots n)$ which is the extension of existing basis function $\frac{x^n}{n!}$, $(n \in \mathbb{N})$ defined in the polynomial function on [0,1]. Besides, we introduce a new sequence $b=(b_n)=$

$\sum_{\nu=n}^{\infty} a_{\nu}$ which can characterize DNA sequence where $a_n \in \{A, C, T, G\}$ and A: Adenine, C: Cytosine, T: Thymine and G: Guanine are four types of nucleotides.

We conclude our thesis by providing conclusions and recommendations in chapter six.

Acknowledgements

I take this opportunity to express my profound gratitude and deep regards to my supervisor Professor Dr. Shailendra Kumar Mishra and co-supervisor Professor Dr. Geeta Bhakta Joshi for their exemplary guidance, monitoring and constant encouragement throughout course of this thesis. The blessing, help and guidance given by them time to time shall carry me long way in the journey of life on which I am about to embark.

I also take this opportunity to express a deep sense of gratitude to Professor Dr. Bharat Raj Pahari, Dean of Institute of Engineering, T.U.; Professor Dr. Subarna Shakya, Assistant Dean of Institute of Engineering, T.U.; Professor Dr. Triratna Bajracharya, Assistant Dean of Institute of Engineering, T. U; Asociate Professor Dr. Arbinda Kumar Mishra, Campus Chief , Central Campus, Pulchowk; Professor Dr. Binod Kumar Bhattarai, Head of Department, Department of Engineering Science and Humanities, Central Campus, Pulchowk; Professor Dr. Bal Krishna Sapkota, Professor Dr. Bhadra Pokharel, Associate Professor Mr. Vidyadhar Sharma, Chairman of Mathematics Instruction Committee, Central Campus, Pulchowk for their cordial support, valuable information and guidance which helped me in completing this thesis in the prescribed time.

I am obliged to my colleagues Associate Professor Nagendra Bahadur Amatya, Dr. Santosh Ghimire, Mr. Jayandra Raj Shrestha, Deputy Head of Department, Department of Engineering Science and Humanities, Central Campus, Pulchowk , Mrs. Dhana Karki (Karmacharya), Administrative Officer , Mr. Ram Kumar Bogati and all teaching and administrative staff members of Department of Engineering Science and Humanities for the valuable information and academic and administrative support provided by them.

My sincere thanks are due to Professor Dr. Sergei Silvestrov, Malardalen University and Ann-Catrin Davis, Project Manager , Erasmus Mundus IDEAS project for their memorable help and hospitality during the time when I was working to prepare this thesis at Malardalen University; Sweden.

Last but not least, I thank my wife Mrs. Indu Prabha Gautam for her constant encouragement, help and support, for which I will be indebted in my entire life, to accomplish this thesis. I would like to appreciate my younger brother Mr. Gyanendra Parajuli, sister- in - law Mrs. Sarita Parajuli, daughter Nischala Parajuli and sons Navin Parajuli and Nitesh Parajuli for their help. My thanks also go to daughter Nirmala Gautam for her valuable support.

Vinod Parajuli

June, 2014

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List of Symbols

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{R}^n$	n- dimensional Euclidean space
$\in$	is a member of
$\notin$	is not a member of
$=$	is equal to
$\neq$	is not equal to
$\geq$	greater than or equal to
$\leq$	less than or equal to
$\cong$	isometric to
$\approx$	isomorphic to
$\Rightarrow$	implies to
$\Leftrightarrow$	implies to and is implied by
$\subseteq, \subset$	is contained in
$\cup$	union
$\cap$	intersection
inf	infimum
sup	supremum
(x_n)or$\{x_k\} \ldots$	sequence
$O(a_n)$ with$a_n > 0$	a quantity that divided by a_n remains bounded
$o(a_n)$ with$a_n > 0$	a quantity that divided by a_n tends to zero as $n \to \infty$
$\overline{E}$	closure of set E
$\rightarrow$	tends to
$\downarrow$	decreases to
Σ	summation

CHAPTER ONE

INTRODUCTION

1.1. Preliminaries and Reviews

The theory of sequence space occupies a very significant position in Analysis. Because of its wide applicability in several branches of mathematics, the study of sequence space is being subject of great interest and central study in Functional analysis. The study of sequence spaces was motivated by the classical results of summability theory which is a tremendous area possessing wide range of application in Functional Analysis. In most of the cases the common general operator from one sequence space into another is, in turn, given by an infinite matrix and therefore the study of matrix transformation go side by side in the study of sequence spaces.

Interest in general matrix transformation theory was, to some extent, stimulated by special results in summability theory which were obtained by Cesaro, Borel, Nörlund and others at the turn of the 20th century. It was however the celebrated German mathematician O. Toeplitz who, in 1911, brought the methods of linear space theory to bear on problems connected with matrix transformation on sequence spaces. Toeplitz characterized all those infinite matrices $A = (a_{nk})$, $n, k \in \mathbb{N}$ which map the convergent sequences into itself, leaving the limit of convergent sequence invariant. The analysis embraced by Toeplitz was classical.

The first application of analysis to the theory of summability was done by Mazur in 1927 when he proved his now famous Mazur's consistency theorem, which won him the prize of university of LWOW [9]. In 1932, Banach, in particular, presented a very short proof of Silverman-Steinhaus theorem. Of course functional analysis was not available to Silverman and Toeplitz in 1911 and they used the only method opened to them, which may be called 'classical' or 'hard' proof. This can be found in Hardy's (1949) classic book " Divergent Series". As mentioned by Maddox [48] with the aid of theorem given by Banach much of the theory became accessible to those who would normally have neither time nor the energy to follow the usual classical approach. The advantage of studying matrix transformation between spaces of sequences over general linear operator is that, in many important cases, the most general linear operator acting between the sequence spaces is actually determined by an infinite matrix.

In 1950 Robinson [6] considered the action of infinite matrices of linear operators from a Banach space of sequences to that space. The classical results of Toeplitz, Kojima- Schur and many more results could be extended to this general setting. A fine account of these results can be found in Maddox [50]. A remarkable contribution and a lot of work have been done in the theory of sequence spaces during last 50+ years. Works of Maddox [44,45,46,47,48,49,50,51,52,54], Lascarides [24,25], Basar [32], Basar and Altay [10,11,12,13,14,33,34,35], Dutta and Reddy [40], Boos and Leiger [55], Cohen and Dunford [64], Sarigol [65], Mursaleen, Gaur and Saif [67], Nanda [80,81], Ahmad and Sarawat [89] can be regarded as milestone in the area of sequence spaces and matrix transformations. It will be difficult to discuss all the aspects of the theory in the thesis. In this context we refer the books of Taylors [2], Wilansky [7,8,9], Limaye [21], Goffman and Pedrick [22], Kreyszig [28] , Zeilder [31] , Reisz and Nagi [36], Diestel [56], Diemling [59], Atosic and Swartz [69], Ahmad and Mursaleen [88], Choudhary and Nanda [18], Maddox [48] , Yosida [63], Kamathan and Gupta [73], Wojtaszczyk [74] ,Cooke [75], Walter [77], Ruckle [85] and Basar [87] to the reader.

In 1981 Kizmaz [41] introduced the notion of difference sequence space. He studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ which have been mentioned in the thesis.The notion was further generalized by Et and Colak [66] by introducing the spaces $l_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. Another type of generalization of sequence spaces is due to Tripathy and Esi [19] , who studied the spaces $l_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$. Tripathy , Esi and Tripathy [20] generalized the above notions and unified these as follows:

Let m, s be non negative integers , then for Z a given sequence space we have

$$Z(\Delta_m^s) = \{x = (x_k) \in \omega : (\Delta_m^s x_k) \in Z\},$$

where

$$(\Delta_m^s x) = (\Delta_m^s x_k) = (\Delta_m^{s-1} x_k - \Delta_m^{s-1} x_{k+m})$$

and $\Delta_m^0 x_k = x_k$ for all k ∈ $\mathbb{N}$; $Z \in \{l_\infty, c, c_0\}$,

which is equivalent to the following binomial representation,

$$\Delta_m^s x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} x_{k+mv}$$

Esi , Tripathy and Sarma [3] showed that $c_0(\Delta_m^s)$, $c(\Delta_m^s)$ and $l_\infty(\Delta_m^s)$ are Banach spaces normed by

$$\|x\| = \sum_{k=1}^{ms} |x_k| + \sup_k |\Delta_m^s x_k|$$

Taking $m = 1$, we get the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [66]. Taking $s = 1$, we get the spaces $l_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [19]. Taking $m = s = 1$, we get the spaces $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [41].

Dutta [39] used the difference operators Δ_r and $\Delta_{(r)}$ to infinite matrices of non-negative real numbers to construct the sequence spaces $(\hat{A}, p, \Delta_{(r)})_0$,$(\hat{A}, p, \Delta_r)_0$, $(\hat{A}, p, \Delta_{(r)})$, $(\hat{A}, p, \Delta_r)$,$(\hat{A}, p, \Delta_{(r)})_\infty$ and $(\hat{A}, p, \Delta_r)_\infty$ respectively.

During last 50+ years in constructing new sequence spaces the matrices that represent difference operators have been considered. The matrices

$$G(u, v) = (g_{nk}) = \begin{cases} u_n v_k, & 0 \le k \le n \\ 0, & k > n \end{cases} \tag{1.1.1}$$

called the generalized weighted mean ;

$$\Delta = (\delta_{nk}) = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n \\ 0, & 0 \le k < n \text{ or } k > n \end{cases} \tag{1.1.2}$$

called the difference operator matrix;

$$S = (s_{nk}) = \begin{cases} 1, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

(1.1.3)

$$R^t = (r_{nk}^t) = \begin{cases} t_k / \sum_{k=0}^{n} t_k\,, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

(1.1.4)

called the Riesz mean ;

the operator Δ_j which can be expressed as a sequence in a double band matrix given by

$$\Delta_j = \begin{pmatrix} 1 & -2 & 0 & 0 & \dots \\ 0 & 2 & -3 & 0 & \dots \\ 0 & 0 & 3 & -4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(1.1.5)

or combination of them have been used to define and construct new sequence spaces. In this endeavor we have constructed new matrices $\lambda = S^n$defined by

$$\lambda = S^n = (\lambda_{nk}) = \begin{cases} n - k + 1, & n \ge k \\ 0, & \text{otherwise} \end{cases}$$

(1.1.6)

which is a lower unitriangular matrix and an operator sparse band matrix λ_j which can be expressed as a sequential double band matrix given by

$$\lambda_j = \begin{pmatrix} \frac{1}{t_1} & -\frac{1}{t_1} & 0 & 0 & \dots \\ 0 & \frac{1}{t_2} & -\frac{1}{t_2} & 0 & \dots \\ 0 & 0 & \frac{1}{t_3} & -\frac{1}{t_3} & \dots \\ 0 & 0 & 0 & \frac{1}{t_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(1.1.7)

to define the new sequence spaces.

1.2. Organization of Chapters

The thesis consists of **six** chapters. The first chapter, where we are in, is introductory in nature.

The chapter two is divided into two parts.

In [12] Altay and Basar have introduced and studied the sequence spaces $\lambda(u,v;p)$; which are derived by generalized weighted mean; defined by

$$\lambda(u,v;p) = \left\{x = (x_k): \left(\sum_{j=0}^{k} u_k v_j x_j\right) \in \lambda(p)\right\}$$

where $\lambda \in \{l_\infty, c, c_0\}$.

If $p_k = 1$ for every $k \in \mathbb{N}$, the sequence spaces $\lambda(u,v;p)$ reduce to $\lambda(u,v)$ as introduced by Malkowski and Savas [29]. The authors have proved that the spaces $\lambda(u,v;p)$ and $\lambda(p)$ where $\lambda \in \{l_\infty, c, c_0\}$ are linearly isomorphic. Besides these they have computed β ,γ- duals of the spaces $\lambda(u,v;p)$ and computed the basis of the spaces $c_0(u,v;p)$ and $c(u,v;p)$. Moreover, they have characterized the classes $(\lambda(u,v;p), \mu)$ and $(\mu, \lambda(u,v;p))$ where μ is any given sequence space.

Further in [13] Altay and Basar have introduced and studied the sequence space $l(u,v;p)$; which is derived by generalized weighted mean ; defined by

$$l(u,v;p) = \left\{x = (x_k): \left(\sum_{j=0}^{k} u_k v_j x_j\right) \in l(p)\right\}$$

The authors have proved that the spaces $l(u,v;p)$ and $l(p)$ are linearly isomorphic, computed β ,γ- duals of the spaces $l(u,v;p)$ and obtained the basis for the spaces $l(u,v;p)$. Further they have characterized the classes $(l(u,v;p), \mu)$ and $(\mu, l(u,v;p))$ where μ is any given sequence space.

Following these works in the first part of the second chapter we have introduced the new sequence spaces $\mu(u,v;p)$ for $\mu \in (w, w_0, w_\infty)$ defined by

$$\mu(u,v;p) = \left\{x = (x_k): \left(\sum_{k=1}^{n} u_n v_k x_k\right) \in \mu(p)\right\}$$

(1.2.1)

We have proved that the sequence spaces $\mu(u,v;p)$ for $\mu \in (w, w_0, w_\infty)$ are complete paranormed space and are isomorphic to the corresponding spaces $\mu(p)$. Further we have obtained β - dual of $w(u,v;p)$ and characterized the matrix classes $(w(u,v;p), l_\infty)$, $(w(u,v;p), c)$ and $(w(u,v;p), c_0)$.

In [78], Demiriz and Cacan have introduced and studied the sequence spaces $\lambda(u,v;p,\Delta)$ for $\lambda \in \{c_{0,} c, l_\infty, l\}$ derived by generalized weighted mean $G(u,v)$ and the difference operator matrix Δ as,

$$\lambda(u,v;p,\Delta) = \left\{x = (x_k): \left(\sum_{k=1}^{n} u_n v_k \Delta x_k\right) \in \lambda\right\}$$

They have proved that these sequence spaces are complete paranormed metric linear spaces and computed their $\alpha-, \beta-, \gamma-$ duals. Moreover they have given the basis for the spaces $\lambda(u,v;p,\Delta)$ for $\lambda \in \{c_{0,} c, l_\infty, l\}$.

Following the work of the authors [10, 11, 15, 29, 33, 45, 78] in the second part of chapter two we have introduced a set of new sequence spaces $\mu(u,v;p,\Delta)$ for $\mu \in \{w, w_{0,} w_\infty\}$ defined by,

$$\mu(u,v;p,\Delta) = \left\{x = (x_k) \in \omega: \left(\sum_{k=1}^{n} u_n v_k \Delta t_k\right) \in \mu(p)\right\}$$

(1.2.2)

where

$$t_k(x) = \frac{1}{k}\sum_{i=1}^{k} x_i$$

and $\Delta t_k = t_k - t_{k-1}$ for all $k \in \mathbb{N}$ with $t_0 = 0$.

We have proved that the sequence spaces $\mu(u,v;p,\Delta)$ for $\mu \in \{w, w_{0,} w_\infty\}$ are linearly isomorphic to $\mu(p)$ and that the sequence spaces are complete paranormed sequence spaces. Moreover we have constructed basis for the space $w(u,v;p,\Delta)$. Besides we have obtained β-dual of $w(u,v;p,\Delta)$ and characterized the matrix classes $(w(u,v;p,\Delta),c)$, $(w(u,v;p,\Delta),c_0)$ and $(w(u,v;p,\Delta),\Omega(t))$. In this chapter our attempt is to fill up existing literature gap in connection with spaces $w(p), w_0$(p) and w_∞(p) with respect to their generalization by means of the generalized weighted mean and the difference operator matrix.

Chapter three is also divided into two parts.

In the first part of chapter three we have introduced new sequence space $l(p,\lambda)$ defined by

$$l(p,\lambda) = \{x = (x_k) \in \omega : \lambda x \in l(p)\}$$

which is generated by infinite lower unitriangular matrix λ defined by

$$\lambda = S^n = (\lambda_{nk}) = \begin{cases} n-k+1, & n \geq k \\ 0, & \text{otherwise} \end{cases}$$

where

$$S = \begin{cases} 1, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

as defined in [15].

We have shown that $l(p) \subseteq \overline{l(p)} \subseteq l(p,\lambda)$; $l(p,\lambda)$ is linearly isomorphic to $l(p)$ and is a complete paranormed sequence space. We have constructed basis for $l(p,\lambda)$. Moreover we have found $\beta-$ dual of $l(p,\lambda)$ and characterized the matrix classes $(l(p,\lambda),\text{c}), (l(p,\lambda),c_0)$and $(l(p,\lambda),l_\infty)$.

In the second part of chapter three we have defined the sequence spaces $X(p,\lambda)$ for $X \in \{l_\infty, c, c_0\}$ as

$$X(p,\lambda) = \{x = (x_k) \in \omega : \lambda x \in X(p)\} \tag{1.2.3}$$

where $\lambda = S^n$ and S are as given in (1.1.6) and (1.1.3) respectively.

We have shown that the sequence spaces $X(p,\lambda)$ are complete paranormed linear metric spacecs and are linearly isomorphic to $X(p)$ for $X \in \{l_\infty, c, c_0\}$. We also have constructed basis for $X(p,\lambda)$ when $X \in \{c, c_0\}$. Further we have obtained $\beta -$ dual of $X(p,\lambda)$ for $X \in \{l_\infty, c, c_0\}$ and have characterized the matrix classes $(l_\infty(p,\lambda), l_\infty)$, $(l_\infty(p,\lambda), c)$ and $(l_\infty(p,\lambda), c_0)$.

Recently in 2013 Baliarsingh [70] has defined the sequence spaces $X(\Delta_j, u, v; p)$ for $X \in \{l_\infty, c, c_0\}$ as,

$$X(\Delta_j, u, v; p) = \left\{ x = (x_k) \in \omega : \left(\sum_{j=1}^{k} u_k v_j \Delta_j x_j \right) \in X(p) \right\}$$

which is derived by using generalized weighted mean $G(u,v)$ and the operator double band matrix Δ_j as defined in (1.1.5) and $\Delta_j x_j$ is defined as

$$\Delta_j(x_j) = j x_j - (j+1) x_{j+1} \quad (j \in \mathbb{N}).$$

The author has proved that the sequence spaces $X(\Delta_j, u, v; p)$ are complete linear metric spaces and that $X(\Delta_j, u, v; p)$ for $X \in \{l_\infty, c, c_0\}$ are linearly isomorphic to the spaces l_∞, c, c_0 respectively. Also, $\alpha-, \beta-, \gamma -$ duals of these spaces have been found and the matrix transformation from these classes to the sequence spaces $l_\infty(q), c(q)$ and $c_0(q)$ have been characterized. Following the work of Baliarsingh [70] in chapter four we have first defined the matrix λ_j and then we have introduced new sequence spaces $X(u, v; p, \lambda_j)$ for $X \in \{l_\infty, c, c_0\}$ as

$$X(u, v; p, \lambda_j) = \left\{ x = (x_k) \in \omega : \left(\sum_{j=1}^{k} u_k v_j \lambda_j x_j \right) \in X(p) \right\} \tag{1.2.4}$$

where $\lambda_j x_j = \frac{1}{t_j} \Delta x_j$; $\frac{1}{t_j} \in (0,1)$ and $\Delta x_j = x_{j-1} - x_j$ with $x_0 = 0$; $(j \in \mathbb{N})$.

We have proved that these spaces are complete linear metric spaces and linearly isomorphic to the corresponding space $X(p)$ for $X \in \{l_\infty, c, c_0\}$. We have constructed the basis for the spaces for $c_0(u, v; p, \lambda_j)$ and $c(u, v; p, \lambda_j)$. We have found $\beta -$dual of the sequence space $l_\infty(u, v; p, \lambda_j)$ and characterized the matrix classes

$\left(l_\infty(u,v;p,\lambda_j),l_\infty\right)$, $\left(l_\infty(u,v;p,\lambda_j),c\right)$, $\left(l_\infty(u,v;p,\lambda_j),c_0\right)$ and $\left(l_\infty(p),l_\infty(u,v;p,\lambda_j)\right)$.

In chapter five we present a practical application of sequence spaces. In [26] Xu and Xu have introduced and studied sequence spaces and function spaces on interval [0,1] for DNA sequencing . Authors have defined the function spaces ,

$$C_{\phi,0}[0,1]=\left\{f(x)=\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}:\lim_{n\to\infty}a_n=0\right\}$$

$$C_{\phi,p}[0,1]=\left\{f(x)=\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}:\sum_{n=0}^{\infty}|a_n|^p<\infty\right\}$$

and

$$C_{\phi,\infty}[0,1]=\left\{f(x)=\sum_{n=0}^{\infty}a_n\frac{x^n}{n!}:\sup_{n\geq 0}|a_n|<\infty\right\}$$

where $a=(a_1,a_2,\ldots\ldots,a_n,\ldots)$ is a DNA sequence and $a_n\in\{A,C,T,G\}$ and A , C, T and G are four types of nucleotide which are linked in different orders in extremely long DNA molecules. The abbreviations A , C, T and G stand for A: Adenine, C: Cytosine, T: Thymine and G: Guanine. Based on the sequence spaces and function spaces on interval [0,1] , we examine the behaviors of sequence generated by DNA. Basically we extend the results of the authors in [26] by introducing a new basis function $\sum_{k=1}^{\nu}\frac{x^k}{k!}$ for $\nu=1,2,3,\ldots,n$ which is the extension of the existing basis function $\frac{x^n}{n!}$ $(n\in\mathbb{N})$ in [26] defined in the polynomial space in [0,1] .Besides, we introduce a new sequence

$$b=(b_n)=\left(\sum_{\nu=n}^{\infty}a_\nu\right) \tag{1.2.5}$$

which can characterize DNA sequence where $a_n\in\{A,C,T,G\}$. Moreover the authors have presented the set inclusion relation as

$$P[0,1]\subset C_{\phi,1}[0,1]\subset C_{\phi,p}[0,1]\subset C_{\phi,0}[0,1]\subset C_{\phi,\infty}[0,1]=C_M^{\infty}\ [0,1],\ 1\leq p<\infty\ .$$

The spaces $C_{\phi,0}\,[0,1]$,$C_{\phi,p}\,[0,1]$ and $C_{\phi,\infty}\,[0,1]$ are isomorphic to c_0 , l_p and l_∞ respectively.

We extend this set inclusion relation to

$P[0,1] \subset C_{\phi,p}[0,1] \subset C_{\psi,p}[0,1] \subset C_{\phi,0}[0,1] \subset C_{\psi,0}[0,1] \subset C_{\phi,\infty}[0,1] \subset C_{\psi,\infty}[0,1] = C_M^\infty\ [0,1]$, $1 \le p < \infty$ where,

$$C_{\psi,0}\,[0,1] = \left\{ g(x) = \sum_{k=1}^{\infty} a_k \left(\sum_{\nu=1}^{k} \frac{x^\nu}{\nu!} \right) : \lim_{n\to\infty} b_n = 0 \right\},$$

$$C_{\psi,p}\,[0,1] = \left\{ g(x) = \sum_{k=1}^{\infty} a_k \left(\sum_{\nu=1}^{k} \frac{x^\nu}{\nu!} \right) : \sum_{n=1}^{\infty} |b_n|^p < \infty \right\} \text{and}$$

$$C_{\psi,\infty}\,[0,1] = \left\{ g(x) = \sum_{k=1}^{\infty} a_k \left(\sum_{\nu=1}^{k} \frac{x^\nu}{\nu!} \right) : \sup_{n\ge1} |b_n| < \infty \right\}$$

which fills the literature gap to the previous set inclusion relation. Further we have established some isomorphism theorems on newly introduced sequence spaces.

Finally in chapter six we wrap up the thesis by providing some conclusive remarks and recommendations.

We now collect some known definitions and results which we shall use in our context.

1.3. Definitions and Useful Results

1.3.1. Metric space and metric linear space

Metric space

Definition: Let X be a non empty set. A metric d on X is a function

$d: X \times X \to \mathbb{R}$ satisfying the following properties for $x, y, z \in X$:

$M1$: $0 \le d(x, y) < \infty$

$M2$: $d(x, y) = 0$ if and only if $x = y$

$M3$: $d(x, y) = d(y, x)$

$M4$: $d(x, z) \leq d(x, y) + d(y, z)$

Any non empty set X together with a metric function d is regarded as a metric space and is denoted by a pair (X, d). The axioms $M2 - M4$ for a metric d are sometimes referred to as Hausdorff postulates. $M4$ is called the triangle inequality.

Metric linear space

Definition: A topological linear space (X, τ) is a linear space with a topology τ on X such that the addition and scalar multiplication are continuous in (X, τ) . If the topology τ on X is given by a metric (respectively semi metric), then we regard X as a metric linear space (respectively semi metric linear space).

1.3.2. Vector space

Definition: A vector space over a field F($\mathbb{R}$ or $\mathbb{C}$) is a set V together with two binary operations; called vector addition i.e. for any vectors $u, v \in V$ their sum $u + v \in V$and scalar multiplication i.e. for any scalar $\lambda \in F$ and a vector $v \in V$, their multiplication$\lambda v \in V$; satisfying the eight conditions listed below for $a, b \in F$ and $u, v, w \in V$:

V1. Associativity of addition

$$u + (v + w) = (u + v) + w$$

V2. Commutativity of addition

$$u + v = v + u$$

V3. Identity element of addition

There exists an element $0 \in V$, called the zero vector, such that

$v + 0 = v$ for all $v \in V$.

V4. Inverse element of addition

For every element $v \in V$ there exists an element $-v \in V$, called the additive inverse of v such that $v + (-v) = 0$,the zero vector of V.

V5. Compatibility of scalar multiplication with field multiplication

$$a(bv) = (ab)v$$

V6. Identity element of scalar addition

$$1.v = v$$

where 1 denotes the multiplicative identity in F.

V7. Distributivity of scalar multiplication with respect to vector addition

$$a(u + v) = av + av$$

V8. Distributivity of scalar multiplication with respect to field addition

$$(a + b)v = av + bv$$

When the scalar field F is real numbers $\mathbb{R}$, the vector space is called a real vector space. When the scalar field F is complex numbers $\mathbb{C}$, the vector space is called a comlpex vector space. $\mathbb{R}^1$, $\mathbb{R}^2$,…,$\mathbb{R}^n$ and $\mathbb{C}^1$, $\mathbb{C}^2$,…,$\mathbb{C}^n$ are the examples of vector spaces.

1.3.3. Topological Vector Space (TVS)

Definition: Suppose that τ is a topology on a vector space X such that

(i) every point of X is a closed set

(ii) the vector space are continuous with respect to τ.

Under these two conditions τ is called vector topology on X and X is called a topological vector space.

1.3.4. Paranorm on a linear space X and Paranormed (total paranormed) space

Definition: A paranorm$g$ on a linear space X over the real field $\mathbb{R}$ is a function $g: X \rightarrow \mathbb{R}$ having the following properties

(i) $g(\theta) = 0$ where θ is the zero vector in X.

(ii) $g(x) = g(-x)$ for all $x \in X$

(iii) $g(x + y) \leq g(x) + g(y)$ for all $x, y \in X$ i.e. g is subadditive in X

(iv) the scalar multiplication is continuous , that is, $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_{n-} \alpha x) \to 0$ for all $\alpha \in \mathbb{R}$ and $x \in X$, $(n \to \infty)$.

A paranormed space is a linear space X together with a paranorm g . A total paranorm is a paranorm such that

(v) $g(x) = 0$ implies $x = 0$

Every Paranormed (total paranormed) space is a semi-metric (metric) linear space. Conversely any semi-metric (metric) linear space can be turned into a paranormed (total paranormed) space. So a paranormed (total paranormed) space and semi-metric (metric) linear spaces are essentially the same.

1.3.5. Norm and Normed Linear Spaces

Norm:

Definition: A norm on a linear space X is a real function $\|.\|: X \to \mathbb{R}$ defined on X such that for every $x, y \in X$ and for all $\lambda \in \mathbb{C}$,

(i) $\|x\| > 0$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

(iii) $\|\lambda x\| = |\lambda| \|x\|$

(iv) $\|x\| = 0$ implies $x = 0$

A **seminorm** is defined by omitting condition (iv) in the definition of a norm. Every seminorm (norm) is a paranorm (total paranorm) but not conversely.

Normed linear space

Definition: A normed space (or normed linear space) is a pair $(X, \|.\|)$, where X is a linear space and $\|.\|$ is a norm on X .

1.3.6. Banach space

Definition: A Banach space $(X, \|.\|)$ is a complete normed linear space where completeness means that for sequence (x_n) in X with $\|x_m - x_n\| \to 0 \quad (m, n \to \infty)$, there exists $x \in X$ such that $\|x_n - x\| \to 0 \quad (n \to \infty)$.

Examples of normed linear space

$\mathbb{R}^n$ is a normed linear space with norm

(a)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

(b)

$$\|x\|_2 = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/2}$$

(c)

$$\|x\|_n = \left[\sum_{i=1}^{n} |x_i|^2\right]^{1/n}$$

(d)

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|$$

(ii) $C[a, b]$ is a normed linear space with norm

$$\|f\| = \sup_{x \in [a,b]} |f(x)|$$

where $C[a, b]$ is the set of continuous functions on $[a, b]$.

(iii) l_∞, c, c_0 are the normed linear spaces with the norm

$\|x\| = sup|x_n|$; but not with $\|x\| = \lim_{n \to \infty} |x_n|$

The word norm is used to denote the function that maps to $\|x\|$. Every normed linear space may be regarded as a metric together with a metric $d(x, y)$, i.e., distance between x and y is $d(x, y)$. In any metric space the open and closed balls with center at x and radius r are the sets

$$B_r(x) = \{y: d(x,y) < r\}$$

and

$$\overline{B_r(x)} = \{y: d(x,y) \leq r\}$$

respectively.

In particular , if X is a normed linear space, the sets

$$B_1(0) = \{x: \|x\| < 1\}$$

and

$$\overline{B_1(0)} = \{x: \|x\| \leq 1\}$$

are called the open unit balls and closed unit balls of X respectively. By declaring a subset of a metric space to be open if it is a (possibly empty) union of open balls, a topology is obtained . It is quite easy to verify that the vector space operations (addition and scalar multiplication) are continuous in this topology if the metric is defined in the form of a norm as above.

1.3.7. Inequalities

We list below some well known inequalities .

(i) Triangle inequality : For any $a, b \in \mathbb{C}$, we have $|a+b| \leq |a| + |b|$.

(ii) Let $p > 1$ and q be that $\frac{1}{p} + \frac{1}{q} = 1$, $a \geq 0$, $b \geq 0$. Then we have $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, with equality if and only if $a^p = b^q$.

(iii) Holder's inequality : Let $p > 1$ and q be that $\frac{1}{p} + \frac{1}{q} = 1$, $a_1, a_2, \dots, a_n \geq 0$ and $b_1, b_2, \dots, b_n \geq 0$. Then

$$\sum_{k=1}^{n} a_k b_k \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}$$

(iv) Minkowski's inequality:

Let $p \geq 1$, $a_1, a_2, \dots, a_n \geq 0$ and $b_1, b_2, \dots, b_n \geq 0$. Then

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}$$

This is called Minkowski's inequality.

1.3.8. Sequence spaces

Definition: Let ω be the family of all complex sequences (x_n) with $x_n \in \mathbb{C}$ and $n \in \mathbb{N}$. The family ω under usual point wise addition and scalar multiplication becomes a linear space over ℂ. Any subspace of ω is called a sequence space.

We shall list some of the sequence spaces which will be frequently used in our context.

(i) l_∞

This is the space of all bounded sequence of $x = (x_n)$ with natural metric

$$d(x, y) = \sup_n |x_n - y_n|$$

and is defined as

$$l_\infty = \{x = (x_k) \in \omega : \sup|x_k| < \infty\}.$$

(ii) The spaces c and c_0

These are the subsets of l_∞ , both having l_∞ metric. c is the space of convergent sequences and c_0 is the space of null sequences $(x_n \to 0)$. In the space c_0 (but not in c) one may actually use $\max|x_n - y_n|$ instead of $\sup|x_n - y_n|$ for the metric. We represent spaces c and c_0 as

$$c = \{x = (x_k) \in \omega : |x_k - l| \to 0 \text{ for some } l \in \mathbb{C}\}$$

$$= \{x = (x_k) \in \omega : |x_k| \to l\ ,\ k \to \infty \text{ for some } l \in \mathbb{C}\}$$

and

$$c_0 = \{x = (x_k) \in \omega : |x_k| \to 0 \ \ as\ k \to \infty\ \}$$

(iii) The space cs

It is the space of all convergent series and is defined as

$$cs = \left\{ x = (x_k) \in \omega \colon \left(\sum_{k=1}^{n} x_k \right)_{n=1}^{\infty} \text{ is convergent} \right\}$$

(iv) The space $l(p)$

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers, so that $0 < p_k \leq \sup p_k = H < \infty$. Then we define the sequence space $l(p)$ as

$$l(p) = \left\{ x = (x_k) \in \omega \colon \sum_{k=1}^{n} |x_k|^{p_k} < \infty \right\}.$$

A natural metric on $l(p)$ is

$$d(x,y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^{p_k} \right)^{\frac{1}{M}}$$

where d is a function

$$d \colon l(p) \times l(p) \to \mathbb{R}$$

As a special case when (p_k) is constant i.e. $p_k = p$, we write l_p for $l(p)$. We note that $p = (p_k)$ is a sequence in case of $l(p)$ whereas p is the number in case of l_p . Explicitly , for $p > 0$, l_p is the set of all sequences such that $\sum_{k=1}^{\infty} |x_k|^p < \infty$. For $p \geq 1$, the metric for l_p is

$$d(x,y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^{p} \right)^{\frac{1}{p}};$$

since $M = p$.

When $0 < p < 1$, since $M = 1$, the metric for l_p is

$$d(x,y) = \sum_{k=1}^{\infty} |x_k - y_k|^{p}.$$

For l_p , the cases $p = 1$ and $p = 2$ are the special case of importance. The metrics for l_1 and l_2 are respectively given by

$$d(x,y) = \sum_{k=1}^{\infty} |x_k - y_k|$$

and

$$d(x,y) = \left(\sum_{k=1}^{\infty} |x_k - y_k|^2\right)^{\frac{1}{2}}.$$

The space l_2 is often called the Hilbert space.

(iv) The space $l_\infty(p)$

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers, so that $0 < p_k \leq \sup p_k = H < \infty$. Then we define the sequence space $l_\infty(p)$ as

$$l_\infty(p) = \left\{x = (x_k): \sup_k |x_k|^{p_k} < \infty\right\}$$

$l_\infty(p)$ is a metric space with the metric

$$d(x,y) = \sup_k |x_k - y_k|^{\frac{p_k}{M}}$$

where $(x,y) \in l_\infty(p)$ and $\mathrm{M} = \max(1, \sup p_k = H)$. If (p_k) is constant i.e. $p_k = p$, we write l_∞ for $l_\infty(p)$. Here l_∞ is the set of all bounded sequences $x = (x_k)$.

(vi) The spaces $c(p)$ and $c_0(p)$

If $p = (p_k)$ be a bounded sequence of strictly positive real numbers, we define

$$c(p) = \{x = (x_k) \in \omega: |x_k - l|^{p_k} \to 0 \text{ as } k \to \infty \text{ for some } l \in \mathbb{C}\}$$

and

$$c_0(p) = \{x = (x_k) \in \omega: |x_k|^{p_k} \to 0 \text{ as } k \to \infty \}$$

These spaces are metric spaces with metric

$$d(x,y) = \sup_k |x_k - y_k|^{\frac{p_k}{M}}$$

where

$$\mathrm{M} = \max(1, \sup p_k = H).$$

If $p = (p_k)$ is constant i.e. $p_k = p$ for all k we write c and c_0 for $c(p)$ and $c_0(p)$ respectively. The spaces c and c_0 represent the sets of all convergent sequence and null sequences respectively. We note that c and c_0 are the metric spaces with the metric

$$d(x, y) = \sup_k |x_k - y_k|$$

(vii) The difference sequences $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$

Kizmaz [41] defined the difference sequences $l_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ as,

$$l_\infty(\Delta) = \{x = (x_k): \Delta x \in l_\infty\}$$

$$c(\Delta) = \{x = (x_k): \Delta x \in c\}$$

$$c_0(\Delta) = \{x = (x_k): \Delta x \in c_0\}$$

where $\Delta x = x_k - x_{k+1}$.

These spaces are Banach spaces with norm

$$\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$$

(viii) The spaces $\Delta l_\infty(p)$ and $l_\infty(\Delta_r p)$

Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers , then we define $\Delta l_\infty(p)$ as

$$\Delta l_\infty(p) = \{x = (x_k): \Delta x \in l_\infty(p)\}.$$

The sequence space $\Delta l_\infty(p)$ is paranormed by

$$g(x) = \sup_k |\Delta x_k|^{\frac{p_k}{M}}$$

Also if $\Delta_r(x) = (k^r \Delta x_k)_{k=1}^\infty$, (r < 1) where $\Delta x = x_k - x_{k+1}$, then we define $l_\infty(\Delta_r p)$ as ,

$$l_\infty(\Delta_r p) = \{x = (x_k): \Delta_r x \in l_\infty(p), r < 1\}$$

(ix) The spaces $w(p)$, $w_0(p)$ and $w_\infty(p)$

If $p = (p_k)$ be a bounded sequence of strictly positive real numbers, Maddox [45] defined the sequence spaces $w(p)$, $w_0(p)$ and $w_\infty(p)$ as:

$$w(p)=\left\{x=(x_k)\in\omega:\frac{1}{n}\sum_{k=1}^{n}|x_k-l|^{p_k}\to 0;\ \ for\ some\ \ l\in C, n\to\infty\right\}$$

$$w_0(p)=\left\{x=(x_k)\in\omega:\frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k}\to 0\ ,\ \ n\to\infty\right\}\text{ and}$$

$$w_\infty(p) = \left\{x = (x_k) \in \omega: \sup_{n\in\mathbb{N}}\frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k} < \infty\right\}$$

The spaces $w(p)$ and $w_0(p)$ are paranormed spaces paranormed by

$$g(x) = \sup\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k}\right)^{\frac{1}{M}}$$

or equivalently

$$g(x) = \sup_r(2^{-r}\Sigma_r|x_k|^{p_k})^{\frac{1}{M}}$$

(1.3.1)

where $\sum_r$ is the sum over the range $2^r \le k < 2^{r+1}$ and $M = (1, \sup p_k)$ as in [44,45]. Further $w_\infty(p)$ is the paranormed space by the paranorm (1.3.1) if and only if $0 < \inf\ p_k \le \sup p_k < \infty$ [44].

(x) The space $\Omega(t)$

The sequence space $\Omega(t)$ was introduced by Fricke and Fridy [38]. For each r in the interval $(0,1)$,

let

$$G(r) = \{x = (x_k) \in \omega : x_k = \mathrm{O}(t_k)\}.$$

We define the set of geometrically dominated sequences as

$$G = \bigcup_{r \in (0,1)} G(r)$$

The analytic sequences are defined by

$$\text{A} = \left\{ x = (x_k) \in \omega : \lim_n \sup |x_n|^{\frac{1}{n}} < \infty \right\}$$

Obviously $G \subseteq A$. Various authors studied matrix transformation from A or G into l_1, c or l_∞ , but the question of mapping from l_1, c or l_∞ into A or G was not considered. To set the stage for general theory, Fricky and Fridy replaced the geometric sequence (r^k) with a nonnegative sequence $t = (t_k)$ and defined the sequence space

$$\Omega(t) = \{ x = (x_k) \in \omega : x_k = \text{O}(t_k) \}.$$

(xi) The sequence space $\overline{l(p)}$

If $p = \{p_k\}$ be a bounded sequence of strictly positive real numbers, then Chodhary and Mishra [15] introduced and studied the sequence space $\overline{l(p)}$ which is defined as

$$\overline{l(p)} = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |t_k(x)|^{p_k} < \infty \right\}$$

where

$$t_k(x) = \sum_{i=1}^{k} x_i \, .$$

If $p = (p_k)$ is constant i.e. $p_k = p$ for all, then we write $\bar{l}_p$ for $\overline{l(p)}$.

1.3.9. Cauchy sequence

A sequence (x_n) in a normed linear space X for every $n \in \mathbb{N}$ is called a Cauchy sequence in X if and only if

$$\|x_n - x_m\| < \varepsilon \, , \ (m, n \to \infty)$$

That is for every $\varepsilon > 0$ there exists $N_0 = N_0(\varepsilon)$ such that $\|x_n - x_m\| < \varepsilon$ for all $m, n > N_0$.

1.3.10. Complete normed linear space

Definition: A normed linear space is said to be complete if every Cauchy sequence in X converges to an element $x \in X$ i.e. for every sequence (x_n) in X with $\|x_n - x_m\| \to 0,\ (m, n \to \infty)$, there exists $x \in X$ such that $\|x_n - x\| \to 0$, $(n \to \infty)$.

We note that a complete normed linear space is called a Banach space. The spaces $\mathbb{R}^n, \mathbb{C}^n, cs,\ l(p), l_\infty,\ c,\ c_0,\ l_p (1 \le p < \infty)$ are the examples of Banach space.

In a normed space convergence and absolute convergence of series may be defined in a natural way. A series $\sum_{k=1}^{\infty} x_k$ with $x_k \in X$ is called convergent to $s \in X$ if and only if $s_n \to s\ (n \to \infty)$, i.e. $\|s_n - s\| \to 0\ (n \to \infty)$ where $s_n = \sum_{k=1}^{n} x_k$. A series $\sum x_k$ is called absolutely convergent if and only if $\sum \|x_k\| < \infty$. In $\mathbb{R}$ and $\mathbb{C}$ it is well known that every absolutely convergent series is convergent , and this result depends upon completeness.

Following theorem gives a nice series characterization of a Banach space.

Theorem: A normed linear space is complete if and only if every absolutely convergent series in X is also convergent in X [48].

1.3.11. Homeomorphisms

Definition: Let X, Y be topological spaces. Then $f: X \to Y$ is called a homeomorphism if and only if it is bijective and bicontinuous. Bicontinuous means that both f and f^{-1} are continuous. Equivalently, f is a homeomorphism if and only if it is bijective, continuous and open.

As an example the open interval and the whole real line $\mathbb{R}$ are homeomorphic with homeomorphism

$$f(x) = \frac{2x - 1}{x(x - 1)}, \qquad x \in (0,1)$$

1.3.12. Isomorphism

Definition: Let X, Y be linear spaces over the same scalar field. A map $f: X \to Y$ is called linear if $f(\lambda x_1 + \mu x_2) = \lambda f(x_1) + \mu f(x_2)$ for all scalars λ, μ and all

$x_1, x_2 \in X$. An isomorphism $f: X \to Y$ is a bijective linear map. We say that X and Y are isomorphic if there is an isomorphism $f: X \to Y$. We regard isomorphic linear spaces as equivalent from the algebraic linear space point of view, for an isomorphism clearly preserves the linear operations.

For an example, the sequence space $\overline{l(p)}$ is isomorphic to the space $l(p)$.

1.3.13. Basis in a paranormed space (X, g)

Definition: Let (X, g) be a paranormed space . A sequence (b_k) of elements of X is called a basis for X if and only if, for each $x \in X$, there exists a unique sequence (λ_k) of scalars such that

$$x = \sum_{k=1}^{\infty} \lambda_k \, b_k$$

that is, such that

$$g\left(x - \sum_{k=1}^{n} \lambda_k \, b_k\right) \to 0 \;\; (n \to \infty).$$

The idea of basis was introduced by Schaulder in 1927 and what we call a basis is often termed as a Schauder basis.

The sequence $(e_k) = (e_{1,} e_2, \ldots.)$ of unit vector is a basis for each of the spaces $l(p)$ and c_0 under their usual paranorms

$$g(x) = \left(\sum |x_k|^{p_k}\right)^{\frac{1}{M}} \;\text{ on }\; l(p)$$

and

$$\|x\| = \sup_k \; |x_k|$$

on c_0 .

The sequence $(e, e_{1,} e_2, \ldots)$ is a basis for the space c of convergent sequences under its natural norm given by

$$\|x\| = \sup_k \; |x_k|$$

for each $x = (x_k) \in c$. By e we denote the sequence $(1,1,1, \ldots)$ and by e_k the k^{th} unit vectors.

Not all normed spaces have a basis. For example, l_∞ , the space of all bounded sequences, with the natural norm $\|x\| = \sup_k |x_k|$ has no basis.

1.3.14. Duals of the sequence space

Definition: For a sequence space X we define

(i)

$$X^\alpha = \left\{a = (a_k): \sum_{k=1}^{\infty} |a_k x_k| < \infty \text{ for every } x \in X\right\}$$

(ii)

$$X^\beta = \left\{a = (a_k): \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in X\right\}$$

(iii)

$$X^\gamma = \left\{a = (a_k): \sup_n \left|\sum_{k=1}^{n} a_k x_k\right| < \infty \text{ for each } x \in X\right\}$$

X^α , X^β and X^γ are called the α- (or Köthe- Toeplitz), β- (or generalized Köthe-Toeplitz [1]) and γ – dual spaces of X respectively. These duals were introduced by Garling [27].

We note that $X^\alpha \subseteq X^\beta \subseteq X^\gamma$. We state below β- duals of the some of the sequence spaces.

Theorem [1].

The β- dual of the sequence spaces c and c_0 is the space l_1 defined by

$$l_1 = \left\{x = (x_k): \sum |x_k| < \infty\right\}$$

Theorem [2].

(i) For $0 < p \leq 1$, the β- dual of the sequence space l_p is the space l_∞ .

(ii) For $1 < p < \infty$, the β- dual of the sequence space l_p is the space l_q where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem [3].

The β- dual of the sequence space l_∞ is $ba(N)$ which is the space of all bounded finitely additive set functions μ defined on the set of all positive integers $\mathbb{N}$.

We note that the β- duals of sequence spaces, c_0 and $l_p(0 < p < \infty)$ are also sequence spaces but that of l_∞ is not a sequence space. This is due to the fact that the sequence space l_∞ has no basis.

Theorem [4].

(i) If $0 < p_k \leq 1$ for every $k \in \mathbb{N}$, then

$$l(p)^\beta = l_\infty(p) \text{ [82]}$$

(ii) If $p_k > 1$ for every $k \in \mathbb{N}$, then

$$l(p)^\beta = \mathcal{M}(p)$$

where

$$\mathcal{M}(p) = \bigcup_{N>1} \left\{ a = (a_k): \sum_{k=1}^{\infty} |a_k|^{q_k} N^{-\frac{q_k}{p_k}} < \infty \right\}$$

with

$$\frac{1}{p_k} + \frac{1}{q_k} = 1 \quad \text{[47].}$$

Theorem [5].

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then

$$l_\infty(p)^\beta = \mathcal{M}_\infty(p)$$

where

$$\mathcal{M}_\infty(p) = \bigcap_{N>1} \left\{ a = (a_k): \sum_{k=1}^{\infty} |a_k| N^{\frac{1}{p_k}} < \infty \right\} \text{ [25].}$$

Theorem [6].

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then

$$c_0(p)^\beta = \mathcal{M}_0(p)$$

where

$$\mathcal{M}_0(p) = \bigcup_{N>1} \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k| N^{-\frac{1}{p_k}} < \infty \right\} \text{ [47]}$$

Theorem [7].

If $0 < p_k \leq 1$ for every $k \in \mathbb{N}$, then

$$w(p)^\beta = \mathcal{M}$$

where

$$\mathcal{M} = \left\{ a = (a_k) : \sum_{r=0}^{\infty} \max\nolimits_r \left[(2^r N^{-1})^{\frac{1}{p_k}} |q_k| \right] < \infty \text{ for some integer } N > 1 \right\}$$

and $\max_r$ is the maximum taken over $2^r \leq k < 2^{r+1}$ [25].

Theorem [8].

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then

$$c(p)^\beta = \mathcal{M}_0(p) \cap cs$$

where

$$\mathcal{M}_0(p) = \bigcup_{N>1} \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k| N^{-\frac{1}{p_k}} < \infty \right\}$$

and

$$cs = \left\{ x \in \omega : \sum_k x_k \text{ converges} \right\} \text{[24].}$$

Theorem [9].

(i) If $0 < p_k \leq 1$ for every $k \in \mathbb{N}$, the β- duals of sequence space $\overline{l(p)}$ is the sequence space $\overline{l_\infty(p)}$ which is defined as

$$l_\infty(p) = \{a = (a_k) : \sum_{k=1}^{\infty} a_k \,(-\sum_{\nu=1}^{k-1} (N^{-2})^{\frac{1}{p_\nu}} + (N^{-2})^{\frac{1}{p_k}} \text{ converges}\}$$

and $sup_k |a_k|^{p_k} < \infty\}$, $N \geq 1$, $\Delta a_k = a_k - a_{k+1}$ [15].

(ii) If $1 < p_k \leq \sup\, p_k < \infty$ for every $k \in \mathbb{N}$, the β- duals of sequence space $\overline{l(p)}$ is the sequence space $\overline{l_\infty(p)} = \overline{M(p)}$ where

$$\overline{M(p)} = \{a = (a_k) : \sum_{k=1}^{\infty} a_k \,(-\sum_{\nu=1}^{k-1} (N)^{-\frac{p_\nu}{q_\nu}} + (N)^{-\frac{p_k}{q_k}} \text{ converges}\}$$

and

$$\sum_{k=1}^{\infty} |\Delta a_k|^{q_k} (N)^{-\frac{p_k}{q_k}} < \infty\,, N > 1 \text{ and } \frac{1}{p_k} + \frac{1}{q_k} = 1 \text{ [15].}$$

1.3.15 Matrix transformations

Definition: Let X and Y be any two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of complex numbers $(n, k = 1,2, \dots)$. We write $Ax = (A_n(x))$ if

$$A_n(x) = \sum_k a_{nk} x_k$$

converges for each $n \in \mathbb{N}$. If $x = (x_k) \in X$ implies that $Ax = (A_n(x)) \in Y$, then we say that A defines a matrix transformation from X into Y and we denote it by writing $A : X \to Y$. The sequence Ax is called the A transform of X. By (X, Y) we mean the classes of the matrices A such that $A: X \to Y$. The matrix A is also called the linear operator. We list below the some of the inclusion theorems on matrix transformation of well known sequence spaces.

Theorem [1]

$A \in (l_\infty, l_\infty)$ if and only if

$$\sup_n \sum_k |a_{nk}| < \infty$$

.

Theorem [2] : Kojima- Schur

$A \in (c, c)$ if and only if

(i)

$$\sup_n \sum_k |a_{nk}| < \infty$$

(ii)

$$\lim_{n\to\infty} a_{nk} = \alpha_k$$

(iii)

$$\lim_{n\to\infty} \sum_k a_{nk} = \alpha$$

Theorem [3]

$A \in (l_\infty(p), l_\infty)$ if and only if

$$\begin{matrix} sup \\ n \end{matrix} \sum_k |a_{nk}| N^{1/p_k} < \infty \text{ for every integer } N > 1$$

Theorem [4]: Schur

$A \in (l_\infty, c)$ if and only if

(i)

$$\sum_{k=1}^{\infty} |a_{nk}|$$

converges uniformly in $n \in \mathbb{N}$.

(ii) There exists

$$\lim_{n\to\infty} a_{nk} = \alpha_k$$

for each $n \in \mathbb{N}$

The class (l_∞, c) was obtained by Schur in 1921. The characterization of this class is known as Schur theorem and the matrices in the class (l_∞, c) are known as Schur matrices.

Theorem [5].

$A \in (l_1, l_p)$ if and only if

(i)

$$M = \sup_k \sum_n |a_{nk}|^p < \infty \qquad (1 \le p < \infty)$$

(ii)

$$\sup_{n,k} |a_{nk}| < \infty (p = \infty) \text{ for } k \in \mathbb{N}.$$

Theorem [6].

Let $1 < p_k < \infty$ and let $A \in (l_\infty, l_\infty) \cap (l_1, l_1)$. Then $A \in (l_p, l_p)$.

Theorem [7]

Let $1 < p_k \le \sup p_k = H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (l(p), l_\infty)$ if and only if there is an integer $B > 1$ such that

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}|^{q_k} B^{-q_k} < \infty$$

where $\frac{1}{p_k} + \frac{1}{q_k} = 1$.

Theorem [8]

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then $A \in (l(p), l_\infty)$ if and only if

$$\sup_n |a_{nk}|^{p_k} < \infty$$

Theorem [9]

Let $1 < p_k \le \sup p_k = H < \infty$ for every $k \in \mathbb{N}$. Then $A \in (l(p), c)$ if and only if

(i) there exists an integer $B > 1$ such that

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}|^{q_k} B^{-q_k} < \infty$$

where

$$\frac{1}{p_k} + \frac{1}{q_k} = 1$$

(ii)

$$a_{nk} \to \alpha_k (n \to \infty)$$

and k is fixed.

Theorem [10]

Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (l(p), c)$ if and only if

(i)

$$\sup_n |a_{nk}|^{p_k} < \infty$$

(ii)

$$a_{nk} \to \alpha_k (n \to \infty)$$

and k is fixed.

Theorem [11].

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $A \in (l_\infty(p), l_\infty)$ if and only if

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| \, N^{\frac{1}{p_k}} < \infty$$

for every integer $N > 1$.

Theorem [12].

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $A \in (l_\infty(p), c)$ if and only if

(i)

$$\sum_{k=1}^{\infty} |a_{nk}| \, N^{\frac{1}{p_k}}$$

converges uniformly in n , for all integers $N > 1$.

(ii)

$$a_{nk} \to \alpha_k (n \to \infty)$$

and k is fixed.

Theorem [13].

Let $(p_k) \in l_\infty$, then $A \in (c(p), c)$ if and only if

(i) there exists an absolute constant $B > 1$ such that

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| \, B^{-\frac{1}{p_k}} < \infty$$

(ii)

$$\lim a_{nk} \to \alpha_k (n \to \infty)$$

and k is fixed.

(iii)

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = \alpha$$

Theorem [14].

Let $(p_k) \in l_\infty$, then $A \in (c_0(p), c)$ if and only if

(i) there exists an absolute constant $B > 1$ such that

$$\sup_n \sum_{k=1}^{\infty} |a_{nk}| \, B^{-\frac{1}{p_k}} < \infty$$

(ii)

$$\lim a_{nk} \to \alpha_k (n \to \infty)$$

exists for every fixed k.

Theorem [15]

Let $0 < p < 1$. Then $A \in (w_p, c)$ if and only if

(i)

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} a_{nk} = \alpha_k$$

k is fixed.

(ii)

$$M(A) = \sup_n \sum_{r=0}^{\infty} 2^{r/p} A_r^1(n) < \infty$$

where

$$A_r^1(n) = \max_r |a_{nk}|$$

for each n .The maximum is taken for k such that

$$2^r \leq k < 2^{r+1}.$$

Theorem [16] .

Let $p \geq 1$. Then $A \in (w_p, c)$ if and only if

(i)

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \alpha_k$$

k is fixed.

(ii)

$$\sup_n \sum_{r=0}^{\infty} 2^{r/p} A_r^p(n) < \infty$$

Theorem [17] .

Let $0 < p_k \leq 1$. Then $A \in (w(p), c)$ if and only if

(i) there exists an integer $B > 1$ such that

$$C = \sup_n \sum_{r=0}^{\infty} max_r \left\{ (2^r B^{-1})^{\frac{1}{p_k}} |a_{nk}| \right\} < \infty$$

(ii)

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = \alpha_k$$

exists for every fixed k .

(iii)

$$\lim_{n\to\infty} \sum a_{nk} = \alpha$$

exists .

1.3.16 Some special types of matrices

(i) Sparse and dense matrices

Definition : A sparse matrix is a matrix populated primarily with zeros as element or entries. On the contrary , if a large number of element differ from zero , then it is common to refer to the matrix as a dense matrix. The fraction of zero elements (or non zero elements) in a matrix is called the sparsity (or density). As an example we can observe that the matrix given by

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 6 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$$

is a sparse matrix which contains only 9 non zero elements out of 35 , with 26 of these elements as zero.

(ii) Band matrix

Definition: A band matrix is a sparse matrix whose non zero entries are confined to a diagonal band , comprising the main diagonal and zero or more diagonals on either side. We may define a band matrix in terms of matrix bandwidth. Consider an $n \times n$ matrix $A = (a_{ij})$. If all matrix elements are zero outside a diagonally bordered band whose range is determined by constants k_1 and k_2 :

$$a_{ij} = 0 \quad \text{if} \quad j < i - k_1 \text{ or } \quad j > i + k_2 \;;\; k_1, k_2 \geq 0$$

then the quantities k_1 and k_2 are called the left and right hand bandwidth respectively. The bandwidth of the matrix is $k_1 + k_2 + 1$. In other words , it is the smallest number of adjacent diagonals to which the non zero elements are confined. In this connection , a matrix is called a band matrix if its bandwidth is reasonably small.

A band matrix with $k_1 = k_2 = 0$ is a diagonal matrix ; a band matrix with $k_1 = k_2 = 1$ is a tridiagonal matrix ; when with $k_1 = k_2 = 2$ one has a pentadiagonal matrix and so on. If one puts $k_1 = 0$, $k_2 = n - 1$, one obtains the definition of an upper triangular matrix. Similarly for $k_1 = n - 1$ and $k_2 = 0$ one obtains a lower triangular matrix.

As an example the matrix

$$\Delta_j = \begin{pmatrix} 1 & -2 & 0 & 0 & \cdots \\ 0 & 2 & -3 & 0 & \cdots \\ 0 & 0 & 3 & -4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a double band matrix.

(iii) Unitriangular matrix

Definition: If the entries of the main diagonal of a (upper or lower) are all 1 , the matrix is called (upper or lower) unitriangular. For example the matrix

$$\lambda = S^n = (\lambda_{nk}) = \begin{cases} n - k + 1, & n \geq k \\ 0, & \text{otherwise} \end{cases}$$

that is

$$\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 2 & 1 & 0 & 0 & \cdots \\ 3 & 2 & 1 & 0 & \cdots \\ 4 & 3 & 2 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is a lower unitriangular matrix.

1.3.17 Infinite matrices as a difference operator

We give brief account of the infinite matrices and difference operators that we have used and taken as a reference in our context.

(i) The infinite matrix S

The matrix $S = (s_{nk})$ introduced in [15] is defined as

$$S = (s_{nk}) = \begin{cases} 1, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

It is an infinite matrix given by

$$S = (s_{nk}) = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & \dots \\ 1 & 1 & 1 & 0 & \dots \\ 1 & 1 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Chaudhary and Mishra [15] have defined the sequence space $\overline{l(p)}$ which consists of all sequences whose S- transform are in $l(p)$ i.e.

$\overline{l(p)} = [l(p)]_S$.

(ii) The matrix R^t

It is the matrix of Riesz mean (R, t_n) and is given by

$$R^t = (r_{nk}^t) = \begin{cases} t_k / \sum_{k=0}^{n} t_k\,, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

where (t_k) is the sequence of positive real numbers.

Altay and Basar [11] have defined the spaces $r_0^t(p)$, $r_c^t(p)$, $r_\infty^t(p)$ and $r^t(p)$ which consists of all sequences whose R^t transforms are in $c_0(p)$, $c(p)$, $l_\infty(p)$ and $l(p)$ respectively, that is,

$r_0^t(p) = [c_0(p)]_{R^t}$, $r_c^t(p) = [c(p)]_{R^t}$, $r_\infty^t(p) = [l_\infty(p)]_{R^t}$ and $r^t(p = [l(p)]_{R^t}$.

(iii) Cesaro matrix of order 1

The matrix defined by

$$C = (c_{nk}) = \begin{cases} \frac{1}{n}, & 1 \le k \le n \\ 0, & k > n \end{cases}$$

is called the Cesaro matrix of order 1 or the matrix of arithmetic mean.

The sequence spaces $w(p), w_0(p)$ and $w_\infty(p)$ which are defined by Maddox [44,45] consists of the sequences whose all C- transforms are in the spaces $l(p)$, $c_0(p)$ and $l_\infty(p)$ respectively, i.e.

$w(p) = [l(p)]_C$, $w_0(p) = [c_0(p)]_C$ and $w_\infty(p) = [l_\infty(p)]_C$.

(iv) The matrix $G(u, v)$

We denote by U the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $u \in U$, let $\frac{1}{u} = \left(\frac{1}{u_n}\right)$. Then we define the matrix $G(u, v)$ which is called the generalized weighted mean or factorable matrix as

$$G(u, v) = (g_{nk}) = \begin{cases} u_n v_k, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

that is

$$G(u,v) = \begin{pmatrix} u_1v_1 & 0 & 0 & 0 & \dots \\ u_2v_1 & u_2v_2 & 0 & 0 & \dots \\ u_3v_1 & u_3v_2 & u_3v_3 & 0 & \dots \\ u_4v_1 & u_4v_2 & u_4v_3 & u_4v_4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Recently in 2006/2007 Altay and Basar [12,13] have defined the sequence spaces $l(u, v, p)$ and $\lambda(u, v, p)$ for $\lambda \in \{l_\infty, c, c_0\}$ which are derived by using the generalized weighted mean $G(u, v)$. The space $l(u, v, p)$ consists of all sequences whose $G(u, v)$ transforms are in $l(p)$ and $\lambda(u, v, p)$ for $\lambda \in \{l_\infty, c, c_0\}$ consist of all sequences whose $G(u, v)$ transforms are in $l(p)$, that is,

$$l(u, v, p) = [l(p)]_{G(u,v)}$$

and

$$\lambda(u, v, p) = [\lambda(p)]_{G(u,v)}$$

for $\lambda \in \{l_\infty, c, c_0\}$.

Using the matrix $G(u, v)$ as the operator we have introduced and studied new sequence spaces $w(u, v, p)$, $w_0(u, v, p)$ and $w_\infty(u, v, p)$.

(v) The difference operator matrix Δ

The difference operator matrix Δ is defined as

$$\Delta = (\delta_{nk}) = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n \\ 0, & 0 \le k < n \text{ or } k > n \end{cases}$$

that is,

$$\Delta = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

It is a double band matrix.

In 2012 Demiriz and Cakan [78] have defined new sequence spaces $\lambda(u,v;p,\Delta)$ for $\lambda \in \{c_0, c, l_\infty, l\}$ by using the operator matrix $G(u,v,\Delta)$ defined by

$$G(u,v,\Delta) = G(u,v)\Delta = \left(g_{nk}^{\Delta}\right) = \begin{cases} u_n(v_k - v_{k+1}), & 0 \le k \le n-1 \\ u_k v_k, & k = n \\ 0, & k > n \end{cases}$$

that is,

$$G(u,v,\Delta) = \begin{pmatrix} u_1 v_1 & 0 & 0 & 0 & \cdots \\ u_2(v_1 - v_2) & u_2 v_2 & 0 & 0 & \cdots \\ u_3(v_1 - v_2) & u_3(v_2 - v_3) & u_3 v_3 & 0 & \cdots \\ u_4(v_1 - v_2) & u_4(v_2 - v_3) & u_4(v_3 - v_4) & u_4 v_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The matrix $G(u,v,\Delta)$ is the combination (product) of the matrices $G(u,v)$ and Δ . The sequence spaces $\lambda(u,v;p,\Delta)$ for $\lambda \in \{c_0, c, l_\infty, l\}$ consist of all sequences whose $G(u,v,\Delta)$ transforms are in λ, that is,

$$\lambda(u,v;p,\Delta) = [\lambda(p)]_{G(u,v,\Delta)} .$$

Using the matrix $G(u,v,\Delta)$ as an operator we have introduced and studied new sequence spaces $(u,v;p,\Delta)$, $w_0(u,v;p,\Delta)$ and $w_\infty(u,v;p,\Delta)$.

(vi) The matrix λ

In our context in chapter three we have defined an infinite matrix λ which is the n'th power of $S = (s_{nk})$.

Thus

$$\lambda = S^n = (\lambda_{nk}) = \begin{cases} n-k+1, & n \geq k \\ 0, & otherwise \end{cases}$$

that is,

$$\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 2 & 1 & 0 & 0 & \dots \\ 3 & 2 & 1 & 0 & \dots \\ 4 & 3 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

It is also a lower unitriangular matrix. Using the matrix λ as the operator we have defined the sequence spaces $l(p,\lambda)$, $l_\infty(p,\lambda)$, $c(p,\lambda)$ and $c_0(p,\lambda)$.

(vii) The matrix λ_j

In our context we have defined an operator matrix λ_j which can be expressed as a sequential double band matrix given by

$$\lambda_j = \begin{pmatrix} \frac{1}{t_1} & -\frac{1}{t_1} & 0 & 0 & \dots \\ 0 & \frac{1}{t_2} & -\frac{1}{t_2} & 0 & \dots \\ 0 & 0 & \frac{1}{t_3} & -\frac{1}{t_3} & \dots \\ 0 & 0 & 0 & \frac{1}{t_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

To construct the matrix λ_j , we have defined a diagonal matrix

$$diag\left(\frac{1}{t_{ij}}\right) = \begin{cases} \frac{1}{t_j}, & i = j \\ 0, & \text{otherwise} \end{cases}$$

that is,

$$\text{diag}\left(\frac{1}{t_{ij}}\right) = \begin{pmatrix} \frac{1}{t_1} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{t_2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{t_3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{t_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where each entry $t = \left(\frac{1}{t_j}\right) \in (0,1)$.

The multiplication of the difference operator matrix Δ and diag $\left(\frac{1}{t_{ij}}\right)$ yields a double band matrix

$$\Delta\,.diag\,\left(\frac{1}{t_{ij}}\right) = \begin{pmatrix} \frac{1}{t_1} & 0 & 0 & 0 & \cdots \\ -\frac{1}{t_1} & \frac{1}{t_2} & 0 & 0 & \cdots \\ 0 & -\frac{1}{t_2} & \frac{1}{t_3} & 0 & \cdots \\ 0 & 0 & -\frac{1}{t_3} & \frac{1}{t_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We have defined the transpose of Δ.diag $\left(\frac{1}{t_{ij}}\right)$ as the matrix λ_j , which is a double band sparse matrix. Using the matrix λ_j together with generalized weighted mean $G(u,v)$ we have defined the new sequence spaces $X(u,v;p,\lambda_j)$ for $X \in \{l_\infty, c, c_0\}$.

CHAPTER TWO

Part One:
Paranormed Sequence Spaces $w(u,v,p)$, $w_0(u,v,p)$ and $w_\infty(u,v,p)$ Generated by Generalized Weighted Mean $G(u,v)$

2.1. Preliminaries

By ω we mean the spaces of all complex valued sequences. A vector subspace of ω is called a sequence space. The usual notations l_∞, c and c_0 represent for the spaces of all bounded, convergent and null sequence respectively. A linear topological space X over the field $\mathbb{R}$ is said to be a paramormed space if

(i) there is a subadditive function

$g: X \to \mathbb{R}$ such that $g(\theta) = 0$, where θ is the zero vector in the linear space X .

(ii) $g(x) = g(-x)$ for all $x \in X$

(iii) scalar multiplication is continuous ,that is, $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$.

If $p = \{p_k\}$ be a bounded sequence of strictly positive real numbers, Maddox [45] defined the sequence spaces $w(p), w_0(p)$ and $w_\infty(p)$ as:

$$w(p) = \left\{x = (x_k) \in \omega : \frac{1}{n}\sum_{k=1}^{n} |x_k - l|^{p_k} \to 0, \text{ for some } l \in \mathbb{C}, \quad n \to \infty\right\}$$

$$w_0(p) = \left\{x = (x_k) \in \omega : \frac{1}{n}\sum_{k=1}^{n} |x_k|^{p_k} \to 0 , \quad n \to \infty\right\} \text{ and}$$

$$w_\infty(p) = \left\{x = (x_k) \in \omega : \sup_{n\in\mathbb{N}} \frac{1}{n}\sum_{k=1}^{n} |x_k|^{p_k} < \infty\right\}$$

It has been shown in [44] that the spaces $w(p)$ and $w_0(p)$ are paranormed spaces paranormed by

$$g(x) = \sup\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^{p_k}\right)^{\frac{1}{M}}$$

or equivalently

$$g(x) = \sup_r\left(2^{-r}\sum_r|x_k|^{p_k}\right)^{\frac{1}{M}}$$

(2.1.1)

where $\sum_r$ is the sum over the range $2^r \le r < 2^{r+1}$ and $M = (1, \sup p_k = H)$. Further $w_\infty(p)$ is the paranorm space paranormed by (2.1.1) if and only if $0 < \inf\ p_k \le \sup\ p_k < \infty$ [44]. Now we shall prove it.

Let us suppose that (2.1.1) is the paranorm for the space $w_\infty(p)$. Then $w_\infty(p)$ is a linear space and so $\sup p_k < \infty$ [44,45]. For a real scalar λ such that $\lambda \to 0$ and a sequence $x \in w_\infty(p)$ such that x is fixed imply $\lambda x \to \theta$, a zero vector of $w_\infty(p)$. This property implies that $\inf p_k > 0$. On the contrary , let us suppose that it is not. Then there exists $k_1 < k_2 < \cdots$ such that $p_{k_i} < \frac{1}{i}$, $(i = 1,2, \ldots)$.

Also k_i must be choosen in such a way that k_1 lies in the interval $2^{r_1} \le k_1 < 2^{r_1+1}$, k_2 lies in the interval $2^{r_2} \le k_2 < 2^{r_2+1}$ …… and so on, where $r_1 < r_2 < \cdots$. Now define

$$\overline{x_k} = \begin{cases} 2^{r_i/p_k}, & k = k_i \\ 0, & \text{otherwise} \end{cases}$$

Then if we write

$$h(x) = \sup_r\left\{\left(\frac{1}{2^r}\right)\sum_r|x_k|^{p_k}\right\}^{1/M}$$

for all $x \in w_\infty(p)$ where $\sum_r$ is the sum over $2^r \le k < 2^{r+1}$, we have

$$\frac{1}{2}\ g(x) \le h(x) \le 2\ g(x)$$

(2.1.2)

where $g(x)$ is as defined in (2.1.1).

Now $h(\bar{x}) = 1$, but for $r = r_i$ and $0 < |\lambda| \leq 1$,

$$\left(\frac{1}{2^r}\right)\sum_r |\lambda\, \overline{x_k}|^{p_k} = |\lambda|^{p_{k_i}} \geq |\lambda|^{1/i} \to 1 \text{ as } i \to \infty \,.$$

Hence for $0 < |\lambda| \leq 1$, we have $h(\lambda\bar{x}) = 1$ and so $g(\lambda\bar{x}) \geq \frac{1}{2}$ by (2.1.2).

But this contradicts the fact that $\lambda \to 0$, $\bar{x} \in w_\infty(p)$imply $\lambda\bar{x} \to \theta$, the zero vector of $w_\infty(p)$. Hence the condition $0 < \inf p_k \leq \sup p_k < \infty$ is necessary.

On the other hand, let us suppose $0 < \inf p_k \leq \sup p_k < \infty$. We need to show (2.1.1) is the paranorm for $w_\infty(p)$. By the definition of g it immediately follows that $g(x) = 0 \Leftrightarrow x = 0$ and $g(x) = g(-x)$ and for $x, y \in w_\infty(p)$ the subadditivity of g follows from Minkowski's inequality. Now it remains to show the continuity of scalar multiplication. For it let us take real scalar λ and $x \in w_\infty(p)$ such that $\lambda \to 0$ and x is fixed. Now,

$$g^M(\lambda x) \leq |\lambda|^m g^M(x) \tag{2.1.3}$$

It holds only $|\lambda| < 1$, $m = \inf p_k > 0$.

From (2.1.3) choosing sufficiently small λ , we have

$g(\lambda x) \to 0$ as $\lambda \to 0$.

This implies $\lambda x \to \theta$, a zero vector of $w_\infty(p)$, thereby showing existence of continuity of scalar multiplication in $w_\infty(p)$.

Hence (2.1.1) is the paranorm for $w_\infty(p)$ if and only if $0 < \inf p_k \leq \sup p_k < \infty$.

Next we shall show that $w(p)$ is complete with its natural paranorm . Let $y > 0$ and $N_r(y)$ for the number of k in $2^r \leq k < 2^{r+1}$ such that $p_k < y$. Two cases are possible:

(i)

$$\inf_{y>0} \lim_{r\to\infty} \sup 2^{-r} N_r(y) = 0$$

(ii)

$$\inf_{y>0} \lim_{r\to\infty} \sup 2^{-r} N_r(y) > 0$$

In case (i) we first let $\varepsilon > 0$. Then there exists $y_0 > 0$ such that

$$\limsup_r 2^{-r}N_r(y_0) < \varepsilon/2,$$

whence $2^{-r}N_r(y_0) < \varepsilon$, for all sufficiently large r. Choose i so large that

$$\left|l - l^{(i)}\right| < \min(1, \varepsilon^{1/y_0}) .$$

This is possible by theorem 5 [46], on the assumption of course that $(x^{(i)})$ is a Cauchy sequence in $w(p)$ with $l^{(i)}$ the strong Cesaro limit of $x^{(i)}$. Now for all sufficiently large ,

$$2^{-r}\sum_r \left|l - l^{(i)}\right|^{p_k} \leq 2^{-r}\sum_{p_k<y_0} 1 + 2^{-r}\sum_{r_{p_k \leq y_0}} \left|l - l^{(i)}\right|^{p_k}$$

$$< 2^{-r}N_r(y_0) + 2^{-r}\sum_{r_{p_k \leq y_0}} \varepsilon$$

$$< 2\varepsilon$$

Hence, $2^{-r}\sum_r \left|l - l^{(i)}\right|^{p_k} \to 0$ $(r \to \infty)$, from which it follows that $w(p)$ is complete.

Now we deal with case (ii). Denote the positive expression in (ii) by $2c$. Then there exists r_1 such that $2^{-r}N_r(1) > c$ for $r = r_1$. Also, there exists $r_2 > r_1$ such that $2^{-r}N_r\left(\frac{1}{2}\right) > c$ for $r = r_2$. Generally we have $2^{-r}N_r\left(\frac{1}{s}\right) > c$ for $r = r_s$, where $r_1 < r_2 < \cdots$. By the argument of theorem 5 [46] , there exists $I = I(c)$ such that $i > I$ implies

$$2^{-r}\sum_r \left|l - l^{(i)}\right|^{p_k} < c/2 \tag{2.1.4}$$

for all sufficiently large r. Now we must have $l^{(i)} = l^{(I)}$ for every $i > I$. For otherwise

$$\left|l^{(i)} = l^{(I)}\right| > 0$$

for some $i > I$ and then , with $r = r_s$,

$$2^{-r}\sum_{r}\left|l^{(i)}-l^{(I)}\right|^{p_k} \geq 2^{-r}\sum_{p_k<1/s}\left|l^{(i)}-l^{(I)}\right|^{p_k}$$

$$\geq 2^{-r}\mathrm{N_r}\left(\frac{1}{\mathrm{s}}\right)\left|l^{(i)}-l^{(I)}\right|^{1/s}$$

$$> \left|l^{(i)}-l^{(I)}\right|^{1/s} > c/2$$

(2.1.5)

for sufficiently large s . The argument above depends on having $\left|l^{(i)}-l^{(I)}\right| \leq 1$, which obviously holds for sufficiently large i, I. Now (2.1.4) and (2.1.5) are contradictory, whence $\left(l^{(i)}\right)$ is ultimately constant. This proves that $w(p)$ is complete.

Let X and Y be any two sequence spaces and $A = (a_{nk})$; $n, k \in \mathbb{N}$ be infinite matrix of complex numbers a_{nk} .Then we say that A defines a matrix mapping X into Y ; and it is denoted by writing $A\colon X \to Y$ if for every sequence $x = (x_k) \in X$, the sequence $\left((Ax)_n\right)$ is in Y , where

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}\, x_k \; ; \quad (n \in \mathbb{N})$$

(2.1.6)

By (X, Y) we denote the class of all matrices A such that $A\colon X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on right side of (2.1.2) converges for each $n \in \mathbb{N}$ and every $x \in X$; and we write,

$$Ax = \{(Ax)_n\}_{n\in\mathbb{N}} \in Y \text{ for all } x \in X .$$

We denote by U for the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $u \in \mathrm{U}$, let $\frac{1}{u} = \left(\frac{1}{u_n}\right)$. Let us define the matrix $G(u, v) = (g_{nk})$ as,

$$g_{nk}=\begin{cases}u_n v_k; & 0\le k\le n\\ 0; & k>n\end{cases}$$

(2.1.7)

for all $n,k\in\mathbb{N}$,where u_n depends only on n and v_k only on k. The matrix $G(u,v)=(g_{nk})$ is called generalized weighted mean or factorable matrix.

2.2. New Sequence Spaces

In the present part of the chapter we shall introduce the sequence spaces $w(u,v;p)$, $w_0(u,v;p)$ and $w_\infty(u,v;p)$. Before introducing these sequence spaces we would like to present some remarks. Malkowsky and Savas [29] have defined the sequence spaces $Z(u,v,X)$ which consists of all sequences whose G(u,v)- transforms are in $X\in\{l_\infty,c,c_0,l(p)\}$ where $u,v\in U$. Chaudhary and Mishra [15] have defined the sequence space $\overline{l(p)}$ which consists of all sequences whose S- transforms are in $l(p)$; where the matrix $S=(s_{nk})$ is defined by

$$S_{nk}=\begin{cases}1; & 0\le k\le n\\ 0; & k>n\end{cases}$$

Moreover Maddox [45] introduced the sequence spaces $w(p)$ of all strongly summable,$w_0(p)$ of strongly summable to zero and $w_\infty(p)$ of bounded sequences which consist of all sequences whose C- transforms are in the spaces $l(p)$, $c_0(p)$ and $l_\infty(p)$ respectively ; where

$$C=(c_{nk})=\begin{cases}\frac{1}{n}; & 1\le k\le n\\ 0; & k>n\end{cases}$$

and $C=(c_{nk})$ is called the Cesaro matrix of order 1 or the matrix of arithmetic mean.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{ x = (x_k) \in \omega : Ax \in X \}$$

(2.2.1)

which is a sequence space.

With the notation of (2.2.1) , we can have the following representations:

$$X(u,v,p) = [X]_Z\,, \quad \text{for } X \in \{ l_\infty, c, c_0, l(p) \}$$

$$\overline{l(p)} = [l(p)]_S\,, \qquad l(u,v;p) = l(p)_{G(u,v)} \text{ [13]}$$

$$w(p) = [l(p)]_C\,,\ w_0(p) = [c_0(p)]_C \text{and } w_\infty(p) = [l_\infty(p)]_C.$$

Following the works of the authors [13,15,29,44] , for $p = \{ p_k \}$ is a bounded sequence of a strictly positive real numbers , we now define the new sequence spaces $\mu(u,v;p)$ for $\mu \in \{ w, w_0, w_\infty \}$ by

$$\mu(u,v;p) = \left\{ x = (x_k) \in \omega : \left(\sum_{k=1}^{\infty} u_n v_k x_k \right) \in \mu(p) \right\}$$

(2.2.2)

Using (2.2.1), we may represent these sequence spaces as ,

$$\mu(u,v;p) = [\mu(p)]_{G(u,v)} \text{ ; for } \mu \in \{ w, w_0, w_\infty \}$$

In other words the sequence spaces $w(u,v;p)$, $w_0(u,v;p)$ and $w_\infty(u,v;p)$ are the sets of all sequences whose $G(u,v)$ transforms are in the spaces $w(p)$, $w_0(p)$ and $w_\infty(p)$ respectively.

If $p_k = 1$ for all $k \in \mathbb{N}$, we write $\mu(u,v)$ instead of $\mu(u,v;p)$ for $\mu \in \{w,\ w_0,\ w_\infty\}$.

It is easy to verify that the sequence spaces $w(u,v;p)$, $w_0(u,v;p)$ and $w_\infty(u,v;p)$ are linear spaces under usual coordinatewise addition and scalar multiplication.

We shall first establish following some simple properties.

Proposition 2.1.1: The sequence spaces $\mu(u,v;p)$ for $\mu \in \{w,\ w_0,\ w_\infty\}$ are complete paranorm space paramormed by

$$h(x)=\sup_{n\in\mathbb{N}}\left\{\frac{1}{n}\sum_{k=1}^{n}\left|u_n v_k x_k\right|^{p_k}\right\}^{\frac{1}{M}};$$

or equivalently

$$h(x)=\sup_r\left(2^{-r}\sum_r\left|u_n v_k x_k\right|^{p_k}\right)^{\frac{1}{M}}$$

(2.2.3)

where $\sum_r$ is the sum over r in the range $2^r \le k < 2^{r+1}$. For the space $w_\infty(u,v;p)$, (2.2.3) is a paranorm if and only if $0<\inf p_k \le \sup p_k < \infty$.

Proof: The proof of this proposition follows from the similar arguments as in the theorems 5,6 in [46] and theorem 2.1 in [13]. If $\{x^n\}$ is a Cauchy sequence in $\mu(u,v;p)$; then $\{G(u,v)x^n\}$ is a Cauchy sequence in μ. Now it is a routine work to show $\mu(u,v;p)$ is complete paranormed space under the usual paranorm.

Proposition 2.1.2: The sequence spaces $\mu(u,v;p)$ are linearly isomorphic to $\mu(p)$ where $\mu\in\{w,w_0,w_\infty\}$.

Proof: We define the transformation

$$T:\mu(u,v;p)\mapsto \mu \text{ by,}$$

$$x\mapsto y=T(x).$$

Linearity of T is obvious. Further, if $Tx=\theta$, then $x=\theta$. Hence T is injective.

Next, let $y=\{y_n\}\in\mu$.

Then

$$y_n = \sum_{k=1}^{n} u_n v_k x_k$$

gives successively

$$y_1 = u_1 v_1 x_1 \text{ or } x_1 = \frac{1}{v_1}\left(\frac{y_1}{u_1}\right)$$

$$y_2 = u_2 v_1 x_1 + u_2 v_2 x_2 \text{or} x_2 = \frac{1}{v_2}\left(\frac{y_2}{u_2} - \frac{y_1}{u_1}\right) \text{ ; using value of } x_1,$$

$$y_3 = u_3 v_1 x_1 + u_3 v_2 x_2 + u_3 v_3 x_3 \text{or} x_3 = \frac{1}{v_3}\left(\frac{y_3}{u_3} - \frac{y_2}{u_2}\right)$$

using value of x_1 and x_2 and so on. Continuing in this way, we have a generalization that

$$x_k = \frac{1}{v_k}\left(\frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}}\right), \qquad k \in \mathbb{N} \tag{2.2.4}$$

where $y_k = 0$ for $k \leq 0$.

Now from (2.2.3)

$$h(x) = \sup_{n\in\mathbb{N}} \left\{\frac{1}{n}\sum_{k=1}^{n} |u_n v_k x_k|^{p_k}\right\}^{1/M}$$

$$= \sup_{n\in\mathbb{N}} \left\{\frac{1}{n}\sum_{k=1}^{n} \left|u_n v_k \frac{1}{v_k}\left(\frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}}\right)\right|^{p_k}\right\}^{1/M}$$

$$= \sup_{n\in\mathbb{N}} \left\{\frac{1}{n}\left(|y_1|^{p_k} + |y_1|^{p_k} + |y_1|^{p_k} + \cdots\right)\right\}^{1/M}$$

$$= \sup_{n\in\mathbb{N}} \left\{\frac{1}{n}\sum_{k=1}^{n} |y_k|^{p_k}\right\}^{1/M}$$

$= g(y)$; using (2.1.1)

Thus , we deduce that $x \in \mu(u,v;p)$ and as a consequence we conclude that T is surjective and is a paranorm preserving. Hence T is a linear bijection and showing that the sequence spaces $\mu(u,v;p)$ are linearly isomorphic to $\mu(p)$.

2.3. Duals

In [25] Lascarides and Maddox have determined the β- dual (the generalized Köthe-Toeplitz the dual) of sequence space $w(p)$ as the space $\mathcal{M}$ given by

$$\mathcal{M} = \left\{ a = (a_k) : \sum_{r=0}^{\infty} \max\nolimits_r \left[(2^r N^{-1})^{\frac{1}{p_k}} |a_k| \right] < \infty \text{ for some integer } N > 1 \right\}$$

for $0 < p_k \leq 1$ and $\max_r$ is the maximum taken over $2^r \leq k < 2^{r+1}$[25].

In this section we obtain the β - dual of $w(u,v;p)$. We recall that if X be a sequence space , we define $\beta -$ dual of X as:

$$X^{\beta} = \left\{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in X \right\}$$

Theorem 2.3.1

Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $w^{\beta}(u.v;p) = \Gamma$ where

$$\Gamma = \left\{ a = (a_k) : \sum_r a_k \left[\frac{1}{v_k} \left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}} \right) \right] \text{converges and} \lim_{m \to \infty} \left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_m}{u_m v_m} = O(1) \right\}$$

.**Proof**: We first assume that the conditions hold. Let $a \in \Gamma$ and $x \in w(u,v;p)$ .Then for $y \in w(p)$, there exists a positive integer $N > 1$ such that

$$\frac{1}{n} \sum_{k=1}^{n} |y_k|^{p_k} < \infty$$

or equivalently

$$\frac{1}{2^r} \sum_r |y_k|^{p_k} < \infty$$

where sum over r runs from $2^r \le k < 2^{r+1}$.

It follows that ,

$$|y_k| \le \left(2^r N^{-1}\right)^{\frac{1}{p_k}} .$$

Now using (2.2.4),we have

$$\left|\sum_{k=1}^{m} a_k x_k\right| = \left|\sum_{k=1}^{m-1} a_k \left[\frac{1}{v_k}\left(\frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}}\right)\right] + \frac{a_m y_m}{u_m v_m}\right|$$

$$\le \left|\sum_{k=1}^{m-1} \frac{a_k}{v_k}\left(\frac{y_k}{u_k} - \frac{y_{k-1}}{u_{k-1}}\right)\right| + \left|\frac{a_m}{u_m v_m}\right| |y_m|$$

$$\le \sum_r \frac{a_k}{v_k}\left|\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right| + \left|\frac{a_m}{u_m v_m}\right| \left(2^r N^{-1}\right)^{\frac{1}{p_m}}$$

$$< \infty .$$

Hence , it follows that $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in w(u,v;p)$ .So, $\Gamma \subseteq w^{\beta}(u,v;p)$.

On the other hand, let $a \in w^{\beta}(u,v;p)$. Then, $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in w(u,v;p)$.

Since ,

$$x = \left\{\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right\} \in w(u,v;p) ;$$

it follows that

$$\sum_{k=1}^{\infty} a_k \left[\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right]$$

converges, which is one of the condition to be proved . Next it remains to show that

$$\lim_{m\to\infty}\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_m}{u_m v_m} = O(1)$$

For it, on the contrary let,

$\lim_{n\to\infty}\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_m}{u_m v_m} \neq O(1)$, which is immediately against the fact that $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in w(u,v;p)$ and $\sum_{k=1}^{\infty} a_k \left[\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right]$ converges.

Hence, we must have,

$$\lim_{m\to\infty}\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_m}{u_m v_m} = O(1)$$

So, we arrive at the result $w^{\beta}(u,v;p) \subseteq \Gamma$; thereby proving $w^{\beta}(u,v;p) = \Gamma$.

2.4. Matrix Transformation

In this section we give characterization for the matrix classes $(w(u,v;p), l_\infty)$, $(w(u,v;p), c)$ and $(w(u,v;p), c_0)$.

Theorem 2.4.1

Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (w(u,v;p), l_\infty)$ if and only if

(i) there exists an integer $N > 1$ such that

$$\sup_n \sum_r \max_r \left[a_{nk} \left\{ \frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k=1}}\right)\right\}\right] < \infty \text{ and}$$

(ii)

$$\lim_{m\to\infty}\left\{\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{a_{nm}}{u_m v_m}\right\}_{n\in\mathbb{N}} = O(1)$$

Proof: Let the conditions be satisfied. Since,

$$\left|\sum_{k=1}^{m} a_{nk} x_k\right| = \left|\sum_{k=1}^{m-1} a_{nk}\left[\frac{1}{v_k}\left(\frac{y_k}{u_k}-\frac{y_{k-1}}{u_{k-1}}\right)\right] + \frac{y_m}{u_m v_m} a_{nm}\right|$$

$$\leq \left|\sum_{k=1}^{m-1} a_{nk}\left\{\frac{1}{v_k}\left(\frac{y_k}{u_k}-\frac{y_{k-1}}{u_{k-1}}\right)\right\}\right| + \left|\frac{a_m}{u_m v_m}\right| |y_m|$$

$$\therefore \sum_{k=1}^{\infty}|a_{nk} x_k| \leq \sum_r \max_r \left|a_{nk}\left\{\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right\}\right| + \left|\frac{a_{nm}}{u_m v_m}\right|\left(2^r N^{-1}\right)^{\frac{1}{p_m}}$$

$$\leq \sup_n \sum_r \max_r \left|a_{nk}\left\{\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right\}\right| + \left(2^r N^{-1}\right)^{\frac{1}{p_m}}\left|\frac{a_{nm}}{u_m v_m}\right|$$

$< \infty$, by using conditions (i) and (ii).

It follows that $A_n \in \Gamma$ and hence $\sum_{k=1}^{\infty} a_{nk} x_k = A_n(x)$ converges for each $x \in w(u,v;p)$ and $n \in \mathbb{N}$. Thus $Ax \in l_\infty$.

On the other hand , let $A \in (w(u,v;p), l_\infty)$. Since ,

$$\left\{\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right\} \in w(u,v;p),$$

the condition (i) holds. In order to show that condition (ii) is necessary, we assume that for $N > 1$,

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}}\ \frac{a_{nm}}{u_m v_m}\right\}_{n\in\mathbb{N}} \neq O(1)$$

that is,

$$\left\{(2^r N^{-1})^{\frac{1}{p_m}}\ \frac{a_{nm}}{u_m v_m}\right\}_{n\in\mathbb{N}} \notin l_\infty.$$

Now, therefore, there exists a sequence $\{N_r\}\to\infty$ such that

$$\sup_n \sum_r \max_r \left[a_{nk}\left\{\frac{1}{v_k}\left(\frac{(2^r N_r^{-1})^{\frac{1}{p_k}}}{u_k} - \frac{(2^r N_r^{-1})^{\frac{1}{p_{k-1}}}}{u_{k-1}}\right)\right\}\right] = o(1)$$

and

$$\lim_{m\to\infty}\left\{\left(2^r {N_r}^{-1}\right)^{\frac{1}{p_m}}\ \frac{a_{nm}}{u_m v_m}\right\}_{n\in\mathbb{N}} = o(1)$$

Hence, $x_k \mapsto o(w(u,v;p))$ but $x_k \mapsto l(w(u,v;p))$. So, we arrive at the contradiction to our assumption $A\in(w(u,v;p),l_\infty)$. Thus, condition (ii) is necessary; thereby completing the proof for the theorem.

By using the arguments as in theorem (2.4.1) it is straight forward matter to prove the following theorems:

Theorem 2.4.2

Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A\in(w(u,v;p),c)$ if and only if

i) there exists an integer $N>1$ such that

$$\sup_n \sum_r \max_r \left[a_{nk}\left\{\frac{1}{v_k}\left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_k}\right)\right\}\right] < \infty \text{ and}$$

ii)

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}}\ \frac{a_{nm}}{u_m v_m}\right\}_{n\in\mathbb{N}} = o(1)$$

iii)

$$\lim_{n\to\infty} a_{nk} = \alpha_k$$

exists for every fixed k.

Theorem 2.4.3

Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in (w(u, v; p), c_0)$ if and only if

i) there exists an integer $N > 1$ such that

$$\sup_n \sum_r \max_r \left[a_{nk} \left\{ \frac{1}{v_k} \left(\frac{\left(2^r N^{-1}\right)^{\frac{1}{p_k}}}{u_k} - \frac{\left(2^r N^{-1}\right)^{\frac{1}{p_{k-1}}}}{u_k} \right) \right\} \right] < \infty$$

(ii)

$$\lim_{m\to\infty} \left\{ (2^r N^{-1})^{\frac{1}{p_m}} \frac{a_{nm}}{u_m v_m} \right\}_{n\in\mathbb{N}} = o(1) \text{ and}$$

(iii)

$$\lim_{n\to\infty} a_{nk} = \alpha_k$$

with $\alpha_k = 0$ for every fixed k.

Part Two:
Paranormed Sequence Spaces $\boldsymbol{w(u,v;p,\Delta)}$, $\boldsymbol{w_0(u,v;p,\Delta)}$ and $\boldsymbol{w_\infty(u,v;p,\Delta)}$ Generated by Combining the Generalized Weighted Mean $\boldsymbol{G(u,v)}$ and the Difference Operator Matrix $\boldsymbol{\Delta}$

2.5. Preliminaries and Reviews

We recall that any subspace of the space ω of all complex valued sequences is called a sequence space. We shall write l_∞ , c and c_0 for the spaces of all bounded, convergent and null sequences respectively. By a paranormed space we mean a linear topological space X over the field $\mathbb{R}$ if there is a sub additive function $g : X \to \mathbb{R}$ such that $g(\theta)=0$, $g(x)=g(-x)$ and scalar multiplication is continuous i.e. $|\alpha_n-\alpha| \to 0$ and $g(x_n-x) \to 0$ imply $g(\alpha_n x_n-\alpha x) \to 0$, for all α's in $\mathbb{R}$ and all x's in X; where θ is the zero vector in the linear space X .

If $p=\{p_k\}$ be a bounded sequence of strictly positive real numbers, Maddox [45] defined the sequence spaces $w(p), w_0(p)$ and $w_\infty(p)$ which are the spaces of strongly summable, strongly summable to zero and bounded sequences respectively. We have shown them in section 2.1.

Let U denote the set of all sequences $u=(u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}$. For $u \in$ U, let $\frac{1}{u}=\left(\frac{1}{u_n}\right)$. Let us define the matrices $G(u,v)=(g_{nk})$ and $\Delta=(\delta_{nk})$ as:

$$g_{nk}=\begin{cases} u_n v_k, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

and

$$\delta_{nk}=\begin{cases} (-1)^{n-k}, & n-1 \le k \le n \\ 0, & 0 \le k < n \text{ or } k > n \end{cases}$$

for all $n,k \in \mathbb{N}$, where u_n depends only on n and v_k only on k. The matrix $G(u,v) = (g_{nk})$ is called generalized weighted mean or factorable matrix and $\Delta = (\delta_{nk})$ is called the difference operator matrix. We denote the combination (product) of $G(u,v)$ and Δ by $G(u,v,\Delta)$ and is given by

$$g_{nk}^{\Delta} = \begin{cases} u_n(v_k - v_{k+1}), & 0 \le k \le n-1 \\ u_k v_k, & k = n \\ 0, & k > n \end{cases} \tag{2.5.1}$$

2.6. Remarks and New Sequence Spaces $w(u,v;p,\Delta)$, $w_0(u,v;p,\Delta)$ and $w_\infty(u,v;p,\Delta)$

In the present part of the chapter we shall introduce the sequence spaces $w(u,v;p,\Delta), w_0(u,v;p,\Delta)$ and $w_\infty(u,v;p,\Delta)$; which are the set of all sequences whose $G(u,v,\Delta)$- transforms are in the spaces $w(p)$, $w_0(p)$ and $w_\infty(p)$ respectively, where $G(u,v,\Delta)$ denotes the matrix as defined in (2.5.1).

Before introducing these sequence spaces we present some remarks. Malkowsky and Savas [29] have defined the sequence spaces Z(u,v,X) which consists of all sequences whose *G(u,v)*- transforms are in $X \in \{l_\infty, c, c_0, l(p)\}$ where $u,v \in U$. Chaudhary and Mishra [15] have defined the sequence space $\overline{l(p)}$ which consists of all sequences whose S- transforms are in $l(p)$;where $S = (s_{nk})$ is defined by

$$s_{nk} = \begin{cases} 1, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

Basar and Altay [33] have defined the space $bs(p)$ as the set of all series whose sequence of partial sums are in $l_\infty(p)$. In [10,11], the authors also have studied the spaces $r^t(p), r_\infty^t(p)$, $r_c^t(p)$ and $r_0^t(p)$.The space $r^t(p)$ consists of all the sequences whose Riesz (R^t) transform are in the space $l(p)$, where the matrix $R^t = (r_{nk}^t)$ of the Riesz mean $(R,t_n) = (r_{nk}^t)$ is given by

$$r_{nk}^{t} = \begin{cases} t_k \Big/ \sum_{k=0}^{n} t_k, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

with the sequence of positive real numbers (t_k).

Moreover the sequence spaces $w(p), w_0(p)$ and $w_\infty(p)$ introduced by Maddox are the set of all sequences whose C- transforms are in the spaces $l(p)$, $c_0(p)$ and $l_\infty(p)$ respectively ; where $C = (c_{nk})$ with

$$c_{nk} = \begin{cases} \frac{1}{n}, & 1 \le k \le n \\ 0, & k > n \end{cases}$$

The matrix $C = (c_{nk})$ is called the Cesaro matrix of order 1 or the matrix of arithmetic mean.

Recently in 2012 Demiriz and Caken [78] have introduced and studied the sequence spaces $c_0(u,v;p,\Delta)$, $c(u,v;p,\Delta)$, $l_\infty(u,v;p,\Delta)$ and $l(u,v;p,\Delta)$ which consists of all sequences whose $G(u,v,\Delta)$-transforms are in $c_0(p), c(p), l_\infty(p)$ and $l(p)$ respectively; where $G(u,v,\Delta)$ is as defined in (2.5.1).

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\} \tag{2.6.1}$$

which is a sequence space.

With the notation of (2.6.1), we have the following representations,

$X(u,v,p) = [X]_Z$, for $X \in \{l_\infty, c, c_0, l(p)\}$ [29]

$\overline{l(p)} = [l(p)]_S$ [15], $bs(p) = [l_\infty(p)]_S$ [33]

$r^t(p) = [l(p)]_{R^t}$, $r_\infty^t(p) = [l_\infty(p)]_{R^t}$, $r_c^t(p) = [c(p)]_{R^t}$, $r_0^t(p) = [c_0(p)]_{R^t}$ [10,11]

$\lambda(u,v;p,\Delta) = [\lambda]_{G(u,v,\Delta)}$ for $\lambda \in \{c_0(p), c(p), l_\infty(p), l(p)\}$ [78].

Following the works of the authors [10,11,15,29,33,45,78] , for $p=\{p_k\}$ a bounded sequence of a strictly positive real numbers , we now define the new sequence spaces $\mu(u,v;p,\Delta)$ for $\mu \in \{w, w_0, w_\infty\}$ by

$$\mu(u,v;p,\Delta) = \left\{x = x_k \in \omega : \left(\sum_{k=1}^{n} u_n v_k \Delta t_k\right) \in \mu(p)\right\} \tag{2.6.2}$$

where $t_k(x)=\frac{1}{k}\sum_{i=1}^{k} x_i$ and $\Delta t_k = t_k - t_{k-1}$ for all $k \in \mathbb{N}$ with $t_0 = 0$. Now, $\mu(u,v;p,\Delta)$ is the set of all sequences whose $G(u,v,\Delta)$ -transforms are in $\mu \in \{w, w_0, w_\infty\}$, that is ,

$$\mu(u,v;p,\Delta)=[\mu(p)]_{G(u,v,\Delta)} .$$

Whenever the matrix $G(u,v,\Delta)$ is defined to be the unit matrix ,

$$d_{nk} = \begin{cases} u_n v_k = 1, & n = k \\ 0, & \text{otherwise} \end{cases}$$

we find that

$w(u,v;p,\Delta)=w(p)$, $w_0(u,v;p,\Delta)=w_0(p)$ and $w_\infty(u,v;p,\Delta)=w_\infty(p)$.

Further if $p_k = p > 0$ for every $k \in \mathbb{N}$, then $w(u,v;p,\Delta)=w^p$, $w_0(u,v;p,\Delta)={w_0}^p$ and $w_\infty(u,v;p,\Delta)=w_\infty^p$ [45].

The sequence $y=(y_m)$ defined as,

$$y_m = \sum_{j=1}^{m} u_m v_j \Delta t_j$$

$$= u_m[v_1\Delta t_1 + v_2\Delta t_2 + v_3\Delta t_3 + \cdots + v_m\Delta t_m]$$

$$= u_m[v_1(t_1 - t_0) + v_2(t_2 - t_1) + v_3(t_3 - t_2) + \dots + v_m(t_m - t_{m-1})]$$

$$\therefore y_m = \sum_{j=1}^{m-1} u_m \, \Delta v_j \, t_j + u_m \, v_m \, t_m \qquad (2.6.3)$$

where $\Delta v_j = v_j - v_{j+1}$; will be frequently used in our context as the $G(u,v,\Delta)$-transform of the sequence $x = (x_k)$.

We shall first establish following some simple properties.

Proposition 2.6.1. The sequence spaces $w(u,v;p,\Delta)$, $w_0(u,v;p,\Delta)$ and $w_\infty(u,v;p,\Delta)$ are linearly isomorphic to $w(p)$, $w_0(p)$ and $w_\infty(p)$ respectively.

Proof: We prove the proposition for the space $w(u,v;p,\Delta)$. For each $x \in w(u,v;p,\Delta)$, we have $G(u,v,\Delta)x \in w(p)$. It is easy to verify that $G(u,v,\Delta)$ is linear and injective. Also the matrix $G(u,v,\Delta)$ has an inverse $H(u,v,\Delta) = (h_{nk})$ given by,

$$h_{nk} = \begin{cases} \dfrac{1}{u_k}\left(\dfrac{1}{v_k} - \dfrac{1}{v_{k+1}}\right), & 0 \le k \le n-1 \\ \dfrac{1}{u_k v_k}, & n = k \\ 0, & k > n \end{cases}$$

Thus $w(u,v;p,\Delta)$ is linearly isomorphic to $w(p)$.

With the similar arguments we can show that $w_0(u,v;p;\Delta)$ and $w_\infty(u,v;p;\Delta)$ are linearly isomorphic to $w_0(p)$ and $w_\infty(p)$ respectively .

Proposition 2.6.2. Let $\zeta_k = \left(G(u,v;\Delta)x\right)_k$ for all $k \in \mathbb{N}$. We define the sequence $h^{(k)} = \left\{h_n^{(k)}\right\}_{n\in\mathbb{N}}$ for every $n,k \in \mathbb{N}$ by

$$h_n^{(k)} = \begin{cases} \dfrac{1}{u_k}\left[\dfrac{1}{v_k} - \dfrac{1}{v_{k+1}}\right], & 0 \le k \le n-1 \\ \dfrac{1}{u_k v_k}, & k = n \\ 0, & k > n. \end{cases}$$

Then, the sequence $h^{(k)} = \left\{h_n^{(k)}\right\}_{n\in\mathbb{N}}$ is a basis for the space $w(u,v;p,\Delta)$ and any $x \in w(u,v;p,\Delta)$ has a unique representation in the form

$$x = \sum_{k=1}^{\infty} \zeta_k h^{(k)} .$$

It can easily be verified.

Proposition 2.6.3. The sequence spaces $\mu(u,v;p,\Delta)$ for $\mu \in \{w, w_0, w_\infty\}$ are complete paranorm sequence spaces paranormed by,

$$h(x) = \sup_{n\in N}\left\{\frac{1}{n}\sum_{k=1}^{n}\left|u_n v_k \Delta t_k\right|^{p_k}\right\}^{\frac{1}{M}}$$

Where

$$M = \max(1, \sup p_k)$$

or equivalently

$$h(x) = \sup_r\left\{2^{-r}\sum_r\left|u_n v_k \Delta t_k\right|^{p_k}\right\}^{\frac{1}{M}} .$$

The summation $\sum_r$ in r runs from the range $2^r \le k < 2^{r+1}$. For the sequence space $w_\infty(u,v;p,\Delta)$, $h(x)$ is a paranorm if and only if $0 < \inf p_k \le \sup p_k < \infty$.

The proof of this proposition follows immediately from the proposition 2.6.1; where $h(x) = P\left(G(u,v,\Delta)x\right)$ and P is the usual paranorm on μ .

2.7. Duals

In this section we find $\beta-$dual of $w(u,v;p;\Delta)$. We recall that if X be a sequence space , we define $\beta-$dual of X as,

$$X^{\beta}=\left\{a=(a_k): \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x\in X\right\}$$

Theorem 2.7.1

Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Then $w^{\beta}(u,v;p,\Delta)=\Gamma$ where Γ is given by ,

$$\Gamma=\left\{a=(a_k):\sum_r \max{}_r\left(2^r N^{-1}\right)^{\frac{1}{p_k}}\left|\frac{1}{u_k}\left\{\frac{a_k}{v_k}+\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{m} a_j\right\}\right|<\infty\right\}$$

and

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}}\ \frac{1}{u_m v_m}\ a_m\right\}=O(1)$$

for some integer $N>1$and $\max_r$ is the maximum taken over $2^r\le k<2^{r+1}$.

Proof: Let $a\in\Gamma$ .Then there exists an integer $N>1$such that,

$$\sum_r \max{}_r\left(2^r N^{-1}\right)^{\frac{1}{p_k}}\left|\frac{1}{u_k}\left\{\frac{a_k}{v_k}+\left(\frac{1}{v_k}-\frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{m} a_j\right\}\right|<\infty$$

and

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}}\ \frac{1}{u_m v_m}\ a_m\right\}=O(1)$$

We take $x\in w(u,v;p,\Delta)$, then $G(u,v,\Delta)x\in w(p)$ for which we shall write $Gx\in w(p)$ in brief.

Hence,

$$\frac{1}{n}\sum_{k=1}^{n}\left|Gx\right|^{p_k}<\infty$$

or equivalently

$$\frac{1}{2^r}\sum_r |Gx|^{p_k} < \infty$$

where the summation over r runs from $2^r \le k < 2^{r+1}$. Also, there exists an integer $N > 1$ such that $|Gx| = |y_m| \le (2^r N^{-1})^{\frac{1}{p_k}}$.

We have,

$$\left|\sum_{k=1}^{m} a_k x_k\right| = \left|\sum_{k=1}^{m-1} \frac{1}{u_k}\left[\frac{a_k}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{m} a_j\right] y_k + \frac{1}{u_m v_m} a_m y_m\right|$$

$$\le \left|\sum_{k=1}^{m-1} \frac{1}{u_k}\left[\frac{a_k}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{m} a_j\right] y_k\right| + \left|\frac{1}{u_m v_m} a_m y_m\right|$$

So, it follows that,

$$\sum_{r=0}^{\infty} |a_k x_k| \le \sum_r \max_r \left[\left(2^r N^{-1}\right)^{\frac{1}{p_k}} \left|\frac{1}{u_k}\left[\frac{a_k}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{m} a_j\right]\right|\right] N\, h(x) + \left(2^r N^{-1}\right)^{\frac{1}{p_m}} \left|\frac{1}{u_m v_m} a_m\right|$$

$< \infty$, where $h(x)$ is as defined in the proposition 2.6.3.

Hence, it follows that $\sum_{k=1}^{\infty} |a_k x_k|$ converges and $\Gamma \subseteq w^{\beta}(u,v;p,\Delta)$. On the other hand, let $a \in w^{\beta}(u,v;p,\Delta)$ but

$$\sum_r \max_r \left(2^r N^{-1}\right)^{\frac{1}{p_k}} \left|\frac{1}{u_k}\left\{\frac{a_k}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{m} a_j\right\}\right| = \infty$$

and

$$\lim_{m\to\infty}\left\{\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{1}{u_m v_m} a_m\right\} \ne O(1)$$

for every integer $N > 1$.

Now, therefore, there exists a sequence $\{N_r\} \to 0$ such that

$$\sum_r \max_r \left(2^r N_r^{-1}\right)^{\frac{1}{p_k}} \left| \frac{1}{u_k} \left\{ \frac{a_k}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \sum_{j=k+1}^{m} a_j \right\} \right| = 0$$

and

$$\lim_{m\to\infty} \left\{ \left(2^r N_r^{-1}\right)^{\frac{1}{p_m}} \frac{1}{u_m v_m} a_m \right\} = o(1)$$

Hence, $x_k \to 0\left(w(u,v;p,\Delta)\right)$; but $x_k \to l\left(w(u,v;p,\Delta)\right)$ which is contradiction to our assumption that $a \in w^{\beta}(u,v;p,\Delta)$.

This implies that $a \in \Gamma$. As a consequence, we get $w^{\beta}(u,v;p,\Delta) \subseteq \Gamma$.

Thus $w^{\beta}(u,v;p,\Delta) = \Gamma$ and this completes the proof. The β -duals for the spaces $w_0(u,v;p,\Delta)$ and $w_\infty(u,v;p,\Delta)$ can be obtained in the similar manner.

2.8. Matrix Transformation

In this section we give characterization for the matrix classes $(w(u,v;p,\Delta), l_\infty)$, $(w(u,v;p,\Delta), c)$, $(w(u,v;p,\Delta), c_0)$ and $\left(w(u,v;p,\Delta), \Omega(t)\right)$.

Theorem 2.8.1

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then $A \in \left(w(u,v;p,\Delta), l_\infty\right)$ if and only if

i) there exists an integer N>1 such that

$$\sup_n \sum_r \max_r \left(2^r N^{-1}\right)^{\frac{1}{p_k}} |c_{nk}| < \infty$$

where

$$c_{nk} = \frac{1}{u_k} \left\{ \frac{a_{nk}}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}} \right) \sum_{j=k+1}^{m} a_{nj} \right\}$$

ii)

$$\lim_{m\to\infty}\left\{\left(2^r N^{-1}\right)^{\frac{1}{p_m}} \frac{1}{u_m v_m} a_{nm}\right\}_{n\in N} = O(1).$$

Proof: Let the condition be satisfied. Since

$$\left|\sum_{k=1}^{m} a_{nk} x_k\right| = \left|\sum_{k=1}^{m-1} c_{nk}\, y_k + \frac{1}{u_m v_m} a_{nm} y_m\right|$$

it follows that,

$$\left|\sum_{k=1}^{\infty} a_{nk} x_k\right| \le \sum_r \max_r \left(2^r N^{-1}\right)^{\frac{1}{p_k}} \left|c_{nk}\right| + \left(2^r N^{-1}\right)^{\frac{1}{p_m}} \left|\frac{1}{u_m v_m} a_{nm}\right|$$

$$\le \sup_n \left\{\sum_r \max_r \left(2^r N^{-1}\right)^{\frac{1}{p_k}} \left|c_{nk}\right|\right\} + \left(2^r N^{-1}\right)^{\frac{1}{p_m}} \left|\frac{1}{u_m v_m} a_{nm}\right|$$

$< \infty$; using conditions (i) and (ii).

It implies that $A_n \in \Gamma$ and hence $\sum_{k=1}^{\infty} a_{nk} x_k = A_n(x)$ converges and belongs to l_∞ for each $x \in w(u,v;p,\Delta)$ and $n \in \mathbb{N}$.

On the other hand, let $A \in \left(w(u,v;p,\Delta), l_\infty\right)$. Hence $\sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $x \in w(u,v;p,\Delta)$ and $n \in \mathbb{N}$. We have,

$$\left|\sum_{k=1}^{m} a_{nk} x_k\right| = \left|\sum_{k=1}^{m-1} c_{nk} y_k + \frac{1}{u_m v_m} a_{nm} y_m\right| \tag{2.8.1}$$

We need to show the existence of conditions (i) and (ii). As a contrary, let us assume that

$$\sup_n \sum_r \max_r \left(2^r N^{-1}\right)^{\frac{1}{p_k}} \left|c_{nk}\right| = \infty$$

and

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}}\ \frac{1}{u_m v_m}\ a_{nm}\right\}_{n\in\mathbb{N}} \neq O(1)$$

Now, therefore, there exists a sequence $\{N_r\}\to\infty$ such that

$$\sum_r \max_r (2^r N_r^{-1})^{\frac{1}{p_k}} |c_{nk}| = 0$$

and

$$\lim_{m\to\infty}\left\{(2^r N_r{}^{-1})^{\frac{1}{p_m}}\ \frac{1}{u_m v_m}\ a_{nm}\right\}_{n\in\mathbb{N}} = o(1)$$

Hence from (2.8.1) $x_k \to 0\big(w(u,v;p,\Delta)\big)$; but $x_k \to l\big(w(u,v;p,\Delta)\big)$ which is contradiction to our assumption that $A\in\big(w(u,v;p,\Delta),l_\infty\big)$. Thus conditions (i) and (ii) must hold. This completes the proof.

By using this theorem 2.8.1, it is now a straight forward matter to prove the following theorem.

Theorem 2.8.2

Let $0<p_k\le 1$ for every $k\in\mathbb{N}$. Then $A\in\big(w(u,v;p,\Delta),c\big)$ if and only if

i) there exists an integer N > 1 such that

$$\sup_n \sum_r \max_r \left(2^r N^{-1}\right)^{\frac{1}{p_k}} |c_{nk}| < \infty$$

where

$$c_{nk} = \frac{1}{u_k}\left\{\frac{a_{nk}}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{m} a_{nj}\right\}$$

ii)

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}}\ \frac{1}{u_m v_m}\ a_{nm}\right\}_{n\in\mathbb{N}} = o(1)$$

and

iii)

$$\lim_{n\to\infty} c_{nk} = \alpha_k$$

exists for every fixed k.

Theorem 2.8.3

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then $A \in \left(w(u,v;p,\Delta), c_0\right)$ if and only if

i) there exists an integer N>1 such that

$$\sup_n \sum_r \max_r \left(2^r N^{-1}\right)^{\frac{1}{p_k}} |c_{nk}| < \infty$$

where

$$c_{nk} = \frac{1}{u_k}\left\{\frac{a_{nk}}{v_k} + \left(\frac{1}{v_k} - \frac{1}{v_{k+1}}\right)\sum_{j=k+1}^{m} a_{nj}\right\}$$

ii)

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}} \frac{1}{u_m v_m} a_{nm}\right\}_{n\in\mathbb{N}} = o(1)$$

and

iii)

$$\lim_{n\to\infty} c_{nk} = \alpha_k$$

exists with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

Fricke and Fridy [38] introduced a new sequence space $\Omega(t)$.We define here $\Omega(t)$ and give some results from [15] which will be used in this section. For each r in the interval (0, 1), let

$$G(r) = \{x = (x_k) \in \omega : x_k = \mathrm{O}(t_k)\}.$$

We define the set of geometrically dominated sequences as

$$G = \bigcup_{r\in(0,1)} G(r)$$

The analytic sequences are defined by

$$\text{A} = \left\{ x = (x_k) \in \omega : \lim_n \sup |x_n|^{\frac{1}{n}} < \infty \right\}$$

Obviously $G \subseteq A$. In [37,71,76], the various authors studied matrix transformation from A or G into $l_1, c \ or \ l_\infty$, but the question of mapping from $l_1, c \ or \ l_\infty$ into A or G was not considered. To set the stage for general theory, Fricky and Fridy replaced the geometric sequence $\left(r^k\right)$ with a nonnegative sequence $t = (t_k)$ and defined ,

$$\Omega(t) = \left\{ x = \left(x_k \right) \in \omega : x_k = \text{O}\left(t_k \right) \right\}.$$

For given infinite matrix A the sequence σ is defined by $\sigma_n = \sum_{k=0}^{\infty} |a_{nk}|$.Further, Fricky and Fridy made the following remarks:

Remark 2.8.1. If one wishes to have a matrix A that transforms every null sequence in to a sequence that converges at least as rapidly as some $t_n \downarrow 0$, then A must satisfy $\sigma \in \Omega\left(t \right)$. Similarly, if t is a nonzero constant sequence, then $\Omega(t) = l_\infty$.

Remark 2.8.2. This remark is about obtaining a “given rate of convergence” by mapping c_0 into $\Omega(t)$. The work in[18,19] has shown that regular matrices cannot accelerate the rate of convergence of every null sequences. Therefore we say that having A map c_0 into $\Omega(t)$ does not say that every sequence in c_0 is accelerated, even if $t_n \downarrow 0$ very rapidly ; some sequences that are already in $\Omega(t)$ may map into other members of $\Omega(t)$ that converge at the same rate or slower.

Theorem 2.8.4

Let $0 < p_k \leq 1$ for every $k \in \mathbb{N}$. Then $A \in \left(w(u,v;p,\Delta), \Omega(t) \right)$ if and only if

$$A \in \left(w(p), \Omega(t) \right)$$

and

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}} \frac{1}{u_m v_m} a_{nm}\right\}_{n\in\mathbb{N}} \in l_\infty$$

Proof: Let $C=(c_{nk})\in\left(w(p),\Omega(t)\right)$ and

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}} \frac{1}{u_m v_m} a_{nm}\right\}_{n\in\mathbb{N}} \in l_\infty$$

Take any $= (x_k) \in w(u,v;p,\Delta)$. As $C\in\left(w(p),\Omega(t)\right)$, then we have $C_n \in w^\beta(p)$ for each $n\in\mathbb{N}$. Hence Cx exists and , therefore, we immediately obtain from the equality

$$\sum_{k=1}^{m} a_{nk}x_k = \sum_{k=1}^{m-1} c_{nk}y_k + \frac{1}{u_m v_m}a_{nm}y_m$$

that Ax exists and

$$A\in\left(w(u,v;p,\Delta),\Omega(t)\right).$$

On the other hand let $A\in\left(w(u,v;p,\Delta),\Omega(t)\right)$ and take any $y=(y_k)\in w(p)$. Then $A_n \in \Gamma$ and , therefore the condition

$$\lim_{m\to\infty}\left\{(2^r N^{-1})^{\frac{1}{p_m}} \frac{1}{u_m v_m} a_{nm}\right\}_{n\in\mathbb{N}} \in l_\infty$$

is necessary.

Moreover we have ,

$$\sum_{k=1}^{m} c_{nk}\, y_k = \sum_{k=1}^{m}\sum_{j=k}^{m} u_k v_k c_{nj} x_k$$

Taking $m\to\infty$, we find that Cy exists and equals to Ax .Hence $C\in\left(w(p),\Omega(t)\right)$. This completes the proof.

CHAPTER THREE

Part One:
Paranormed Sequence Space $\boldsymbol{l(p,\lambda)}$ Generated by Lower Unitriangular Matrix λ

3.1. Preliminaries

By ω we mean the spaces of all complex valued sequences. A vector subspace of ω is called a sequence space. We shall write l_∞, c and c_0 for the spaces of all bounded, convergent and null sequence respectively. A linear topological space X over the field R is said to be a paramormed space if there is a subadditive function $g(x): X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous i.e. $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha x_n - \alpha x) \to 0$, for all α's in R and all x's in X, where θ is the zero vector in the linear space X. Maddox [44,45] has introduced the sequence space

$$l(p) = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty \right\},$$

where $p = \{p_k\}$ is a bounded sequence of strictly positive real numbers. Latter Chaudhary and Mishra [15] introduced and studied the sequence space

$$\overline{l(p)} = \left\{ x = (x_k) \in \omega : \sum_{k=1}^{\infty} |t_k(x)|^{p_k} < \infty \right\}$$

where

$$t_k(x) = \sum_{i=1}^{k} x_i .$$

The sequence space $\overline{l(p)}$ is a complete metric linear space paranormed by,

$$g(x) = \left(\sum_{k=1}^{\infty} |t_k(x)|^{p_k} \right)^{\frac{1}{M}}$$

where

$$M = \max(1, \sup_k p_k) .$$

Let X and Y be any two sequence spaces and $A=(a_{nk})$; $n, k \in \mathbb{N}$ be an infinite matrix of complex numbers a_{nk} . Then we say that A defines a matrix mapping X into Y ; and it is denoted by writing $A: X \to Y$ if for every sequence $x=(x_k) \in X$, the sequence $((Ax)_n)$ is in Y , where

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k \ , n \in \mathbb{N}$$

(3.1.1)

By (X,Y) we denote the class of all matrices A such that $A: X \to Y$. Thus, $A \in (X,Y)$ if and only if the series on right side of (3.1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$; and we write,

$$Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y \text{ for all } x \in X .$$

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\} ,$$

(3.1.2)

which is a sequence space.

With the notation as in (3.1.2) , we can have the following representation,

$$\overline{l(p)} = [l(p)]_S$$

(3.1.3)

In other words the sequence space $\overline{l(p)}$ which is the set of all sequences whose S-transforms are in the sequence space $l(p)$ [15], where $S = (s_{nk})$ is an infinite matrix given by

$$S = (s_{nk}) = \begin{cases} 1, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

(3.1.4)

In expanded form

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The multiplication S with itself to n factors produces an infinite matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 2 & 1 & 0 & 0 & \ldots \\ 3 & 2 & 1 & 0 & \ldots \\ 4 & 3 & 2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which we denote by λ.

Thus,

$$\lambda = S^n = (\lambda_{nk}) = \begin{cases} n - k + 1, & n \ge k \\ 0, & \text{otherwise} \end{cases}$$

(3.1.5)

It is a lower unitriangular matrix.

Using λ as the operator , we now introduce a new sequence space $l(p,\lambda)$ as

$$l(p,\lambda) = \{x = (x_k) \in \omega : \lambda x \in l(p)\}$$

(3.1.6)

where λ is as defined in (3.1.5)

Thus, $l(p,\lambda)$ is now the set of all sequences $\{u_k\}$ whose $\lambda = S^n$- transforms are in the sequence space $l(p)$. Using the notation as in (3.1.2) $l(p,\lambda)$ can be represented as

$$l(p,\lambda)=[l(p)]_{\lambda}$$

where the sequences,

$$\{\Delta u_k = u_k - u_{k-1}\} \in \overline{l(p)} \text{ with } u_0 = 0$$

and

$$\{u_k\} = \left\{ \sum_{i=1}^{k} (k-i+1)\, x_i \right\}.$$

(3.1.7)

We shall first establish some simple propositions for $l(p,\lambda)$.

Proposition 3.1.1. We have,

$$l(p) \subseteq \overline{l(p)} \subseteq l(p,\lambda).$$

Proof : We have

$$l(p)=\left\{x=(x_k)\in\omega : \sum_{k=1}^{\infty} |x_k|^{p_k} < \infty\right\}$$

and

$$\overline{l(p)}=\left\{x=(x_k)\in\omega : \sum_{k=1}^{\infty} |t_k(x)|^{p_k} < \infty\right\}$$

where

$$t_k(x)=\sum_{i=1}^{k} x_i \,.$$

It follows immediately by using the definitions of the sequence spaces $l(p), \overline{l(p)}$ and $l(p,\lambda)$ that

$$l(p) \subseteq \overline{l(p)} \subseteq l(p,\lambda).$$

Proposition 3.1.2. The sequence space $l(p,\lambda)$ is linearly isomorphic to $l(p)$.

Proof: For each $x \in l(p,\lambda)$, we have $\lambda x \in l(p)$ where λ is as defined in (3.1.5). It is easy to verify that λ is linear and bijective. Also the matrix λ has an inverse given by

$$\mu = (\mu_{nk}) = \begin{cases} 1, & k = n, \quad n \geq 3 \text{ and } k \leq n-2 \\ 0, & k > n, \quad n \geq 4 \text{ and } k \leq n-3 \\ -2, & n \geq 2 \text{ and } k \leq n-1 \end{cases}$$

(3.1.8)

that is,

$$\mu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ -2 & 1 & 1 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & \ldots \\ 0 & 1 & -2 & 1 & 0 & \ldots \\ 0 & 0 & 1 & -2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Thus, the sequence space $l(p,\lambda)$ is linearly isomorphic to $l(p)$.

Proposition 3.1.3. The sequence space $l(p,\lambda)$ is a complete paranormed sequence space paranormed by,

$$g(x) = \left\{ \sum_{k=1}^{n} |u_k|^{p_k} \right\}^{1/M}$$

(3.1.9)

where

$$M = \max(1, \sup p_k)$$

Proof: The proof of this proposition follows immediately from the proposition 3.1.2; where $g(x) = P(\lambda x)$ and P is the usual paranorm on $l(p)$.

Proposition 3.1.4: Let $\zeta_k = (\lambda x)_k$ for all $k \in \mathbb{N}$. We define the sequence $\mu^k = \left\{ \mu_n{}^{(k)} \right\}$ for every $k \in \mathbb{N}$ as in (3.1.8). Then the sequence $\{\mu^{(k)}\}_{k\in\mathbb{N}}$ is the basis for the sequence space $l(p,\lambda)$ and any $x \in l(p,\lambda)$ and has a unique representation

$$x = \sum_{k=1}^{\infty} \zeta_k \mu^{(k)} .$$

It is easy to verify.

3.2. Duals

In this section we find the β-dual of the sequence space $l(p,\lambda)$ for $0 < p_k \le 1$ and $1 < p_k \le \sup p_k < \infty$ for every $k \in \mathbb{N}$. Recall that if X be a sequence space by $\beta-$dual of X, we mean the space X^β defined as

$$X^\beta = \left\{a = (a_k): \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in X\right\}$$

We shall begin the section with the following lemmas [25] to prove the following theorems.

Lemma 3.2.1. If $0 < p_k \le 1$for every $k \in \mathbb{N}$, then $l(p)^\beta = l_\infty(p)$ where

$$l_\infty(p) = \left\{x = (x_k): \sup_k |x_k|^{p_k} < \infty\right\} \text{ [82].}$$

Lemma 3.2.2. If $p_k > 1$ for every $k \in \mathbb{N}$, then $l(p)^\beta = \mathcal{M}(p)$ where

$$\mathcal{M}(p) = \bigcup_{N>1} \left\{a = (a_k): \sum_{k=1}^{\infty} |a_k|^{q_k} N^{-\frac{q_k}{p_k}} < \infty\right\}$$

with

$$\frac{1}{p_k} + \frac{1}{q_k} = 1 \text{ [24,47] .}$$

Theorem 3.2.1

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then

$$l^\beta(p,\lambda) = \overline{l_\infty(p,\lambda)}$$

where

$$\overline{l_\infty(p,\lambda)} = \{a = (a_k): sup_{\,k} |\Delta^2 a_k|^{p_k} < \infty\}$$

and

$$\Delta^2 a_k = \Delta a_k - \Delta a_{k+1}$$

Proof: Let $a \in \overline{l_\infty(p,\lambda)}$. Then there exists an integer $N \ge 1$ such that

$$\sup_k \left|\Delta^2 a_k\right|^{p_k} < \infty$$

where

$$\Delta^2 a_k = \Delta a_k - \Delta a_{k+1}$$

Take $x \in l(p,\lambda)$, then $\lambda x \in l(p)$.

Hence,

$$\sum_{k=1}^{\infty} |\lambda x|^{p_k} < \infty.$$

So, there exists an integer $N \geq 1$ such that

$$|\lambda x| \leq (N^{-2})^{\frac{1}{p_k}}$$

We have,

$$\left|\sum_{k=1}^{m} a_k x_k\right| = \left|\sum_{k=1}^{m} a_k (u_k - 2u_{k-1} + u_{k-2})\right|$$

where

$$u_k = \sum_{i=1}^{k} (k - i + 1) x_i$$

with

$$u_k = 0 \text{ for } k \leq 0$$

Now,

$$\left|\sum_{k=1}^{m} a_k x_k\right| = |a_1 u_1 + a_2(u_2 - 2u_1) + a_3(u_3 - 2u_2 + u_1) + \cdots$$

$$+ a_m(u_m - 2u_{m-1} + u_{m-2})|$$

$= |(a_1 - 2a_2 + a_3)u_1 + (a_2 - 2a_3 + a_4)u_2 + \cdots + (a_m - 2a_{m+1} + a_{m+2})u_m|$

$=|(\Delta a_1 - \Delta a_2)u_1 + (\Delta a_2 - \Delta a_3)u_2 + \cdots + (\Delta a_m - \Delta a_{m+1})u_m|$;

where

$$\Delta a_j = a_j - a_{j+1}$$

$$\therefore \left|\sum_{k=1}^{m} a_k x_k\right| = \left|\sum_{k=1}^{m} \Delta^2 a_k u_k\right|$$

$$\leq \sum_{k=1}^{m} |\Delta^2 a_k||u_k|$$

$$\leq \left|\sum_{k=1}^{m} \Delta^2 a_k\right| (N^{-2})^{1/p_k}$$

Since, $\left|\Delta^2 a_k\right|^{p_k}$ is bounded, so that for some M > 0, $\left|\Delta^2 a_k\right|^{p_k} \leq M$.

We remark that the sequence

$$\{N^{-2}\}^{\frac{1}{p_k}} \in l(p)$$

and if

$$\sum_{k=1}^{\infty} M^{\frac{1}{p_k}} (N^{-2})^{\frac{1}{p_k}} > \infty \,,$$

then

$$\left\{M^{\frac{1}{p_k}}\right\} \notin l^{\beta}(p) \,.$$

Therefore,

$$\sum_{k=1}^{\infty} \left|a_k x_k\right| \leq \sum_{k=1}^{\infty} M^{\frac{1}{p_k}} (N^{-2})^{\frac{1}{p_k}} < \infty \,.$$

Hence it follows that $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in l(p,\lambda)$ and $\overline{l_\infty(p,\lambda)} \subseteq l^{\beta}(p,\lambda)$.

On the other hand, let $a \in l^{\beta}(p,\lambda)$. Then, $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in l(p,\lambda)$. It needs to show that

$$\sup_k \left|\Delta^2 a_k\right|^{p_k} < \infty \,.$$

On the contrary, let

$$\sup_k \left|\Delta^2 a_k\right|^{p_k} = \infty \,.$$

Then,

$$\left\{\Delta^2 a_k\right\} \notin l^{\beta}(p) = l_{\infty}(p) .$$

Hence, there exists a sequence $y=\{y_k\} \in l(p)$ such that $\sum_{k=1}^{\infty} \Delta^2 a_k y_k$ does not converge. Although if we define the sequence $\mu=\{\mu_k\}$ by

$$\mu_k = y_{k-2} - 2y_{k-1} + y_k ; \quad y_j = 0$$

for $j \le 0$, then

$$\mu \in l(p,\lambda)$$

and

$$\sum_{k=1}^{\infty} a_k \mu_k = \sum_{k=1}^{\infty} \Delta^2 a_k y_k .$$

It follows that the series $\sum_{k=1}^{\infty} a_k \mu_k$ does not converge; which is contradiction to our assumption that $a \in l^{\beta}(p,\lambda)$. Hence we must have

$$\sup_k \left|\Delta^2 a_k\right|^{p_k} < \infty$$

which shows $l^{\beta}(p,\lambda) \subseteq \overline{l_{\infty}(p,\lambda)}$ and completes the proof.

Theorem 3.2.2

Let $1 < p_k \le \sup p_k < \infty$ for every $k \in \mathbb{N}$. Then $l^{\beta}(p,\lambda) = \mathrm{M}(p,\lambda)$ where

$$M(p,\lambda) = \left\{ a = (a_k) : \sum_{k=1}^{\infty} |\Delta^2 a_k|^{q_k} N^{-\frac{q_k}{p_k}} \text{ converges where } \frac{1}{q_k} + \frac{1}{p_k} = 1 \text{ and } N > 1 \right\}$$

Proof: Let $a \in \mathrm{M}(p,\lambda)$ and $x \in l(p,\lambda)$. From the inequality

$$|b_k y_k| \le |b_k|^{q_k} + |y_k|^{p_k}$$

we obtain,

$$\left|a_k x_k\right| = \left|\Delta^2 a_k u_k\right| \le \left|\Delta^2 a_k\right|^{q_k} N^{-\frac{q_k}{p_k}} + N\left|u_k\right|^{p_k} \tag{3.2.1}$$

where N is the integer associated with $a \in \mathrm{M}(p,\lambda)$ and $\frac{1}{q_k}+\frac{1}{p_k}=1$.

Now,

$$\begin{aligned}
\left|\sum_{k=1}^{m} a_k x_k\right| &= \left|\sum_{k=1}^{m} \Delta^2 a_k u_k\right| \\
&\le \sum_{k=1}^{m}\left|\Delta^2 a_k u_k\right| \\
&\le \sum_{k=1}^{m}\left\{\left|\Delta^2 a_k\right|^{q_k} N^{-\frac{q_k}{p_k}} + N\left|u_k\right|^{p_k}\right\} \\
&\le \sum_{k=1}^{m}\left\{\left|\Delta^2 a_k\right|^{q_k} N^{-\frac{q_k}{p_k}} + N g^m(x)\right\} \\
&< \infty
\end{aligned}$$

It follows that $\sum_{k=1}^{\infty} a_k x_k$ converges and $\mathrm{M}(p,\lambda) \subseteq l^{\beta}(p,\lambda)$.

On the other hand, let $a \in l^{\beta}(p,\lambda)$. Then, $\sum_{k=1}^{\infty} a_k x_k$ converges for each $x \in l(p,\lambda)$. As a contrary, let $a \notin \mathrm{M}(p,\lambda)$. Then,

$$\sum_{k=1}^{\infty}\left|\Delta^2 a_k\right|^{q_k} N^{-\frac{q_k}{p_k}}$$

does not converge.

Since ,

$$x = \{x_k\} = \left\{N^{-\frac{q_k}{p_k}}\right\} \in l(p), \text{ then } \{\Delta^2 a_k\} \notin \mathrm{M}(p) = l^{\beta}(p).$$

Now , there exists a sequence $y = \{y_k\} \in l(p)$; such that $\sum_{k=1}^{\infty} \Delta^2 a_k y_k$ does not converse.

However , if we define $\mu = \{\mu_k\}$ by ,

$$\mu_k = y_{k-2} - 2y_{k-1} + y_k$$

with $y_j = 0$ for $j \le 0$, then

$$\mu \in l(p,\lambda)$$

and

$$\sum_{k=1}^{\infty} a_k \mu_k = \sum_{k=1}^{\infty} \Delta^2 a_k y_k \ .$$

It follows that the series $\sum_{k=1}^{\infty} a_k \mu_k$ does not converge which is contradiction to our assumption that $a \in l^{\beta}(p,\lambda)$. Hence we must have the series $\sum_{k=1}^{\infty} \left|\Delta^2 a_k\right|^{q_k} N^{-\frac{q_k}{p_k}}$ converges and $l^{\beta}(p,\lambda) \subseteq \mathrm{M}(p,\lambda)$. This completes the proof.

3.3. Matrix Transformation

In this section we give characterization for the classes $(l(p,\lambda), l_{\infty})$, $(l(p,\lambda), c)$ and $(l(p,\lambda), c_0)$.

Theorem 3.3.1

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then, $A \in (l(p,\lambda), l_{\infty})$ if and only if

$$\sup_{n,k} \left|\Delta^2 a_{nk}\right|^{p_k} < \infty \ .$$

Proof: Let the conditions hold. Then we have,

$$\sup_{n,k} \left|\Delta^2 a_{nk}\right|^{p_k} < \infty \ .$$

Take $x \in l(p,\lambda)$. Then $\lambda x \in l(p)$ and hence $\sum_{k=1}^{\infty} \left|\lambda x\right|^{p_k} < \infty$.

So, there exists an integer $N \ge 1$ such that

$$\left|\lambda x\right| \le (N^{-2})^{\frac{1}{p_k}} \ .$$

We have ,

$$\left|\sum_{k=1}^{\infty} a_{nk} x_k\right| = \left|\sum_{k=1}^{\infty} (\Delta a_{nk} - \Delta a_{n,k+1}) u_k\right| \text{ ,where } u_k = \sum_{i=1}^{k} (k-i+1) x_i$$

$$= \left|\sum_{k=1}^{\infty} \Delta^2 a_{nk} u_k\right|$$

$$\leq \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right| \left|u_k\right|$$

$$\leq \sup_{n,k} \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right|^{p_k} \left|N^{-2}\right|^{\frac{1}{p_k}}$$

$$< \infty .$$

Hence it follows that $\sum_{k=1}^{\infty} a_{nk}\, x_k$ converges and $Ax \in l_\infty$.

Conversely, let $A \in (l(p,\lambda), l_\infty)$. Then $\sum_{k=1}^{\infty} a_{nk}\, x_k$ converges for each $x = (x_k) \in l(p,\lambda)$ and $n \in \mathbb{N}$. We need to show that

$$\sup_{n,k} |\Delta^2 a_{nk}|^{p_k} < \infty$$

Now , since $\sum_{k=1}^{\infty} a_{nk} x_k$ converges , we have $\{a_{nk}\}_{k\in\mathbb{N}} \in l^{\beta}(p,\lambda)$ for every $n \in \mathbb{N}$. It implies that

$$\sup_{n,k} |\Delta^2 a_{nk}|^{p_k} < \infty$$

which is as desired.

Theorem 3.3.2

Let $1 < p_k \leq \sup p_k < \infty$ for every $k \in \mathbb{N}$. Then $A \in \left(l(p,\lambda), l_\infty\right)$ if and only if

$$\sup_{n} \sum_{k=1}^{\infty} |\Delta^2 a_{nk}|^{q_k} N^{-q_k/p_k} < \infty$$

where

$$\frac{1}{q_k}+\frac{1}{p_k}=1.$$

Proof: Let the conditions hold i.e. $1<p_k\le\sup p_k<\infty$. Take $x\in l(p,\lambda)$. Then $\lambda x\in l(p)$ and hence

$$g(x)=\sum_{k=1}^{\infty}|\lambda x|^{p_k}<\infty$$

Then there exists an integer $N\ge 1$ such that

$$|\lambda x|\le (N^{-1})^{\frac{1}{p_k}}.$$

Now,

$$\left|\sum_{k=1}^{\infty}a_{nk}x_k\right| = \left|\sum_{k=1}^{\infty}\Delta^2 a_{nk}u_k\right|$$

$$\le\sum_{k=1}^{m}\left\{\left|\Delta^2 a_{nk}\right|^{q_k} N^{-\frac{q_k}{p_k}}+N\left|u_k\right|^{p_k}\right\}$$

(3.3.1)

$$\le\sup_n\sum_{k=1}^{m}\left\{\left|\Delta^2 a_{nk}\right|^{q_k} N^{-\frac{q_k}{p_k}}+Ng^m(x)\right\}$$

$$<\infty$$

Hence it follows that $\sum_{k=1}^{n}a_{nk}x_k$ converges and $Ax\in l_\infty$.

Conversely, let $A\in\left(l(p,\lambda),l_\infty\right)$. Then $\sum_{k=1}^{n}a_{nk}x_k$ converges for each $x=(x_k)\in l(p,\lambda)$ and $\{A_n x\}\in l_\infty$. We need to show,

$$\sup_n\sum_{k=1}^{\infty}\left|\Delta^2 a_{nk}\right|^{q_k} N^{-\frac{q_k}{p_k}}<\infty$$

where

$$\frac{1}{p_k}+\frac{1}{q_k}=1.$$

Since, $\sum_{k=1}^{\infty} a_{nk}x_k$ converges for each $x \in l(p,\lambda)$, then

$$\{a_{nk}\}_{k\in\mathbb{N}} \in l^{\beta}(p,\lambda)$$

for every $n \in \mathbb{N}$.

Hence, $\sum_{k=1}^{\infty}\left|\Delta^2 a_{nk}\right|^{q_k} N^{-\frac{q_k}{p_k}} < \infty$; where $\frac{1}{p_k}+\frac{1}{q_k}=1$.

Further, since $\{A_n x\} \in l_\infty$, $\sup_n \left|\sum_{k=1}^{n} a_{nk}x_k\right| < \infty$ for $n \in \mathbb{N}$, it follows immediately from (3.1.1) that

$$\sup_n \sum_{k=1}^{\infty}\left|\Delta^2 a_{nk}\right|^{q_k} N^{-\frac{q_k}{p_k}} < \infty$$

where

$$\frac{1}{p_k}+\frac{1}{q_k}=1.$$

Hence it completes the proof.

Theorem 3.3.3

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then, $A \in (l(p,\lambda),c)$ if and only if

i)

$$\sup_{n,k}\left|\Delta^2 a_{nk}\right|^{p_k} < \infty \text{ and}$$

ii)

$$\lim_{n\to\infty} \Delta^2 a_{nk} = \Delta^2 \alpha_k$$

for every fixed k .

Proof: Let the conditions (i) and (ii) hold. Take any $x \in l(p,\lambda)$. Then $\lambda x \in l(p)$.

Hence,

$$\sum_{k=1}^{\infty} |\lambda x|^{p_k} < \infty .$$

Again, there exists an integer $N \geq 1$, such that

$$|\lambda x| \leq (N^{-2})^{\frac{1}{p_k}}$$

We have,

$$\left|\sum_{k=1}^{\infty} a_{nk}\, x_k\right| = \left|\sum_{k=1}^{\infty} \Delta^2 a_{nk} u_k\right| < \infty$$

as in theorem 3.3.1.

Also, by using condition (ii) ,

$$\left|\sum_{k=1}^{\infty} \Delta^2 a_{nk}\, u_k\right| = \left|\sum_{k=1}^{\infty} \Delta^2 \alpha_k u_k\right| < \infty .$$

Therefore, $\left\{\Delta^2 \alpha_k\right\}_{k\in N} \in l^{\beta}(p)$ and since the sequence $\{a_{nk}\}_{k\in\mathbb{N}} \in l^{\beta}(p,\lambda)$;the series $\sum_{k=1}^{\infty} a_{nk}\, x_k$ and $\sum_{k=1}^{\infty} \Delta^2 \alpha_k u_k$ converge for every $n \in \mathbb{N}$ and every $x=(x_k) \in l(p,\lambda)$. Hence $Ax \in c$.

Conversely, let $A \in \left(l(p,\lambda), c\right)$. Then $\sum_{k=1}^{\infty} a_{nk} x_k$ for $x=(x_k) \in l(p,\lambda)$ and $n \in \mathbb{N}$. We need to show that the conditions (i) and (ii) hold.

Moreover , $\left\{A_n x\right\} \in c$ for $n \to \infty$ and for some fixed k . Then,

$$\lim_{n\to\infty} \left|\sum_{k=1}^{\infty} a_{nk} x_k\right| = \lim_{n\to\infty} \left|\sum_{k=1}^{\infty} \Delta^2 a_{nk} u_k\right| .$$

So, $\lim_{n\to\infty} \Delta^2 a_{nk} = \Delta^2 \alpha_k$, where $\alpha_k = \lim_{n\to\infty} a_{nk}$ for some fixed k. Further it remains to show that

$$\sup_{n,k} \left| \Delta^2 a_{nk} \right|^{p_k} < \infty .$$

Since, $\sum_{k=1}^{\infty} a_{nk} x_k$ converges , we have

$$\{a_{nk}\}_{k \in \mathbb{N}} \in l^{\beta}(p, \lambda)$$

and hence $\sup_{n,k} \left| \Delta^2 a_{nk} \right|^{p_k} < \infty$; which is as desired.

Theorem 3.3.4.

Let $1 < p_k \leq \sup p_k < \infty$ for every $k \in \mathbb{N}$. Then $A \in \left(l(p, \lambda), c \right)$ if and only if

i)

$$\sup_n \sum_{k=1}^{\infty} \left| \Delta^2 a_{nk} \right|^{q_k} N^{-\frac{q_k}{p_k}} < \infty$$

where

$$\frac{1}{q_k} + \frac{1}{p_k} = 1 \text{ and}$$

ii)

$$\lim_{n \to \infty} \Delta^2 a_{nk} = \Delta^2 \alpha_k$$

for every fixed k .

Proof: Let the conditions (i) and (ii) hold. Take any $x \in l(p, \lambda)$. Then $\lambda x \in l(p)$. We have,

$$g(x) = \sum_{k=1}^{\infty} |\lambda x|^{p_k} < \infty$$

Again, there exists an integer $N > 1$, such that

$$\left| \lambda x \right| \leq (N^{-1})^{\frac{1}{p_k}} .$$

We have,

$$\left|\sum_{k=1}^{\infty} a_{nk}\, x_k\right| = \left|\sum_{k=1}^{\infty} \Delta^2 a_{nk}\, u_k\right| < \infty$$

as in theorem 3.3.2.

Also by using condition (ii)

$$\left|\sum_{k=1}^{\infty} \Delta^2 a_{nk}\, u_k\right| = \left|\sum_{k=1}^{\infty} \Delta^2 \alpha_k u_k\right| < \infty \,.$$

Now, using the same argument as in theorem (3.3.3), we arrive at the result $Ax \in c$.

Conversely, let $A \in \left(l(p,\lambda), c\right)$; then $\sum_{k=1}^{\infty} a_{nk}\, x_k$ converges for each $x = (x_k) \in l(p,\lambda)$. We need to show that conditions (i) and (ii) hold.

Moreover , $\left\{A_n x\right\} \in c$ for $n \to \infty$ and for some fixed k. Then,

$$\lim_{n\to\infty} \left|\sum_{k=1}^{\infty} a_{nk} x_k\right| = \lim_{n\to\infty} \left|\sum_{k=1}^{\infty} \Delta^2 a_{nk} u_k\right| \;.$$

So,

$\lim_{n\to\infty} \Delta^2 a_{nk} = \lim_{n\to\infty} \Delta(\Delta a_{nk}) = \lim_{n\to\infty} \Delta(\Delta a_{nk}) = \lim_{n\to\infty} \Delta(a_{n,k+1} - a_{n,k}) = \Delta(\alpha_{k+1} - \alpha_k)$
$= \Delta^2 \alpha_k$ for some fixed k, which is condition (ii). Now it remains to show that condition (i) holds.

Since, $\sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $x \in l(p,\lambda)$, then we have

$$\{a_{nk}\}_{k\in\mathbb{N}} \in l^{\beta}(p,\lambda)$$

for every $n \in \mathbb{N}$.

Hence ,

$$\sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right|^{q_k} N^{-\frac{q_k}{p_k}} < \infty$$

where

$$\frac{1}{p_k} + \frac{1}{q_k} = 1 \,.$$

It follows from the same arguments given in the proof for theorem 3.3.2 that

$$\sup_{n} \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right|^{q_k} N^{-\frac{q_k}{p_k}} < \infty .$$

It completes the proof.

By using the arguments as in the theorems (3.3.3) and (3.3.4), it is straight forward matter to prove the following theorems:

Theorem 3.3.5 :

Let $0 < p_k \le 1$ for every $k \in \mathbb{N}$. Then, $A \in (l(p,\lambda), c_0)$ if and only if

i)

$$\sup_{n,k} \left|\Delta^2 a_{nk}\right|^{p_k} < \infty \text{ and}$$

ii)

$$\lim_{n\to\infty} \Delta^2 a_{nk} = \Delta^2 \alpha_k$$

with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

Theorem 3.3.6 :

Let $1 < p_k \le \sup p_k < \infty$ for every $k \in \mathbb{N}$. Then $A \in \left(l(p,\lambda), c_0\right)$ if and only if

i)

$$\sup_{n} \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right|^{q_k} N^{-\frac{q_k}{p_k}} < \infty$$

where

$$\frac{1}{q_k} + \frac{1}{p_k} = 1 \text{ and}$$

ii)

$$\lim_{n\to\infty} \Delta^2 a_{nk} = \Delta^2 \alpha_k$$

with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

Finally we remark that the sequence, $\lambda = \{(1,0,0,\dots),(-2,1,0,0,\dots),(1,-2,1,0,0,\dots),\dots\}$ is not $l(p)$ convergent but it is λ-$l(p)$, that is, $l(p,\lambda)$ convergent.

Part Two:
Paranormed Sequence Spaces $X(p,\lambda)$ for $X \in \{l_\infty, c, c_0\}$ Generated by Lower Unitriangular Matrix

3.4. Preliminaries

By ω we mean the space of all complex valued sequences and any vector subspace of ω is referred as a sequence space. The symbols l_∞, c and c_0 stand for the spaces of all bounded, convergent and null sequence respectively. By a paranormed space we mean a linear topological space X over the field $\mathbb{R}$ if there is a subadditive function $g: X \to \mathbb{R}$ such that $g(\theta)=0$, $g(x)=g(-x)$ and scalar multiplication is continuous i.e. $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$, for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X . If $p=(p_k)$ be a bounded sequence of strictly positive real numbers, Maddox [44,45] defined the sequence spaces $l_\infty(p)$, $c(p)$ and $c_0(p)$ as follows:

$$l_\infty(p)=\left\{x=(x_k)\in\omega: \sup_k |x_k|^{p_k} < \infty\right\},$$

$$c(p) = \left\{x = (x_k) \in \omega: \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{C}\right\}$$

$$c_0(p)=\left\{x=(x_k)\in\omega: \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}.$$

The space $c_0(p)$ is a complete paranormed space paranormed by

$$h(x)=\sup_k |x_k|^{\frac{p_k}{M}}$$

and the spaces $l_\infty(p)$ and $c(p)$ are complete paranorm spaces paranormed by $h(x)$ if and only if $\inf p_k > 0$ [44,45,46].

Let X and Y be any two sequence spaces and $A=(a_{nk})$; $n, k\in\mathbb{N}$ be infinite matrix of complex numbers a_{nk} . Then we say that A defines a matrix mapping X into Y ; and it is denoted by writing ,

$$A: X \to Y$$

if for every sequence $x = (x_k) \in X$, the sequence $((Ax)_n)$ is in Y , where

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k , \quad n \in \mathbb{N} \tag{3.4.1}$$

By (X, Y) we denote the class of all matrices A such that $A: X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on right side of (3.4.1) converges for each $n \in \mathbb{N}$ and every $x \in X$; and we write,

$$Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in Y \tag{3.4.2}$$

for all $x \in X$.

We now introduce new sequence spaces $X(p, \lambda)$ for $X \in \{l_\infty, c, c_0\}$ as,

$$X(p, \lambda) = \{x = (x_k) : \lambda x \in X(p)\} \tag{3.4.3}$$

where

$$\lambda = (\lambda_{nk}) = S^n = \begin{cases} n - k + 1, & n \geq k \\ 0, & \text{otherwise} \end{cases}$$

as defined in (3.1.5) in section 3.1

and

$$S = (s_{nk}) = \begin{cases} 1, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

as defined in (3.1.4) in section 3.1.

We recall that the matrix domain X_A of an infinite matrix *A* in a sequence space *X* is defined by

$$X_A = \{ x = (x_k) \in \omega : Ax \in X \}$$

(3.4.4)

which is a sequence space.

Using the notation (3.4.4) , we can represent $X(p,\lambda)$ as

$$X(p,\lambda) = [X(p)]_\lambda .$$

$X(p,\lambda)$ can also be defined as the set of all sequences $\{u_k\}$ whose $\lambda = S^n$ transforms are in the sequence space $X \in \{l_\infty, c, c_0\}$ where the sequence $\{u_k\}$ is given by

$$\{u_k\} = \left\{ \sum_{i=1}^{k} (k-i+1)x_i \right\}$$

(3.4.5)

We shall now establish some propositions.

Proposition 3.4 1 : Sequence space $c_0(p,\lambda)$ is linear metric space paranormed by g, defined by

$$g(x) = \sup_k |\lambda x|^{\frac{p_k}{M}} \text{, where } M = \max(1, \sup_k p_k)$$

$$= \sup_k |u_k|^{\frac{p_k}{M}} .$$

(3.4.6)

Proof: From the definition of g it is clear that $g(x) = 0 \Leftrightarrow x = 0$ and $g(-x) = g(x)$ for all $x \in c_0(p,\lambda)$. To show linearity of $c_0(p,\lambda)$ with respect to coordinate-wise addition and scalar multiplication, let us take any two sequences $x, y \in c_0(p,\lambda)$ and scalars $,\beta \in \mathbb{R}$. Since λ is linear operator by [48] ,we note that

$$g(\alpha x+\beta y)=\sup_k \left|\lambda(\alpha x+\beta y)\right|^{\frac{p_k}{M}}$$

$$\leq \max\{1,|\alpha|\} \sup_k |\lambda x|^{\frac{p_k}{M}} + \max\{1,|\beta|\} \sup_k |\lambda y|^{\frac{p_k}{M}}$$

$$= \max\{1,|\alpha|\}\, g(x)+\max\{1,|\beta|\}\, g(y)$$

This follows the subadditivity of g , i.e.

$$g(x+y)\leq g(x)+g(y) \tag{3.4.7}$$

Now it remains to show the continuity of scalar multiplication in $c_0(p,\lambda)$. For it, let $\{x^n\}$ be any sequence of the points in $c_0(p,\lambda)$ such that

$$g(x^n-x)\to 0$$

and $\{\alpha_n\}$ be sequence of real scalars such that $\alpha_n \to \alpha$. Now by using (3.4.7), we have

$$g(x^n)\leq g(x)+g(x^n-x)$$

Further,

$$g(\alpha_n x^n-\alpha x) = \sup_k \left|\lambda(\alpha_n x^n-\alpha x)\right|^{\frac{p_k}{M}}$$

$$\leq (|\alpha_n-\alpha|^{\frac{p_k}{M}}\, g(x^n)+|\alpha_n-\alpha|^{\frac{p_k}{M}}\, g(x^n-x))<\infty \tag{3.4.8}$$

for all n.

Since $\{g(x^n)\}$ is bounded, we find from (3.4.8) that

$g(\alpha_n x^n-\alpha x)<\infty$

for all $n \in \mathbb{N}$.

That is, the scalar multiplication for g is continuous and therefore g is a paranorm on the sequence space $c_0(p,\lambda)$.

It can easily be verified that g is the paranorm for the spaces $l_\infty(p,\lambda)$ and $c(p,\lambda)$ if and only if $\inf p_k > 0$.

Proposition 3.4.2 : The sequence spaces $X(p,\lambda)$ for $X \in \{l_\infty, c, c_0\}$ are complete metric spaces paranormed by g , defined as in proposition 3.4.1.

Proof: We prove this proposition for $c_0(p,\lambda)$. Take a Cauchy sequence $\{x^n\}$ in the space $c_0(p,\lambda)$, where

$$x^n = \{x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots\}.$$

Now for given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that,

$$g(x^n - x^m) < \varepsilon \text{ for all } m,n \geq n_o(\varepsilon).$$

Also, from the definition of g for each fixed $k \in \mathbb{N}$, we have

$$\left|\{\lambda x^n\}_k - \{\lambda x^m\}_k\right|^{\frac{p_k}{M}}$$

$$\leq \sup_k \left|\{\lambda x^n\}_k - \{\lambda x^m\}_k\right|^{\frac{p_k}{M}}$$

$$< \varepsilon$$

for all $m,n \geq n_o(\varepsilon)$.

Now , this implies that, $\{(\lambda x^0)_k, (\lambda x^1)_k, (\lambda x^2)_k, \ldots\}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, the sequence $\{\lambda x^n\}_k$ converges and let

$$\{\lambda x^n\}_k \to \{\lambda x\}_k \text{ as } n \to \infty .$$

For each fixed $k \in \mathbb{N}$, $m \to \infty$ and $n \geq n_o(\varepsilon)$, it is clear that

$$\left|\{\lambda x^n\}_k - \{\lambda x\}_k\right|^{\frac{p_k}{M}} \leq \frac{\varepsilon}{2}$$

(3.4.9)

Since ,

$$x^n = \{x_k^{(n)}\} \in c_0(p,\lambda)$$

we have,

$$\left|\{\lambda x^n\}_k\right|^{\frac{p_k}{M}} \leq \frac{\varepsilon}{2}$$

(3.4.10)

for each fixed $k \in \mathbb{N}$.

Combining (3.4.9) and (3.4.10), we obtain that

$$\left|\{\lambda x\}_k\right|^{\frac{p_k}{M}}$$

$$\leq \left|\{\lambda x^n\}_k - \{\lambda x\}_k\right|^{\frac{p_k}{M}} + \left|\{\lambda x^n\}_k\right|^{\frac{p_k}{M}}$$

$$\leq \varepsilon$$

for all $n \geq n_o(\varepsilon)$.

Hence , the sequence $\{\lambda x\} \in c_0(p)$. Since $\{x^n\}$ was an arbitrary Cauchy sequence in $c_0(p,\lambda)$, we conclude that the space $c_0(p,\lambda)$ is complete. It completes the proof.

Proposition 3.4.3: The sequence spaces $X(p,\lambda)$ for $X \in \{l_\infty, c, c_0\}$ are linearly isomorphic to the respective spaces *X*.

Proof: For each $x \in X(p,\lambda)$, we have $\lambda x \in X(p)$, where λ is as defined in section 3.1.5 . It is easy to verify that λ is linear and bijective. Also the matrix λ has an inverse given by,

$$\mu = (\mu_{nk}) = \begin{cases} 1, & k = n, \quad n \geq 3 \text{ and } k \leq n-2 \\ 0, & k > n, \quad n \geq 4 \text{ and } k \leq n-3 \\ -2, & n \geq 2 \text{ and } k \leq n-1 \end{cases}$$

(3.1.11)

Thus, the sequence spaces $X(p,\lambda)$ is linearly isomorphic to the corresponding spaces $X(p)$ for $X \in \{l_\infty, c, c_0\}$.

Proposition 3.4.4: Let $\zeta_k = (\lambda x)_k$ and $0 < p_k \leq \sup_k p_k < \infty$ for all $k \in \mathbb{N}$. We define the sequence $\mu^k = \left\{\mu_n^{(k)}\right\}_{n\in\mathbb{N}}$ for every fixed $k \in \mathbb{N}$ as in proposition 3.4.3. Then,

(i) the sequence

$$\left\{\mu_n^{(k)}\right\}_{n\in\mathbb{N}}$$

is the basis for the sequence space $c_0(p,\lambda)$ and any $x \in c_0(p,\lambda)$ has a unique representation $x = \sum_{k=1}^{\infty} \zeta_k \mu^{(k)}$ and

(ii) the set

$$\{\upsilon, \mu^{(k)}\}$$

is a basis for the space $c(p,\lambda)$ and any $x \in c(p,\lambda)$ has a unique representation in the form

$$x = l\upsilon + \sum_{k=1}^{\infty} (\zeta_k - l)\mu^{(k)}$$

where $l = \lim_{k\to\infty} (\lambda x)_k$ and $\upsilon^T = (1,3,0,0,...)$

It is easy to verify this proposition.

3.5. Duals

In this section we find the generalized Köthe-Toeplitz dual i.e. β-dual of the sequence spaces $l_\infty(p,\lambda)$, $c_0(p,\lambda)$ and $c(p,\lambda)$. If X be a sequence space , we define $\beta -$dual of X as

$$X^{\beta} = \left\{ a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent for each } x \in X \right\}$$

Theorem 3.5.1

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $l_{\infty}^{\beta}(p,\lambda) = M_{\infty}(p,\lambda)$ where

$$M_{\infty}(p,\lambda) = \bigcap_{N=2}^{\infty} \left\{ a = (a_k) : \sum_{k=1}^{\infty} \left|\Delta^2 a_k\right| N^{\frac{1}{p_k}} < \infty \right\}$$

and

$$\Delta^2 a_k = \Delta a_k - \Delta a_{k+1} .$$

Proof: Let $a \in M_{\infty}(p,\lambda)$ and $x \in l_{\infty}(p,\lambda)$. We choose an integer $N > \max(1, \sup_k \left|u_k\right|^{p_k})$. Then we have,

$$\left|\sum_{k=1}^{m} a_k x_k\right| = \left|\sum_{k=1}^{m} (\Delta a_k - \Delta a_{k+1}) u_k\right| \text{ ; where } u_k = \sum_{i=1}^{k} (k-i+1) x_i$$

$$= \left|\sum_{k=1}^{m} \Delta^2 a_k u_k\right|$$

$$\leq \sum_{k=1}^{\infty} \left|\Delta^2 a_k\right| \left|u_k\right|$$

$$\leq \sum_{k=1}^{\infty} \left|\Delta^2 a_k\right| N^{\frac{1}{p_k}}$$

$$< \infty .$$

Hence,

$$M_{\infty}(p,\lambda) \subseteq l_{\infty}^{\beta}(p,\lambda) .$$

On the other hand , let $a \in l_{\infty}^{\beta}(p,\lambda)$ but $a \notin M_{\infty}(p,\lambda)$. Then there exists an integer N >1 such that

$$\sum_{k=1}^{\infty} \left|\Delta^2 a_k\right| N^{\frac{1}{p_k}} = \infty .$$

Then, $\{\Delta^2 a_k\} \notin l_\infty^\beta\ (p) = M_\infty(p)$, where

$$M_\infty(p) = \bigcap_{N>1} \left\{ a = (a_k) : \sum_{k=1}^{\infty} |a_k| N^{\frac{1}{p_k}} < \infty \right\} \ [25]$$

Hence , there exists a sequence $y = \{y_k\} \in l_\infty(p)$ such that $\sum_{k=1}^{\infty} \Delta^2 a_k y_k$ does not converge. Although if we define the sequence $\mu = \{\mu_k\}$ by

$$\mu_k = y_{k-1} - 2y_k + y_{k+1}$$

with $y_j = 0$ for $j \le 0$,then $\mu \in l_\infty(p, \lambda)$ and ,therefore ,

$$\sum_{k=1}^{\infty} a_k \mu_k = \sum_{k=1}^{\infty} \Delta^2 a_k y_k \ .$$

Hence it follows that the series $\sum_{k=1}^{\infty} a_k \mu_k$ does not converge; which is contradiction to our assumption that $a \in l_\infty^\beta(p, \lambda)$. Hence we must have,

$$\sum_{k=1}^{\infty} \left| \Delta^2 a_k \right| N^{\frac{1}{p_k}} < \infty$$

This shows that $l_\infty^\beta(p, \lambda) \subseteq M_\infty(p, \lambda)$. It completes the proof.

Theorem 3.5.2

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $c_0^\beta(p, \lambda) = M_0(p, \lambda)$ where

$$M_0(p, \lambda) = \bigcup_{N>1}^{\infty} \{ a = (a_k) : \sum_{k=1}^{\infty} \left| \Delta^2 a_k \right| N^{-\frac{1}{p_k}} < \infty \} \ .$$

Proof: Let $a \in M_0(p, \lambda)$ and $x \in c_0(p, \lambda)$. Then

$$\sum_{k=1}^{\infty} \left| \Delta^2 a_k \right| N^{-\frac{1}{p_k}} < \infty$$

for some N >1 and

$$|u_k|^{p_k} < \frac{1}{N}$$

for all sufficiently large k ; whence for such k ,

$$\sum_{k=1}^{\infty}|a_k x_k| = \sum_{k=1}^{\infty}|\Delta^2 a_k u_k|$$

$$\leq \sum_{k=1}^{\infty}|\Delta^2 a_k||u_k|$$

$$\leq \sum_{k=1}^{\infty}|\Delta^2 a_k| N^{-\frac{1}{p_k}} < \infty$$

Hence,

$$M_0(p,\lambda) \subseteq c_0^{\beta}(p,\lambda) .$$

On the other hand, let $a \in c_0^{\beta}(p,\lambda)$ but $a \notin M_0(p,\lambda)$. Then the convergence of $\sum_{k=1}^{\infty} a_k x_k$ for all $x \in c_0(p,\lambda)$ implies that $a \in M_0(p,\lambda)$. For otherwise, as in the proof of theorem 3.5.1, we can easily construct a sequence $\mu \in c_0(p,\lambda)$ such that $\sum_{k=1}^{\infty} a_k \mu_k$ does not converge; which becomes contradiction.

Hence,

$$c_0^{\beta}(p,\lambda) \subseteq M_0(p,\lambda) .$$

This completes the proof.

Corollary 3.5.1. Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $c^{\beta}(p,\lambda) = M_0(p,\lambda) \cap cs$, where cs is the set of convergent series. The proof of this corollary is the direct consequence of the theorem 3.5.2.

3.6. Matrix transformation

In this section we characterize the classes $(l_\infty(p,\lambda), l_\infty)$, $(l_\infty(p,\lambda), c)$ and $(l_\infty(p,\lambda), c_0)$.

Theorem 3.6.1

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $A \in (l_\infty(p,\lambda), l_\infty)$ if and only if

$$\sup_n \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right| N^{\frac{1}{p_k}} < \infty$$

for every integer N > 1.

Proof: Let the condition holds. Then we have,

$$\sup_n \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right| N^{\frac{1}{p_k}} < \infty .$$

Take $x \in l_\infty(p,\lambda)$. Then $\lambda x \in l_\infty(p)$ and hence $\sup_k \left|\lambda x\right|^{p_k} < \infty$. So there exists an integer $N \geq 1$ such that

$$\left|\lambda x\right| \leq N^{\frac{1}{p_k}} .$$

Then,

$$\left|\sum_{k=1}^{\infty} a_{nk} x_k\right| = \left|\sum_{k=1}^{\infty} \Delta^2 a_{nk} u_k\right| \text{ where } u_k = \sum_{i=1}^{k} (k-i+1) x_i$$

$$\leq \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right| \left|u_k\right|$$

$$\leq \sup_n \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right| N^{-\frac{1}{p_k}}$$

$$< \infty.$$

Hence it follows that $\sum_{k=1}^{\infty} a_{nk} x_k$ converges for each $n \in \mathbb{N}$ and $Ax \in l_\infty$.

On the other hand , let $A \in (l_\infty(p,\lambda), l_\infty)$. As a contrary let us assume that there exists an integer such that

$$\sup_n \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right| N^{\frac{1}{p_k}} = \infty .$$

Then the matrix $(\Delta^2 a_{nk}) \notin (l_\infty(p), l_\infty)$, as in theorem 3 [25] and so there exists a $y = (y_k) \in l_\infty(p)$ with $\sup_k |y_k| = 1$ such that

$$\sum_k \Delta^2 a_{nk} y_k \neq O(1)$$

Although if we define the sequence $\mu = \{\mu_k\}$ by

$$\mu_k = y_{k-2} - 2y_{k-1} + y_k$$

with $y_j = 0$ for $j \leq 0$, then $\mu \in l_\infty(p, \lambda)$ and therefore

$$\sum_{k=1}^{\infty} a_{nk} \mu_k = \sum_{k=1}^{\infty} \Delta^2 a_{nk} y_k \ .$$

It follows that the sequence $\{A_n(\mu)\} \notin l_\infty$; which is contradiction to our assumption. Thus,

$$\sup_n \sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right| N^{\frac{1}{p_k}} < \infty$$

and it completes the proof.

Theorem 3.6.2 :

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $A \in (l_\infty(p, \lambda), c)$ if and only if

(i)

$$\sum_{k=1}^{\infty} \left|\Delta^2 a_{nk}\right| N^{\frac{1}{p_k}}$$

converges uniformly in n for all integer $N > 1$.

(ii)

$$\lim_{n\to\infty} \Delta^2 a_{nk} = \Delta^2 \alpha_k$$

for some fixed k.

Proof: Let the conditions (i) and (ii) hold. We first state a lemma due to Lascarides and Maddox [25].

Lemma 3.6.1: Let $p_k > 0$ for every k . Then $A \in (l_\infty(p), c)$ if and only if

(i)

$$\sum |a_{nk}| N^{1/p_k}$$

converges uniformly in n , for all integers $N > 1$.

(ii)

$$a_{nk} \to \alpha_k$$

$(n \to \infty$, k fixed$)$.

Now, since the conditions (i) and (ii) hold , using lemma 3.6.1 we have the matrix

$$(\Delta^2 a_{nk}) \in (l_\infty(p), c) .$$

By using,

$$\sum_{k=1}^{\infty} a_{nk}\, x_k = \sum_{k=1}^{\infty} \Delta^2 a_{nk}\, u_k \tag{3.6.1}$$

we have ,

$$(A_n(x)) \in (l_\infty(p, \lambda), c)$$

for every $n \in \mathbb{N}$.

Hence, $A \in (l_\infty(p, \lambda), c)$.

On the other hand let $A \in (l_\infty(p, \lambda), c)$. Then from (3.6.1) it follows that

$$(\Delta^2 a_{nk}) \in (l_\infty(p), c)$$

Hence from the lemma 3.6.1 , we arrive at the result that the conditions (i) and (ii) hold. This proves the theorem.

Using the same arguments as in the theorems (3.6.1) and (3.6.2), it is straight forward matter to prove the theorem:

Theorem 3.6.3 :

Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $A \in (l_\infty(p,\lambda), c_0)$ if and only if

(i)

$$\sum_{k=1}^{\infty} \left| \Delta^2 a_{nk} \right| N^{\frac{1}{p_k}}$$

converges uniformly in n for all integers $N > 1$ and

(ii)

$$\lim_{n\to\infty} \Delta^2 a_{nk} = \Delta^2 \alpha_k \text{ with } \alpha_k = 0 \text{ for all } k \in \mathbb{N} \; .$$

Finally we remark that the sequence,

$b = (b_k) = \{(1,0,0,\ldots),(-2,1,0,0,\ldots),(1,-2,1,0,0,\ldots)\} \notin l_\infty(p)$ but $\in l_\infty(p,\lambda)$.

CHAPTER FOUR

SOME PARANORMED SEQUENCE SPACES GENERATED BY COMBINING SPARSE MATRIX λ_j AND GENERALIZED WEIGHTED MEAN $G(u,v)$ THAT GUARANTEES THE GIVEN RATE OF CONVERGENCE

4.1. Preliminaries

By ω we denote the space of all complex valued sequences. Any vector subspace of ω is regarded as a sequence space. We shall write l_∞, c, c_0 and cs for the spaces of all bounded, convergent, null and convergent series respectively.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g: X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous , that is, $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_{n-}\alpha x) \to 0$ for all $\alpha \in \mathbb{R}$ and $x \in X$; where θ is the zero vector in the linear space X.

Let X, Y be any two sequence spaces, and let $A = (a_{nk})$ be any infinite matrix of real number a_{nk} where $, n, k \in \mathbb{N}$. Then we say that A defines a matrix mapping from X into Y by writing $A: X \to Y$, if for every sequence $x = (x_k) \in X$, the sequence $Ax = (A_n(x))$, called the A- transform of x, is in Y, where

$$A_n(x) = \sum_k a_{nk}\, x_k \qquad (n \in \mathbb{N}) \tag{4.1.1}$$

By (X, Y) , we denote the class of all matrices A such that $A: X \to Y$. Thus, $A \in (X, Y)$ if and only if the series on the right hand sided of (4.1.1) converges for each $n \in \mathbb{N}$ and every $x \in X$, and we have $Ax \in Y$ for all $x \in X$.

We shall assume here and after that $\{p_k\}$ is a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max[1, H]$. Then, I.J. Maddox [44,45] have defined the following sequence spaces $c(p)$, $c_0(p)$ and $l_\infty(p)$as ,

$$c(p) = \left\{x = (x_k): \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \text{ for some } l \in \mathbb{C}\right\}$$

$$c_0(p) = \left\{x = (x_k): \lim_{k\to\infty} |x_k|^{p_k} = 0\right\}$$

and

$$l_\infty(p) = \left\{x = (x_k): \sup_{k\in\mathbb{N}} |x_k|^{p_k} < \infty\right\}$$

The space $c_0(p)$ is a complete paranorm space paranormed by

$$g(x) = \sup_{k\in\mathbb{N}} |x_k|^{\frac{p_k}{M}} \tag{4.1.2}$$

The spaces $l_\infty(p)$ and $c(p)$ are complete paranormed space paranormed by $g(x)$ if and only if $\inf p_k > 0$.

For simplicity in notation, here and in what follows, the summation without limit runs from 1 to ∞. Let (X, g) be a paranormed space. A sequence (b_k) of elements of X is called a basis for X if and only if , for each $x \in X$, there exists a unique sequence (α_k) of scalars such that

$$g\left(x - \sum_{k=1}^{n} \alpha_k b_k\right) \to 0$$

as $n \to \infty$.

The series $\sum_{k=1}^{\infty} \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and is written as

$$x = \sum_{k=1}^{\infty} \alpha_k b_k$$

In this chapter we introduce a set of new paranormed sequence spaces $l_\infty(u, v; p, \lambda_j)$, $c(u, v; p, \lambda_j)$ and $c_0(u, v; p, \lambda_j)$ generated by the combination of sparse matrix λ_j and the generalized weighted mean matrix $G(u, v)$. We establish some topological properties, obtain bases for $c(u, v; p, \lambda_j)$ and $c_0(u, v; p, \lambda_j)$ and find β- duals. Furthermore, we characterize the matrix classes $(l_\infty(u, v; p, \lambda_j), l_\infty)$, $(l_\infty(u, v; p, \lambda_j), c)$ and $(l_\infty(u, v; p, \lambda_j), c_0)$. Besides, we give characterization theorem for the case of mapping from the sequence space $l_\infty(p)$ to new sequence space $l_\infty(u,v;p,\lambda_j)$ that guarantees the given rate of convergence.

4.2. Remarks

Several authors have defined many new sequence spaces by using a generalized weighted mean (or a factorable) matrix $G(u,v)$ and the difference operator matrix Δ or by combining them. The difference sequence spaces were first studied by Kizmaz in 1981 [41]. Since then many authors have defined and studied new difference sequence spaces by considering the matrices that represent the difference operator. Some of the example, are as follows:

Malkowsky and Savas [29] have defined the sequence spaces $Z(u,v,X)$ which consists of all sequences such that $G(u,v)-$ transform are in $\in \{l_{\infty,}c, c_0, l_p\}$. Choudhary and Mishra [15] have defined the sequence space $\overline{l(p)}$ whose S – transform are in $l(p)$. Altay and Basar [10] have studied the space $r^t(p)$ which consists of all sequences whose Riesz transforms (R^t) are in the space $l(p)$. Recently, Demiriz and Caken [78] have defined the sequence spaces $\lambda(u,v;p,\Delta)$ for $\lambda \in \{c_0, c, l_\infty, l\}$ by combining the matrix

$$G(u,v) = (g_{nk}) = \begin{cases} u_n v_{k,} & 0 \le k \le n \\ 0\,, & k > n \end{cases} \tag{4.2.1}$$

and the difference operator matrix

$$\Delta = (\delta_{nk}) = \begin{cases} (-1)^{n-k}, & n-1 \le k \le n \\ \\ 0, & 0 \le k < n \text{ or } k > n \end{cases} \tag{4.2.2}$$

Most recently Baliarsingh [70] has introduced the spaces $X(\Delta_j, u, v, p)$ for $X \in \{l_\infty, c, c_0\}$ by combining the matrix $G = (g_{nk})$ as given in (4.2.1) and a double band matrix

$$\Delta_j = \begin{pmatrix} 1 & -2 & 0 & 0 & \cdots \\ 0 & 2 & -3 & 0 & \cdots \\ 0 & 0 & 3 & -4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(4.2.3)

For a sequence space X, the matrix domain X_A of infinite matrix A is defined by

$$X_A = \{x = (x_k) : Ax \in X\}$$

(4.2.4)

Using the notation (4.2.4), the sequence spaces introduced by the authors stated above can be represented as

$Z(u,v,p) = [X]_{G(u,v)}$, $\overline{l(p)} = [l(p)]_S$, $r^t(p) = [l(p)]_{R^t}$,

$\lambda(u,v;p,\Delta) = [\lambda]_{G(u,v,\Delta)}$ and $X(\Delta_j,u,v,p) = [X]_{G(u,v,\Delta_j)}$

Can now we make generalization in constructing new sequence spaces by introducing the operator matrix which guarantees the fast rate of convergence? The answer, we claim, is yes. Before introducing the new sequence spaces, we construct a new double band sparse matrix λ_j . For this we begin with a diagonal matrix ,

$$diag\left(\frac{1}{t_{ij}}\right) = \begin{cases} \frac{1}{t_j}, & i = j \\ 0, & \text{otherwise} \end{cases}$$

In expanded form , $$diag\left(\frac{1}{t_{ij}}\right) = \begin{pmatrix} \frac{1}{t_1} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{t_2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{t_3} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{t_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(4.2.5)

where

$$t = \left(\frac{1}{t_j}\right) \in (0,1).$$

The multiplication of this matrix with the difference operator matrix Δ yields a double band matrix ,

$$\Delta.diag\left(\frac{1}{t_{ij}}\right) = \begin{pmatrix} \frac{1}{t_1} & 0 & 0 & 0 & \cdots \\ -\frac{1}{t_1} & \frac{1}{t_2} & 0 & 0 & \cdots \\ 0 & -\frac{1}{t_2} & \frac{1}{t_3} & 0 & \cdots \\ 0 & 0 & -\frac{1}{t_3} & \frac{1}{t_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We denote the transpose of $\Delta.diag\left(\frac{1}{t_{ij}}\right)$ by λ_j. Thus,

$$\lambda_j = \begin{pmatrix} \frac{1}{t_1} & -\frac{1}{t_1} & 0 & 0 & \cdots \\ 0 & \frac{1}{t_2} & -\frac{1}{t_2} & 0 & \cdots \\ 0 & 0 & \frac{1}{t_3} & -\frac{1}{t_3} & \cdots \\ 0 & 0 & 0 & \frac{1}{t_4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{4.2.6}$$

We use λ_j together with $G(u,v)$ to define our new sequence spaces.

We write by U the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for $n \in \mathbb{N}$. For $u \in U$, let

$$\frac{1}{u} = \left(\frac{1}{u_n}\right)$$

Let $u, v \in U$ and let us take the matrix $G(u,v)$ as defined in (4.2.1) for all $n,k \in \mathbb{N}$; where u_n depends only on n and v_k only on k. The matrix $G(u,v)$ is called generalized weighted mean or factorable matrix. We shall now define the matrix $G(u,v,\lambda_j)$ as,

$$G(u,v,\lambda_j) = G(u,v)\lambda_j = \left(g_{nk}^{\lambda_j}\right) = \begin{cases} u_n\left(\frac{v_k}{t_k} - \frac{v_{k-1}}{t_{k-1}}\right), & k \le n \\ -\frac{1}{t_n}u_n v_n, & k = n+1 \\ 0, & \text{otherwise} \end{cases}$$

(4.2.7)

We use the matrix $G(u,v,\lambda_j)$ to define new sequence spaces.

4.3. The Paranormed Sequence Spaces $X(u,v;p,\lambda_j)$ for $X \in \{l_\infty, c, c_0\}$

Following [10,11,12,13,15,29,42,70,78],we define the sequence spaces $X(u,v;p,\lambda_j)$ for $X \in (l_\infty, c, c_0)$ by

$$X(u,v;p,\lambda_j) = \left\{x = (x_k) : \left(\sum_{j=1}^{k} u_k v_j \lambda_j x_j\right) \in X(p)\right\}$$

(4.3.1)

where $\lambda_j x_j$ is defined as follows

$$\lambda_j x_j = \frac{1}{t_j}\Delta x_j , \qquad (j \in \mathbb{N})$$

and $\Delta x_j = x_{j-1} - x_j$ with $x_0 = 0$, $(j \in \mathbb{N})$. $\lambda_j$is a sequential double band matrix as defined in (4.2.6). Using the notation as in (4.2.4), we may represent the sequence spaces $X(u,v;p,\lambda_j)$ in (4.3.1) as

$$X(u,v;p,\lambda_j) = [X(p)]_{G(u,v,\lambda_j)}$$

for $\in (l_\infty, c, c_0)$.

In other words $X(u,v;p,\lambda_j)$ are the sequence spaces which consist of all sequences whose$G(u,v,\lambda_j)$- transforms are in $X(p)$.

Here and after we use the convention that any term with negative and zero subscript is equal to zero. In the following propositions we prove that these spaces are complete paranormed linear metric spaces and isomorphic to the spaces $l_\infty(p), c(p)$ and $c_0(p)$ respectively. Moreover, we establish basis for the spaces $c(u, v; p, \lambda_j)$ and $c_0(u, v; p, \lambda_j)$. Since the proof may also be obtained in the similar way for the other spaces, we give the proof only for one of these spaces in order to avoid the repetitions of the similar statements.

Proposition 4.3.1 : Sequence space $c_0(u, v; p, \lambda_j)$ is a linear metric space paranormed by g, defined by ,

$$g(x) = \sup_k \left| \sum_{j=1}^{k} u_k v_j \lambda_j x_j \right|^{\frac{p_k}{M}} \tag{4.3.2}$$

Proof: We shall check the properties that g should satisfy. From the definition it is clear that $g(x) = 0 \Leftrightarrow x = 0$ and $g(x) = g(-x)$ for all $x \in c_0(u, v; p, \lambda_j)$. To show the linearity of $c_0(u, v; p, \lambda_j)$ with respect to coordinatewise addition and scalar multiplication, let us take any two elements $x, y \in c_0(u, v; p, \lambda_j)$ and scalar $\alpha, \beta \in \mathbb{R}$. Since λ_j is a linear operator from Maddox [25] , we note that

$$g(\alpha x + \beta y) = \sup_k \left| \sum_{j=1}^{k} u_k v_j \lambda_j (\alpha x_j + \beta y_j \right|^{\frac{p_k}{M}}$$

$$\leq max\{1, |\alpha|\} \sup_k \left| \sum_{j=1}^{k} u_k v_j \lambda_j x_j \right|^{\frac{p_k}{M}} + \; max\{1, |\beta|\} \sup_k \left| \sum_{j=1}^{k} u_k v_j \lambda_j y_j \right|^{\frac{p_k}{M}}$$

$$= max\{1, |\alpha|\} g(x) + \; max\{1, |\beta|\} g(y)$$

This follows the subaddivity of g, that is,

$$g(x + y) \leq g(x) + g(y) \tag{4.3.3}$$

Now, it remains to show the continuity of scalar multiplication in $c_0(u,v;p,\lambda_j)$. For it, let $\{x^n\}$ be any sequence of points in $c_0(u,v;p,\lambda_j)$ such that $g(x^n - x) \to 0$ and $\{\alpha_n\}$ be sequence of real numbers such that $\alpha_n \to \alpha$. Now by using (4.3.3), we have

$$g(x^n) \leq g(x) + g(x^n - x)$$

Further,

$$g(\alpha_n x^n - \alpha x) = \sup_k \left| \sum_{j=1}^{k} u_k v_j \lambda_j (\alpha_n x_j^n - \alpha x_j \right|^{\frac{p_k}{M}}$$

$$\leq \left(|\alpha_n - \alpha|^{\frac{p_k}{M}} g(x^n) + |\alpha|^{\frac{p_k}{M}} g(x^n - x) \right) < \infty \tag{4.3.4}$$

for all $n \in \mathbb{N}$

Since $\{g(x^n)\}$ is bounded , we find from (4.3.4)that

$$g(\alpha_n x^n - \alpha x) < \infty$$

for all $n \in \mathbb{N}$.

That is, the scalar multiplication for g is continuous and therefore g is paranorm on the sequence space $c_0(u,v;p,\lambda_j)$.

It can easily be verified that g is the paranorm for the spaces $l_\infty(u,v;p,\lambda_j)$ and $c(u,v;p,\lambda_j)$ if and only if $\inf p_k > 0$.

Proposition 4.3.2: The sequence spaces $X(u,v;p,\lambda_j)$ for $X \in \{l_\infty, c, c_0\}$ are complete metric spaces paranormed by g, defined as in proposition 4.3.1.

Proof: We prove the proposition for the sequence space $c_0(u,v;p,\lambda_j)$. Take a Cauchy sequence $\{x^n\}$ in the sequence space $c_0(u,v;p,\lambda_j)$, where

$$x^n = \left\{ x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots \right\}$$

Now, since $\{x^n\}$ is a Cauchy sequence, for given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that,

$$g(x^n - x^m) < \varepsilon$$

for all $m, n \geq n_0(\varepsilon)$.

Also from the definition of g for each fixed $n \in \mathbb{N}$, we have

$$\left|\left\{G\big(u, v, \lambda_j\big)x^n\right\}_k - \left\{G\big(u, v, \lambda_j\big)x^m\right\}_k\right|^{\frac{p_k}{M}}$$

$$\leq \sup_k \left|\left\{G\big(u, v, \lambda_j\big)x^n\right\}_k - \left\{G\big(u, v, \lambda_j\big)x^m\right\}_k\right|^{\frac{p_k}{M}} < \varepsilon$$

for all $m, n \geq n_0(\varepsilon)$.

This implies that

$$\left\{\left(G\big(u, v, \lambda_j\big)x^0\right)_k, \left(G\big(u, v, \lambda_j\big)x^1\right)_k, \dots\right\}$$

is a Cauchy sequence in $\mathbb{R}$ for each fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete the sequence $\left\{G\big(u, v, \lambda_j\big)x^n\right\}_k$ converges and let

$$\left\{G\big(u, v, \lambda_j\big)x^n\right\}_k \rightarrow \left\{G\big(u, v, \lambda_j\big)x\right\}_k$$

as $n \rightarrow \infty$.

For each fixed $k \in \mathbb{N}$, $m \rightarrow \infty$ and $n \geq n_0(\varepsilon)$, it is clear that

$$\left|\left\{G\big(u, v, \lambda_j\big)x^n\right\}_k - \left\{G\big(u, v, \lambda_j\big)x\right\}_k\right|^{\frac{p_k}{M}} \leq \frac{\in}{2} \tag{4.3.5}$$

Since $x^n = \left\{x_k^{(n)}\right\}$ is a Cauchy sequence in $c_0(u, v; p, \lambda_j)$ we have

$$\left|\left\{G\big(u, v, \lambda_j\big)x^n\right\}_k\right|^{\frac{p_k}{M}} \leq \frac{\in}{2} \tag{4.3.6}$$

for each fixed $k \in \mathbb{N}$.

Therefore by combining (4.3.5) and (4.3.6) we obtain that

$$\left|\left\{G(u,v,\lambda_j)x\right\}_k\right|^{\frac{p_k}{M}}$$

$$\leq \left|\left\{G(u,v,\lambda_j)x^n\right\}_k - \left\{G(u,v,\lambda_j)x\right\}_k\right|^{\frac{p_k}{M}} + \left|\left\{G(u,v,\lambda_j)x^n\right\}_k\right|^{\frac{p_k}{M}} \leq \varepsilon$$

for all $n \geq n_0(\varepsilon)$.

Hence, we have the sequence $\{G(u,v,\lambda_j)x\} \in c_0(u,v;p,\lambda_j)$. Since $\{x^n\}$ was taken as an arbitrary Cauchy sequence, the space $c_0(u,v;p,\lambda_j)$ is complete. This completes the proof.

Proposition 4.3.3: The sequence spaces $X(u,v;p,\lambda_j)$ for $X \in (l_\infty, c, c_0)$ are linearly isomorphic to the spaces $X(p)$.

Proof: For each $x \in X(u,v;p,\lambda_j)$ we have $G(u,v,\lambda_j)x \in X(p)$ where λ_j as defined in (4.2.6). It is easy to verify that λ_j is linear and bijective. Also the matrix λ_j has an inverse given by,

$$\eta = (\eta_{nk}) = \begin{cases} \sum_{j=k}^{n-1}\left(\frac{t_{j+1}}{v_{j+1}} - \frac{t_j}{v_j}\right)\frac{1}{u_j}, & 1 \leq k \leq n-1 \\ -\sum_{j=k}^{n}\frac{t_j}{u_j v_j}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

for all $n,k \in \mathbb{N}$. Thus, the sequence spaces $X(u,v;p,\lambda_j)$ for $X \in (l_\infty, c, c_0)$ are linearly isomorphic to the spaces $X(p)$.

Proposition 4.3.4: Let $\mu_k = \left(G(u,v,\lambda_j)x\right)_k$ for all $k \in \mathbb{N}$. Now for fixed $n \in \mathbb{N}$ we define the sequence $\alpha^{(k)} = \left\{\alpha_n^{(k)}\right\}_{n\in\mathbb{N}}$ by

$$\alpha_n^{(k)} = \begin{cases} \displaystyle\sum_{j=k}^{n-1}\left(\frac{t_{j+1}}{v_{j+1}} - \frac{t_j}{v_j}\right)\frac{1}{u_j}, & 1 \le k \le n-1 \\ \displaystyle -\sum_{j=k}^{n}\frac{t_j}{u_j v_j}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$. Then,

(i) The sequence $\{\alpha^{(k)}\}_{k\in\mathbb{N}}$ is the basis for the sequence space $c_0(u, v; p, \lambda_j)$ and any $x \in c_0(u, v; p, \lambda_j)$ can uniquely be represented as

$$x = \sum_k \mu_k \alpha^{(k)}$$

(ii) The set $\{z, \alpha^{(k)}\}_{k\in\mathbb{N}}$ is the basis for the sequence space $c(u, v; p, \lambda_j)$ and any $x \in c(u, v; p, \lambda_j)$ can uniquely be represented as

$$x = \ell z + \sum_k (\mu_k - \ell)\alpha^{(k)}$$

where

$$\ell = \lim_{k\to\infty}\left(G(u, v, \lambda_j)x\right)_k$$

$$z = (z_k)$$

and

$$z_k = \frac{1}{v_k}\sum_{j=1}^{k}\left(\frac{t_{j-1}}{u_{j-1}} - \frac{t_j}{u_j}\right)$$

The proof of the proposition is straight forward.

4.4. Duals

In this section we determine β- dual of the spaces $X(u,v;p,\lambda_j)$ for $X \in \{l_\infty, c, c_0\}$. We recall that if X be a sequence space , we define β- dual of X as,

$$X^\beta = \left\{a = (a_k): \sum_{k=1}^{\infty} a_k x_k \text{is convergent for each } x \in X\right\}$$

Theorem 4.4.1

Define the sets $d_1(p)$, $d_2(p)$, $d_3(p)$ and $d_4(p)$ as follows:

$$d_1(p) = \bigcap_{N>1} \left\{a = (a_k): \sup_n \sum_k \left| \sum_{j=k}^{n-1} \left[\sum_{i=1}^{j-1} \left(\frac{t_{i+1}}{v_{i+1}} - \frac{t_i}{v_i} \right) \frac{1}{u_i} - \frac{t_j}{u_j v_j} \right] a_j \right| N^{\frac{1}{p_k}} < \infty \right\}$$

$$d_2(p) = \bigcup_{N>1} \left\{a = (a_k): \sup_n \sum_k \left| \sum_{j=k}^{n-1} \left[\sum_{i=1}^{j-1} \left(\frac{t_{i+1}}{v_{i+1}} - \frac{t_i}{v_i} \right) \frac{1}{u_i} - \frac{t_j}{u_j v_j} \right] a_j \right| N^{-\frac{1}{p_k}} < \infty \right\}$$

$$d_3(p) = \cup_{N>1} \left\{a = (a_k): \left(\sum_{j=k}^{n-1} \left[\sum_{i=1}^{j-1} \left(\frac{t_{i+1}}{v_{i+1}} - \frac{t_i}{v_i} \right) \frac{1}{u_i} - \frac{t_j}{u_j v_j} \right] a_j \, N^{-\frac{1}{p_k}} \right) \in l_\infty \right\} \text{ and}$$

$$d_4(p) = \left\{a = (a_k): \lim_{n\to\infty} \left(\sum_{j=k}^{n-1} \left[\sum_{i=1}^{j-1} \left(\frac{t_{i+1}}{v_{i+1}} - \frac{t_i}{v_i} \right) \frac{1}{u_i} - \frac{t_j}{u_j v_j} \right] a_j \right) \text{exists} \right\}$$

Then,

$$\{l_\infty(u,v;p,\lambda_j)\}^\beta = d_1(p) \cap cs$$

$$\{c_0(u,v;p,\lambda_j)\}^\beta = d_2(p) \cap d_3(p) \text{ and}$$

$$\{c(u,v;p,\lambda_j)\}^\beta = d_2(p) \cap d_3(p) \cap d_4(p)$$

Proof: We find the β-dual of the sequence space $l_\infty(u,v;p,\lambda_j)$ only. Before giving proof we state the following lemma which we will use latter.

Lemma 4.4.1 [61,70]: Let $p_k > 0$ for every $k \in \mathbb{N}$. Then $A \in (l_\infty(p), c(q))$ if and only if $\sup_n \sum_k |a_{nk}| N^{\frac{1}{p_k}} < \infty$ for all integers $N > 1$ and there exists $\alpha_k \in \omega$ such that

$$\lim_{n\to\infty}\left(\sum_k |a_{nk} - a_k| N^{\frac{1}{p_k}}\right)^{q_n} = 0$$

for all integers $N > 1$.

Now for the sequence $a = (a_n) \in \omega$, we define the infinite matrix,

$$D = (d_{nk}) = \begin{cases} \sum_{j=k}^{n-1}\left(\frac{t_{j+1}}{v_{j+1}} - \frac{t_j}{v_j}\right)\frac{a_j}{u_j}, & 1 \le k \le n-1 \\ -\sum_{j=k}^{n}\frac{t_j}{u_j v_j} a_j, & k = n \\ 0, & \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$.

For any $x = (x_k) \in l_\infty(u, v; p, \lambda_j)$; we have

$$\sum_{k=1}^{n} a_k x_k = \sum_{k=1}^{n}\left(\sum_{i=1}^{k}\frac{t_i}{v_i}\left(\frac{y_{i-1}}{u_{i-1}} - \frac{y_i}{u_i}\right)\right)a_k$$

$$= \sum_{j=k}^{n-1}\left[\sum_{i=1}^{j-1}\left(\frac{t_{i+1}}{v_{i+1}} - \frac{t_i}{v_i}\right)\frac{y_i}{u_i} - \frac{t_j}{u_j v_j} y_j\right]a_j = (Dy)_n \qquad (n \in \mathbb{N}) .$$

Thus we observe that the sequence $(a_n x_n) \in cs$ whenever $(x_n) \in l_\infty(u, v; p, \lambda_j)$ if and only if $Dy \in c$ and $y \in l_\infty(p)$. This implies that $a = (a_n) \in l_\infty(u, v; p, \lambda_j)^\beta$ if and only if $D \in (l_\infty(p), c)$. Hence from the lemma 4.4.1 we conclude that

$$\{l_\infty(u, v; p, \lambda_j)\}^\beta = d_1(p) \cap cs$$

4.5. Matrix transformation

In first part of this section, we give the characterization of the classes $\left(l_\infty(u, v; p, \lambda_j), l_\infty\right)$, $\left(l_\infty(u, v; p, \lambda_j), c\right)$ and $\left(l_\infty(u, v; p, \lambda_j), c_0\right)$. Define a matrix $C = (c_{nk})$ by

$$c_{nk} = \sum_{j=k}^{\infty} \left[\sum_{i=1}^{j-1} \left(\frac{t_{i+1}}{v_{i+1}} - \frac{t_i}{v_i} \right) \frac{1}{u_i} - \frac{t_j}{u_j v_j} \right] a_{nj}$$

(4.5.1)

Then we have the following characterization theorems.

Theorem 4.5.1

$A \in \left(l_\infty(u, v; p, \lambda_j), l_\infty \right)$ if and only if

$$\sup_n \left(\sum_k |c_{nk}| N^{\frac{1}{p_k}} \right) < \infty$$

for all integers $N > 1$.

Theorem 4.5.2

$A \in \left(l_\infty(u, v; p, \lambda_j), c \right)$if and only if

(i)

$$\sup_n \left(\sum_k |c_{nk}| N^{\frac{1}{p_k}} \right) < \infty$$

for all integers $N > 1$ and

(ii)

$$\lim_{n \to \infty} \left(\sum_k |c_{nk} - \alpha_k| N^{\frac{1}{p_k}} \right) = 0, \ \alpha = (\alpha_k) \in \omega \text{ and } N > 1.$$

Theorem 4.5.3: $A \in \left(l_\infty(u, v; p, \lambda_j), c_0 \right)$ if and only if

(i)

$$\sup_n \left(\sum_k |c_{nk}| N^{\frac{1}{p_k}} \right) < \infty$$

for all integers $N > 1$ and

(ii)

$$\lim_{n\to\infty}\left(\sum_k |c_{nk}-\alpha_k| N^{\frac{1}{p_k}}\right)=0\ ,\ \alpha=(\alpha_k)\in\omega \text{ and } N>1 \text{ and}$$

(iii)

$$\lim_{n\to\infty} c_{nk}=\alpha_k \text{ exists with } \alpha_k=0 \text{ for all } k\in\mathbb{N}.$$

In the second part, we give some remarks before characterization of new class. Various authors, including us, have studied matrix transformation from new sequence spaces ,for example, $X(u,v;p,\Delta)$ to X or $X(p)$. However, the cases of mapping from X or $X(p)$to the new sequence space $X(u,v;p,\Delta)$ have not been considered. In this connection we give the following characterization theorem.

Theorem 4.5.4

$A\in\left(l_\infty(p),l_\infty(u,v;p,\lambda_j)\right)$ if and only if

$$(e_{nk})_{n=1}^{\infty}=\left(\sum_{j=k}^{\infty}\left\{u_j\left(\sum_{i=1}^{k-1}\frac{v_{i+1}}{t_{i+1}}-\frac{v_i}{t_i}\right)-\frac{u_jv_j}{t_j}\right\}a_{nj}\right)_{n=1}^{\infty}\in l_\infty^{\beta}(u,v;p,\lambda_j).$$

Proof: First suppose that $A\in\left(l_\infty(p),l_\infty(u,v;p,\lambda_j)\right)$ but $(e_{nk})\notin l_\infty^{\beta}(u,v;p,\lambda_j)$ for every $n\in\mathbb{N}$. So there exists an $x\in l_\infty(u,v;p,\lambda_j)$ such that

$$\sum_k e_{nk}x_k\neq O(1)$$

for each $n\in\mathbb{N}$.

However if we define a sequence $y=(y_k)$ by

$$y_k=,u_k\left[\sum_{i=1}^{k-1}\left(\frac{v_{i+1}}{t_{i+1}}-\frac{v_i}{t_i}\right)x_i-\frac{v_kx_k}{t_k}\right], \tag{4.5.2}$$

then it is clear that $y\in l_\infty(p)$ and that

$$\sum_k a_{nk}y_k=\sum_k e_{nk}x_k\neq O(1).$$

This contradicts the fact that

$$A \in \left(l_\infty(p), l_\infty\left(u, v; p, \lambda_j\right)\right)$$

Hence, we must have

$$(e_{nk}) \in l_\infty^\beta(u, v; p, \lambda_j)$$

for each $n \in \mathbb{N}$.

Next, suppose that the given condition is satisfied. Then it follows immediately from the fact

$$\sum_k a_{nk} y_k = \sum_k e_{nk} x_k$$

that, $Ay \in l_\infty\left(u, v; p, \lambda_j\right)$ for arbitrary $y \in l_\infty(p)$.Thus $A \in \left(l_\infty(p), l_\infty\left(u, v; p, \lambda_j\right)\right)$. This completes the proof.

CHAPTER FIVE

ON EXPLORATION OF SEQUENCE SPACES AND FUNCTION SPACES ON INTERVAL [0,1] FOR DNA SEQUENCING

5.1. Preliminaries

John Maynard Smith in 1970 first introduced the notion of sequence space for protein evolution. He proposed a "sequence space" where all possible proteins are arranged in a protein space in which neighbors can be interconnected by single mutation [23]. These problems are not only unique to protein structures but relevant to many other areas such as DNA sequence, brain imaging, climate data, financial data and others. In these area of interest the data have common features that: data are enormous, information is multi dimensional and complex, the sample size is relevantly small, they posses finitely many non zero elements in the sequence and some elements in the sequence repeat many times. For instance, four types of nucleotide A, T, G and C are linked in different orders in extremely long DNA molecules. It now becomes a continuing challenge for scientists, engineers, mathematicians and others to record and preserve data in these endeavors.

When the data received from the reservoir to obtain some information have lower dimension and samples have larger size, the statistical methods such as that the covariance matrix [4, 68], dot matrix [57] and position weight matrix [83,86] can deal with the cases promptly in a simplified way. However, when data have multidimensional character and the sample size is smaller, the statistical methods may lead to errors [26].

In this connection authors [26] have pointed out the necessity of the new definition of norm to fit a given data 'a' in a of set some class samples S as follws:

Let us consider a simple example from a classification problem. Set S as a set of some class samples and a as a given data. Is a close to someone of S or a new class?A simpler approach is to consider problem $\inf_{s\in S}\|a - s\|_p$, where p denotes the norm in ℓ_p space. In most cases, there is at least one $s_0 \in S$ such that

$$\|a - s_0\|_p = \inf_{s\in S}\|a - s\|_p.$$

We denote by (a) the feasible set. Can we say that a is close to some $s_0 \in F(a)$? To see disadvantage, we divide sequence $s \in S$ into three segments (s_1, s_2, s_3); the first segment s_1 is composed of the first n_1 elements, the second segment s_2 is made of the next n_2 elements, and the third is composed of the others. Similarly, we also divide a into corresponding three parts (a_1, a_2, a_3). Now, we reconsider

$$\inf_{S_1}\|a_1 - s_1\|_p, \quad \inf_{S_2}\|a_2 - s_{21}\|_p, \quad \inf_{S_3}\|a_3 - s_3\|_p$$

Perhaps we would find that $F(a_1) \cap F(a_2) \cap F(a_3) = 0$. Can one say that a is a new class? From this example, we see that we need a new definition of the norm to fit application. Motivated by these questions, we revisit the sequence spaces and function spaces defined on [0, 1] .Here, the sequence spaces we work on are different from the existing spaces. In the present chapter, we shall introduce our idea and the resulted sequence space and function spaces on [0,1].

Based on the sequence spaces and function spaces on interval [0,1], in the present chapter we examine the behaviors of sequences generated by DNA nucleotides. It has been aimed to extend the results of authors [26] by: introducing new function space in [0,1], extending the basis function $\frac{x^n}{n!}$, introducing a new sequence $b = (b_n) = (\sum_{v=n}^{\infty} a_v)$ which can characterize DNA sequence , obtaining some new completion results among the existing spaces in [0,1] and formulating strongly p- summation method.

Definitions and Notations

The following definitions and notations will be useful in further discussion.

(i) **DNA**

Definition: DNA stands for Deoxyribonucleic acid which is the chemical stuff it is made of. Structurally DNA is polymer – a larger structure that is made up of repeating parts of smaller structure – like a brick wall is made up not just one brick but of many similar bricks all closely joined.

(ii) **DNA Nucleotides**

Definition: In the DNA polymer, the tiny repeating structure are called Nucleotides. In other words, nucleotides are organic molecules that serve as the monomers or subunits of DNA . The millions of tiny unit nucleotides together form the entire DNA polymer which is called a DNA strand having double helix structure. There are four types of nucleotides. They are:

A = Adenine , C = Cytosine, G = Guanine, T = Thymine

(iii) **Sequence alignment**

Definition: Sequence alignment is the procedure of comparing two (pair-wise alignment) or more multiple sequences by searching for a series of individual characters or patterns that are in the same order in the sequences. There are two types of alignment: local and global. In global alignment, an attempt is made to align the entire sequence. If two sequences have approximately the same length and are quite similar, they are suitable for the global alignment. Local alignment concentrates on finding stretches of sequences with high level of matches.

(iv) **DNA sequence**

Definition: A DNA sequence is a specific sequence of all little bases each base is either Adenine (A), Cytosine (C), Thymine (T) or Guanine (G).

(v) **DNA sequencing**

Definition: DNA sequencing is the process of determining the precise order of nucleotides within a DNA molecule. It includes any method or technology that is used to determine the order of the four bases – adenine, cytosine, thymine and guanine – in a strand of DNA.

5.2. Sequence Spaces and Function Spaces on [0,1] for DNA Sequencing

We discuss the existing function space on [0,1], basis function representation theorem and the set inclusion relation as in [26].

Let $a = (a_{1,}a_{2,}a_{3,} \dots , a_{n,} \dots)$ be a DNA sequence where $a_n \in \{A, C, T, G\}$ and

$$a(x) = Ap_1(x) + Cp_2(x) + Tp_3(x) + Gp_4(x)$$

(5.2.1)

Clearly, for different DNA sequence, we have different polynomials $p_j(x)$ .It is a simpler reserve form. To extend it into a sequence of infinitely many non zero terms, we take $x \in [0,1]$. Here, $a(x)$ is called the generation function in the classical queuing theory. We remark that the generation function is not continuous function defined in [0, 1] . Hence in order to find out a feasible form of $a(x)$ we integrate first and then differentiate.

Denoting by L the integral operation and performing it for constant 1 leads to,

$$L^1(1)(x) = \int_0^x 1\, dx = x$$

$$L^2(1)(x) = \int_0^x L^1(1)\, xdx = \frac{x^2}{2}$$

.

.

.

Generalizing we get,

$$L^n(1)(x) = \frac{x^n}{n!}$$

(5.2.2)

for all $n \in \mathbb{N}$.

For any polynomial $p_n(x)$ of order n, it can be written as

$$p_n(x) = a_0.1 + a_1 x + a_2 \frac{x^2}{2!} + \cdots + a_n \frac{x^n}{n!}$$

$$= \left[\sum_{k=0}^{n} a_k L^k\right](1)(x)$$

(5.2.3)

Next , we consider the differential operator D for the for the function $\frac{x^n}{n!}$ which yields

$D^1\left(\frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!}$, $D^2\left(\frac{x^n}{n!}\right) = \frac{x^{n-2}}{(n-2)!}$, …

In general for $1 \le k \le n$, it holds that

$$D^k\left(\frac{x^n}{n!}\right) = \frac{x^{n-k}}{(n-k)!}$$

(5.2.4)

Therefore the coefficient sequence is given by

$$(a_0, a_1, a_2, \dots, a_n) = (D^0, D^1, D^2, \dots, D^n)p_n|_{x=0}$$

(5.2.5)

and $\frac{x^n}{n!}$ is defined to be the basis function.

Moreover, the polynomial space over [0, 1] , denoted by P[0,1], is a normed space with the norm

$$\|p\|_{\phi} = \sup_{n\ge 0}\left\{\|D^n p\|_{\infty}\right\}$$

(5.2.6)

where

$$\|f\|_{\infty} = \max_{0\le x\le 1}|f(x)|$$

In this space, the integral and differential operations are bounded linear operators. To extend to an infinite sequence, we take a subset $C_M^{\infty}[0,1]$ of $C^{\infty}[0,1]$ defined by

$$C_M^{\infty} = \left\{f \in C^{\infty}[0,1]\text{: } \sup_{n\ge 0}\|D^n f\|_{\infty} < \infty\right\}$$

(5.2.7)

$C_M^{\infty}[0,1]$ is a Banach space. Now for the function space on interval [0,1], there exist the following set inclusion relations

$$P[0,1] \subset C_M^{\infty}[0,1] \subset C^{\infty}[0,1] \subset C^k[0,1] \subset C[0,1] \subset L^{\infty}[0,1] \subset L^p[0,1] \subset L^1[0,1]$$

(5.2.8)

But the completion of $(P[0,1], \|.\|_{\emptyset})$ is not the space $(C_M^{\infty}[0,1], \|.\|_{\emptyset})$. For the completion of the space $(P[0,1], \|.\|_{\emptyset})$ authors have defined the following spaces on [0,1] :

$$C_{\phi,0}[0,1]=\left\{ f(x)=\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}: \lim_{n\to\infty} a_n=0 \right\},$$

$$C_{\phi,p}[0,1]=\left\{ f(x)=\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}: \sum_{n=0}^{\infty} |a_n|^p<\infty \right\} \text{for } p \geq 1 \text{ and}$$

$$C_{\phi,\infty}[0,1]=\left\{ f(x)=\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}: \sup_{n\geq 0} |a_n|<\infty \right\}$$

These spaces are isomorphic to c_0 , l_p and l_∞ respectively [26].

Obviously $P[0,1] \subset C_{\emptyset,0}[0,1] \subset C_M^{\infty}[0,1]$ and authors have shown the following set inclusion relations:

$$P[0,1] \subset C_{\phi,1}[0,1] \subset C_{\phi,p}[0,1] \subset C_{\phi,0}[0,1] \subset C_{\phi,\infty}[0,1] = C_M^{\infty}\ [0,1],\ \ 1 \leq p < \infty$$

(5.2.9)

5.3. New Function Space and Sequence Space on [0,1] for DNA Sequencing and New Set Inclusion Relations

We define for any $x \in [0,1]$, a polynomial function of order n

$$p_n(x)=\sum_{\nu=1}^{n} a_\nu \left(\sum_{k=1}^{\nu} L^k(1)(x) \right),\ a_0=0.$$

$$= \sum_{\nu=1}^{n} a_\nu \left(\sum_{k=1}^{\nu} \frac{x^k}{k!} \right)$$

(5.3.1)

where L is integral operator and

$$\sum_{k=1}^{\nu} \frac{x^k}{k!} \text{ for } \nu = 1, 2, 3, \ldots, n$$

(5.3.2)

is new basis function defined in the polynomial function P[0,1] which illustrates better approximation to the problem. Further by using differential operator for the basis function

$$\sum_{k=1}^{\nu} \frac{x^k}{k!} \text{ for } \nu = 1, 2, 3, \ldots, n$$

we find that ,

$$D^k \left[\sum_{i=1}^{\nu} \frac{x^i}{i!} \right] = \frac{x^{\nu-k}}{(\nu-k)!}, \; 1 \le k \le \nu$$

Obviously ,

$$D^1 p_n(0) = a_1 + a_2 + \ldots + a_n$$

$$D^2 p_n(0) = a_2 + a_3 + \ldots + a_n$$

.

.

$$D^n p_n(0) = a_n$$

Therefore the coefficient sequence $b = (b_n)$ is given by

$$(a_1 + a_2 + \ldots + a_n, a_2 + a_3 + \ldots + a_n, \ldots, a_n) = (D^1, D^2, \ldots, D^n)\, p_n(x)\big|_{x=0}$$

(5.3.3)

Thus we obtained new coefficient sequence to characterize DNA sequence. With the coefficient sequence $b = (b_k)$ defined by $b_k = \sum_{\nu=k}^{n} a_\nu$, for all k ; we can characterize DNA sequence and the result is helpful to explore for the possible application in DNA sequencing. The following table shows the distribution of the coefficient sequence $b = (b_k)$ with all possible alignments of DNA nucleotides.

Table 1. Distribution of the coefficient sequence $b = (b_k)$

b_1 =	$a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 \dots a_n$
b_2 =	$a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 \dots a_n$
b_3 =	$a_3 a_4 a_5 a_6 a_7 a_8 a_9 \dots a_n$
b_4 =	$a_4 a_5 a_6 a_7 a_8 a_9 \dots a_n$
……….	……… …….. ……..
……….	……… …….. ……..
……….	……… …….. ……..
$b_{n=}$	a_n

where $a_n \in \{A, C, T, G\}$. In computational process, if we input a DNA sequence , BLAST (Basic Local Alignment Search Tool) will display all possible gene matches with closure similarities between the existing DNA sequence in Gene Bank and the input sequence. The most likely matches will be displayed from top to bottom sequence alignments.

The polynomial space P[0,1] is now a normed space normed by,

$$\|p\|_\psi = \sup_{n\geq 1} \left\| \sum_{k=1}^{n} (D^k p - D^{k-1} p) \right\|_\infty$$

To extend the case to an infinite dimension, consider a subset of function space $C^\infty[0,1]$ defined by

$$C_M^\infty[0,1] = \left\{ f \in C^\infty[0,1] : \sup_{n\geq 0} \left\| D^n f \right\|_\infty < \infty \right\}$$

which is a linear space.

The authors [26] have shown the sets inclusion relations as

$$P[0,1] \subset C_{\phi,1}[0,1] \subset C_{\phi,p}[0,1] \subset C_{\phi,0}[0,1] \subset C_{\phi,\infty}[0,1] = C_M^\infty\ [0,1],\ 1 \leq p < \infty\ .$$

Let the completion of the space $C_{\phi,0}[0,1]$ be $C_{\psi,0}[0,1]$. Then we have the following representation theorem.

Theorem 5.3.1

The space

$$C_{\psi,0}[0,1]=\left\{g(x)=\sum_{k=1}^{\infty}a_k(\sum_{\nu=1}^{k}\frac{x^\nu}{\nu!}):\lim_{n\to\infty}b_n=0\right\}$$

is isomorphic to the space $C_{\phi,0}[0,1]$, where

$$b=(b_n)=\left(\sum_{\nu=n}^{\infty}a_\nu\right).$$

Proof: We define an operator

$$T:C_{\psi,0}[0,1]\to C_{\phi,0}[0,1]$$

by

$$(b_n)\mapsto(a_n)=T((b_n)).$$

The linearity of T is obvious. Now,

$$\begin{aligned}
T((b_n)) &= g(x)\\
&= \sum_{n=1}^{\infty}b_n\frac{x^n}{n!}=b_1\frac{x}{1!}+b_2\frac{x^2}{2!}+b_3\frac{x^3}{3!}+...\\
&= (a_1+a_2+a_3+...)\frac{x}{1!}+(a_2+a_3+a_4+...)\frac{x^2}{2!}+...\\
&= a_1\frac{x}{1!}+a_2(\frac{x}{1!}+\frac{x^2}{2!})+a_3(\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!})+...\\
&= \sum_{n=1}^{\infty}a_n(\sum_{k=1}^{n}\frac{x^k}{k!})
\end{aligned}$$

Hence T is bijective. Thus T is isomorphism mapping and $C_{\psi,0}[0,1]$ is isomorphic to $C_{\phi,0}[0,1]$.

Now for $p \geq 1$ we define new norm on the space P[0,1] by

$$\|g\|_{\psi,p} = \left\{ \sum_{k=1}^{\infty} \left\| \sum_{\nu=1}^{k} (D^{\nu}p - D^{\nu-1}p) \right\|_{\infty}^{p} \right\}^{\frac{1}{p}}$$

Let $C_{\psi,p}[0,1]$ be the completion of the space $C_{\phi,p}[0,1]$. Then we have the following representation theorem.

Theorem 5.3.2

The space

$$C_{\psi,p}[0,1] = \left\{ g(x) = \sum_{k=1}^{\infty} a_k \left(\sum_{\nu=1}^{k} \frac{x^{\nu}}{\nu!} \right) : \sum_{n=0}^{\infty} |b_n|^p < \infty \right\}$$

is isomorphic to the space $C_{\phi,p}[0,1]$.

The proof of the theorem follows immediately by using isomorphism operator defined as in the proof of theorem 5.3.1.

Further, letting $p \to \infty$ we define new norm on the space P[0,1] by

$$\|g\|_{\psi,\infty} = \sup_{n \geq 1} \left\| \sum_{k=1}^{n} (D^{k}f - D^{k-1}f) \right\|_{\infty}$$

Then we have the following theorem.

Theorem 5.3.3

The space

$$C_{\psi,\infty}[0,1] = \left\{ g(x) = \sum_{k=1}^{\infty} a_k \left(\sum_{\nu=1}^{k} \frac{x^{\nu}}{\nu!} \right) : \sup_{n \geq 0} |b_n| < \infty \right\}$$

is isomorphic to the space $C_{\phi,\infty}[0,1]$.

The proof is similar to the proof of theorem 5.3.1 .

We, therefore, observe the following sets inclusion relations:

$P[0,1] \subset C_{\phi,p}[0,1] \subset C_{\psi,p}[0,1] \subset C_{\phi,0}[0,1] \subset C_{\psi,0}[0,1] \subset C_{\phi,\infty}[0,1] \subset C_{\psi,\infty}[0,1] = C_M^{\infty}\ [0,1]$, $1 \leq p < \infty$

(5.3.4)

Moreover the spaces $C_{\psi,0}[0,1]$, $C_{\psi,p}[0,1]$ and $C_{\psi,\infty}[0,1]$ are respectively equivalent to $C_{\phi,0}[0,1]$, $C_{\phi,p}[0,1]$ and $C_{\phi,\infty}[0,1]$. Hence $C_{\psi,0}[0,1]$, $C_{\psi,p}[0,1]$ and $C_{\psi,\infty}[0,1]$ are Banach spaces with their natural norms.

5.4. Strongly Summation Method

Let (b_n) be a sequence of real or complex numbers and satisfy $\lim_{n\to\infty} b_n = 0$. We define a new strongly p- summation method for the sequence (b_n) as

$$s_{0,p} = |b_n| = \left(|b_n|^p\right)^{\frac{1}{p}}$$

$$s_{1,p} = \left(|b_{n-1}|^p + |b_{n-2}|^p\right)^{\frac{1}{p}}$$

$$s_{2,p} = \left(|b_{n-2}|^p + |b_{n-3}|^p + |b_{n-4}|^p\right)^{\frac{1}{p}}$$

.

.

.

$$s_{k,p} = \left(\sum_{j=0}^{k} |b_{n-k-j}|^p\right)^{\frac{1}{p}}$$

We , therefore, obtained a new non negative sequence $\left(s_{k.p}; k \geq 0\right)$; where

$s_{k,p}^m \leq s_{k.p} \leq s_{k,p}^M$ and $s_{k,p}^m$ and $s_{k,p}^M$

are the values in decreasing and increasing queuing.

Then it is a normed space normed by

$$\|(b_n)\|_{H,p} = \sup_{k\geq 0} s_{k.p}$$

(5.4.1)

where H is the generalized strongly summation and p is the p-norm in finite dimensional space .

In particular when $\{b_n\} \in l^p$, $s_{k,p} \to 0$, as $k \to \infty$,hence

$$\|(b_n)\|_{H,p} < \left(\sum_{n\geq 1}^{\infty} |b_n|^p\right)^{\frac{1}{p}} < \infty .$$

Finally, we define the sequence spaces by

$$c_{H,p,M} = \left\{(b_n) : \sup_{k\geq 0} s_{k,p} < \infty\right\},$$

$$c_{H,p} = \left\{(b_n) : \lim_{k\to\infty} s_{k,p} = 0\right\}$$

These spaces are evidently Banach spaces with their norm as defined in (5.4.1).

CHAPTER SIX

CONCLUSIONS AND RECOMMENTATIONS

6.1. Conclusions

We have presented our results in chapter two to chapter five. The results in each chapter posses their own significance, specific characteristics and applications. In chapters two, three and four the role of infinite matrices has been considered as operators to construct new sequence spaces. In chapter five we have presented a practical application of sequence - function space on [0,1] to characterize DNA sequencing.

In chapter two, we have considered the role of infinite matrices $G(u,v)$, called generalized weighted mean and the difference operator matrix Δ to introduce the new sequence spaces. In the first part of chapter two, generalized weighted mean $G(u,v)$ has been introduced to construct the new sequence spaces $w(u,v,p)$, $w_0(u,v,p)$ and $w_\infty(u,v,p)$, which are the set of all sequences whose $G(u,v)$ transforms are in $w(p)$, $w_0(p)$ and $w_\infty(p)$ respectively. Any generalization of the sequence spaces of Maddox i.e. $w(p)$, $w_0(p)$ and $w_\infty(p)$ by the application of generalized weighted mean $G(u,v)$ have not been considered yet. In this regard our work leads to the extension of the work of Maddox [44,45]. In order to provide comprehensiveness to the work, we have established some properties and characterized the matrix classes $(w(u,v,p), l_\infty)$, $(w_0(u,v,p), c)$, and $(w_\infty(u,v,p), c_0)$.

In the second part of chapter two, the role of the matrix $G(u,v,\Delta)$ which is the combination of generalized weighted mean $G(u,v)$ and the difference operator matrix Δ has been applied to introduce the new sequence spaces $w(u,v;p,\Delta)$, $w_0(u,v;p,\Delta)$ and $w_\infty(u,v;p,\Delta)$, which are the set of all sequences whose $G(u,v,\Delta)$ transforms are in $w(p)$, $w_0(p)$ and $w_\infty(p)$ respectively. This work is the continuation of our work in the first part of chapter two. It focuses on the extension of the work of Maddox by the application of the matrix $G(u,v,\Delta)$. To complete the work in concrete form we have discussed essential properties along with characterization of the matrix classes $(w(u,v;p,\Delta), c)$, $(w_0(u,v;p,\Delta), c)$, $(w_\infty(u,v;p,\Delta), c_0)$ and $(w(u,v;p,\Delta), \Omega(t))$.

In chapter three, we have constructed a new matrix $S^n = \lambda$;

where

$$S = \begin{cases} 1, & 0 \leq k \leq n \\ 0, & k > n \end{cases}$$

and

$$\lambda = \begin{cases} n - k + 1, & n \geq k \\ 0, & \text{otherwise} \end{cases}$$

as in (3.1.4) to define the new sequence spaces $l(p,\lambda)$ in the first part and $l_\infty(p,\lambda)$, $c(p,\lambda)$ and $c_0(p,\lambda)$ in the second part. The sequence space $l(p,\lambda)$ is the set of all sequences whose λ- transform are in the sequence space $l(p)$. Similarly the sequence spaces $l_\infty(p,\lambda)$, $c(p,\lambda)$ and $c_0(p,\lambda)$ are the set of all sequences whose λ- transform are in the sequence space $l_\infty(p)$, $c(p)$ and $c_0(p)$ respectively. Our work is expected to lead a remarkable contribution in constructing new sequence spaces by generalizing the spaces $l(p)$, $l_\infty(p)$, $c(p)$ and $c_0(p)$ using a lower unitriangular matrix λ . Moreover along with the establishment of some properties , we have characterized the matrix classes $(l(p,\lambda),c)$, $(l(p,\lambda),c_0)$ and $(l(p,\lambda),l_\infty)$ in the first part and $(l_\infty(p,\lambda),l_\infty)$, $(l_\infty(p,\lambda),c)$ and $(l_\infty(p,\lambda),c_0)$ in the second part of chapter three.

In chapter four, we have constructed a new operator sparse band matrix λ_j which we combined with $G(u,v)$ to define the new sequence spaces $l_\infty(u,v;p,\lambda_j)$, $c(u,v;p,\lambda_j)$ and $c_0(u,v;p,\lambda_j)$. By the nature of construction, the sequence spaces $l_\infty(u,v;p,\lambda_j)$, $c(u,v;p,\lambda_j)$ and $c_0(u,v;p,\lambda_j)$ are the set of all sequences whose λ_j transforms are in the sequence spaces $l_\infty(p)$, $c(p)$ and $c_0(p)$ respectively. Furthermore we have characterized the matrix classes $(l_\infty(u,v;p,\lambda_j),l_\infty)$, $(l_\infty(u,v;p,\lambda_j),c)$ and $(l_\infty(u,v;p,\lambda_j),c_0)$. Besides, we have given the characterization theorem for the case of mapping from the sequence space $l_\infty(p)$ to the newly defined sequence space $l_\infty(u,v;p,\lambda_j)$ that guarantees the given rate of convergence.

We remark that the matrices $G(u,v)$, $G(u,v,\Delta)$, λ and λ_j that have been used as operators in different chapters to construct the new sequence spaces are all distinct and posses different characteristics. We expect that the rate of convergence improves by the application of these matrices in comparison to the earlier generalizations in the corresponding spaces.

In chapter five, we have presented a practical application of sequence spaces for DNA sequencing in the field of bioinformatics. Based on the function spaces and sequence spaces on interval [0,1], in chapter five we have examined the behaviors of sequence spaces generated by DNA nucleotides. We have extended the results of authors [7] by introducing a new coefficient sequence $b = (b_n) = (\sum_{v=n}^{\infty} a_v)$ where $a_n \in \{A, C, T, G\}$ on [0,1] and extending the basis function $\frac{x^n}{n!}(n \in \mathbb{N})$ in [26] into $\sum_{k=1}^{n} \frac{x^k}{k!}$ $(n \in \mathbb{N})$ as a new basis function.

We have also established some isomorphism theorems on newly introduced function spaces and obtained some new completion results between the existing spaces in [26].

6.2. Recommendations

Summability theory has very wide applications in functional analysis. It is not possible to discuss all the properties and aspects of newly introduced sequence spaces in the present thesis. Regarding the results found in this thesis, further generalizations can be done to fill the gap in existing literature. We list below some of the future works which one may carry out:

1. Finding α and γ duals of the spaces.

2. Finding further characterization classes of the spaces.

3. Studying further properties of the spaces.

4. Finding dual spaces for function spaces in chapter five.

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