MODELS OF FINSLER SPACES

THESIS

SUBMITTED FOR THE DEGREE

OF

Doctor of Philosophy

IN

MATHEMATICS

By

DHIRENDRA THAKUR

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS & STATISTICS

D.D.U. GORAKHPUR UNIVERSITY

GORAKHPUR- 273009 (U.P).

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Dedicated

To

My Reverend Late Grand-mother

Shrimati Sita Devi

D. D. U. Gorakhpur University, Gorakhpur,India

Candidate's Declaration

I hereby declare that the work which is being presented in the thesis entitled "*MODELS OF FINSLER SPACES*" in fulfillment of the requirement for the award of the degree of Doctor of Philosophy in Mathematics of D.D.U. Gorakhpur University, Gorakhpur is an authentic record of my own work carried out during a period from April 2009 to January 2012 under the supervision of Dr. T. N. Pandey, Professor in Mathematics, Department of Mathematics & Statistics, D.D.U. Gorakhpur University, Gorakhpur,India.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute/University.

(Dhirendra Thakur)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Dated: 2nd February, 2012

(Prof. T. N. Pandey)

Forwarded

Prof. R. S. Srivastava
Professor & Head
Department of Mathematics & Statistics
D. D. U. Gorakhpur University, Gorakhpur, India

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I am extremely grateful to my supervisor Professor **T.N. Pandey**, Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur, India who has been a constant source of inspiration and support. He is not only my supervisor but also my local guardian who always took care of my personal problems .He motivated me to do whatever I can achieve academic excellence in Mathematics. I can never forget the long hours in night at his place when I had to stay back to work on my thesis. It was precisely during these tireless periods that Mrs. Rajkumari Pandey and the members of her family stood me in good stead. It was a relief from serious work. But for her motherly care, it would have been difficult for me to stand that ordeal. This is an obligation, I shall, I know, will never be able to repay.

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Finally, my thanks are also due to all those who helped me directly or indirectly for the completion of the work.

(Dhirendra Thakur)

PREFACE

The present thesis is an outcome of my investigations in the Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur under the supervision of Prof. T.N.Pandey. The purpose of the present thesis is to study models of Finsler spaces .

The whole thesis is divided into seven chapters and each chapter is subdivided into various sections. Throughout this thesis ordered -3 tuples of positive integers (a.b.c) are used to locate equations ,lemmas, theorems etc .The first integers a locates the chapter , b locate the section , c tell you the number of the equation, proposition, lemma or theorem .The array a.b indicates the section . For instance, Theorem 2.6.3 is the third theorem of section 2.6, but there is also equation 2.6.3 , it is the third equation of section 2.6 .The numbers in the square bracket in a chapter correspond to the references given at the end of the chapter.

The symbols ∂ and $\dot{\partial}$ denote the partial derivative with respect to x^i and y^i respectively. Small and long vertical lines (I and |) stands for h and v- covariant derivative respectively.

First chapter is an introductory in nature and consists of preliminary details. Some useful results and definitions such as Finsler space, some connections like the Berwald, Carton and Runds have been mentioned therein.

The second chapter deals with the relation between Carton's connections of two Finsler spaces (M^n, L) and (M^n, $\overline{L}$) where $\overline{L}$ obtained from L by h-Randers change. It has been obtained the conditions under which this change is projective. It also deals the conditions under which Douglas space, Landsberge space or Weakly Berwald space becomes invariant.

The third chapter is devoted to study for Finsler spaces $\bar{F}^n$ obtained by Randers Conformal change of Finsler spaces F^n of Douglas type to be also of Douglas type and vice versa .It has been also worked out the condition under which the said transformation is projective.

In the fourth chapter we discuss the Finsler space $\bar{F}^n = (M^n, \bar{L}(\alpha, \beta))$ obtained by Conformal Randers change of Finsler space $F^n = (M^n, L(\alpha, \beta))$ of Douglas type remains to be Douglas type and vice versa .

The fifth chapter is devoted to investigate the Berwald connection, condition for projectively flatness of Finsler space with 2nd approximated exponential (α, β) metric $L = \alpha e^{\frac{\beta}{\alpha}} + \beta$ and the conditions under which said space is Douglas type.

In the sixth chapter, we investigate condition that the Finsler space with (α,β) – metric like Randers metric, Kropina metric and Matsumoto metric become Weakly Landsberg space. We also give an example for Weakly Landsberg space which is not Landsberg space.

The seventh chapter is the last chapter of my thesis and is devoted to study the S_4- likeness of Quartic Rander's change of a Finsler space and the relation between V-curvature tensor of Quartic Rander's changed Finsler space.

<u>RESEARCH ACTIVITIES</u>

(A) *Published / Communicated Research Papers*

1. *Conformal change of a Finsler Space with* (α,β) – *Metric of Douglas Type,* The Nepali Math .Sc. Report Vol.30, No.1 & 2 ,2010, 123 – 130.
2. *H- Randers Change of Finsler space,* Inv*estigations in* mathematical sciences, Vol.1, September 2011, 67 -78.
3. *Randers Conformal Change of a Finsler space of Douglas Type*, Journal of Rajasthan Academy of Physical Sciences Rajasthan ISSSN 0972 – 6306,

   Vol. 11, No.2 (2012) .

4. *Conformal Randers change of a Finsler Space with* (α,β) *Metric of Douglas Type,* Bulletin of Pure and Applied Mathematics (Jodhpur).
5. *Quartic Randers Change of Finsler Metric*, South East Asian Journal of Mathematics and Mathematical Science.
6. *Exponential with* (α,β)-*Metric*, Far East journal of Mathematical Science (FJMS).
7. *On Weakly – Landsberg Space of Some* (α,β) – Metric, Bulletin of the Iranian Mathematical Society.

B- Conferences attended / paper presented

1. National Conference of Mathematics , January 20 -22 ,2012
   Organized by Nepal Mathematical Society, Nepal
2. University Grants Commission Academic Staff College University of Lucknow ,November 22 – 26 ,2011
3. 4th Annual Conference of the Tensor Society on Application of Tensor and Differential Geometry in Engineering and physical Sciences, October, 8-9, 2011, SRMGPC, LUCKNOW
4. Conference on Recent Trends in Pure & Applied Mathematics, 24- 25 June, 2011, Department of Mathematics and Statistics, Deen Dayal Upadhyay Gorakhpur University, Gorakhpur
5. National Conference on Mathematics and its Applications (NCMA 2011), 13-14 January, 2011, Department of Mathematics, Jadavpur University.
6. 12th International Conference of International Academy of Physical Sciences (CONIAPS XII) on Emerging Interfaces of Physical Sciences, December 22-24, 2010 ,(University of Rajasthan , Jaipur – 302004.
7. Conference on Recent Trends in Pure & Applied Mathematics, 24- 25 July, 2010, Department of Mathematics and Statistics, Deen Dayal Upadhyay Gorakhpur University, Gorakhpur
8. Conference on Recent Trends in Pure & Applied Mathematics, 11- 12 July,

2009, Department of Mathematics and Statistics, Deen Dayal Upadhyay Gorakhpur University, Gorakhpur.

(C) **Workshop attend**

1. Instructional Workshop on Differential Geometry (January 15- 18, 2011), Department of Pure Mathematics, University of Calcutta.
2. University Grants Commission Academic Staff College University of Lucknow, Workshop on Latex and Scilab November 22 – 26, 2011.

(D) **Libraries attended**

1. Indian institute of Technology, Kanpur 208016,P. K. Kelkar library
2. Banaras Hindu University Library Varanasi -221005
3. 4shard .com
4. Gigapedia.com
5. Mendeley.com

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TABLE OF SYMBOLS IN FINSLER GEOMETRY

For easy reference, the below table lists symbols in Finsler geometry,

their definitions and homogeneity.

Name	Notation	Homogeneity in directional argument
coordinates of point	$x = (x^i)$	
Tangent space of M at p	T_pM	
Tangent vector at point P	y	
The Einstein summation convention	$y = y^i \frac{\partial}{\partial x^i}$	
Tangent bundle	$TM = \bigcup_{x\in M} T_xM$	
Local coordinates of a point in TM	(x, y)	
Co –tangent bundle	T_x^*M	

A basis of T_pM	$\left(\frac{\partial}{\partial x^i}\right)$	
A Finsler space of dimension n	$F^n = (M^n , L)$	
Normalized supporting element	$l^i = \frac{y^i}{L(x,\ y)}$	0
		0
Fundamental metric tensor	& $l_i = \frac{\partial L}{\partial y^i}$	0
Inverse of g_{ij}	$g_{ij} = \frac{1}{2}\dot{\partial}_i \dot{\partial}_j L^2(x,\ y),$	0
Angular metric tensor	g^{ij}	0
	$h_{ij} = g_{ij} - l_i l_j = L \frac{\partial^2 L}{\partial y^i \partial y^j}$	
Christoffel symbol		0
	$\gamma^i_{jk} = \frac{1}{2} g^{ih}[\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial}{\partial}$	
Cartan tensor		-1
	$C_{ijk}\ \frac{1}{2}\frac{\partial g_{ij}}{\partial y^i \partial y^j}$	-1
Geodesic coefficient	$C^i_{jk} = g^{is} C_{sjk}$	2
Geodesic spray	$2G^i = \gamma^i_{jk} y^j y^k$	
	$G = y^i \frac{\partial}{\partial x^i} - G^i \frac{\partial}{\dot{\partial} y^i}$	1
Non-linear connection	$N^i_j = \frac{\partial G^i}{\partial y^j}$	
Horizontal basis vector	$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$	

Set of vector field on M Horizontal connection component vertical connection component	ℵ (M) F^i_{jk}, C^i_{jk}	0 **-1**

Chapter 1

Introduction

1.1 A brief historical of development of Finsler Geometry

Finsler geometry is a kind of differential geometry, which was originated by P. Finsler [13] in 1918. Main focus of Finsler in his dissertation was to geometrize calculus of variation the idea given by his teacher Caratheodory. The creator of this geometry is really L. Berwald in 1925. The germs of Finsler geometry were present in the epoch-making lecture of B. Riemann which he delivered in June 1854, at Gottingen University. In the lecture Riemann had discussed various possibilities by means of which an n-dimensional space may be equipped with the metric before coming to the square root metric. He has thought over cubic and quartic also, but he give up it ,due to the difficulty of geometrical meaning to various differential invariants, furthermore the computation is very complicated. Consequently he concludes that the theory of such generalized metrics (Cubic and Quartic) would hardly contribute to the progress of geometry. It is usually considered as a generalization of the Riemannian geometry in which the space consists of tangent bundles instead of collection of points. Finsler spaces differ from Riemannian spaces by the fact that metric depends on direction also. Finally

Riemannians main attention was on a metric, where the distance *ds* between two neighbouring points represented by the co-ordinates x^i and $(x^i + dx^i)$ defined by,

$$ds = \sqrt{g_{ij}(x)dx^i dx^j} \quad (I,\ j = 1, 2, 3,\ldots\ldots,n)$$

where the coefficients g_{ij} are functions of coordinates x^i and

$\det(g_{ij}) \neq 0$ .This quadratic differential form is called a Riemannian metric and space with such metric called Riemannian space.

There are two approaches of Finsler Space out of which one is considered as Riemannian metric generalization. Finsler Space is a Space where metric function has been taken as

$$ds = L(x^1, x^2, x^3, \ldots\ldots x^n, dx^1, dx^2, dx^3, \ldots\ldots\ldots, dx^n) = L(x,y) \ (y = dx) \ .$$

We are concerned with the generalized metric $ds = L(x, y)$ which gives the distance between two points(x^i) and $(x^i + dx^i)$. Riemann has also discuss that the positive fourth root of a fourth order differential form ($ds^4 = g_{ijmn}\, dx^i\, dx^j\, dx^m\, dx^n$) might serve as a metric .These functions have three properties in common

(i) they are positive definite;

(ii)they are homogeneous of first degree in differential;

(iii) they are convex in differentials.

It would seem natural Therefore, to introduce a further generalization to the affect that the distance between two neighbouring points (x^i) and $(x^i + dx^i)$ be defined by some function $L(x^i, dx^i)$,where

$$ds = L(x^i, dx^i)$$

and it satisfies above three properties. Riemann asserted that the differential geometry based on such generalized metric will be able to develop in a way similar to the case of Riemannian metric. It will be difficult to give suitable geometrical meanings to differential invariants and further the computation for it will be very complicated. Consequently he concluded that the theory of such generalized metric will hardily contribute to the progress of geometry. He put it in the following words.

"Investigation of this more general class would actually require no essential different principles but it would be rather time consuming and throw relatively no light on the study of space, especially since results cannot be expressed geometrically."

Due to Riemann's comments, mathematicians did not try to study such spaces for more than 60 years. In 1918, 24 years old German, Paul Finsler [9] tried to study such spaces and submitted his thesis to Gottingen University. His approach of study this geometry was based on calculus of variation. He put the idea of calculus of variation with special reference to new geometrical background, which was given by his teacher Caratheodory .Finsler geometry is a kind of differential geometry which is usually considered as a generalization of Riemannian geometry. The history of development of Finsler geometry can be divided into four periods.

1. 1924 – 1933
2. 1934 – 1950
3. 1951 – 1963
4. 1963 – up to now

1. 1924 – 1933

The first period of the history of Finsler geometry began in 1924, when the three geometrician, J. H. Taylor [32], J. L. Synge [31] and L. Berwald [4], [5] simultaneously started the work in this field. Berwald is the first man who has introduced the concept of connection in the theory of Finsler spaces. He was the creator of Finsler geometry and, what was more, the founder. He had developed a theory with particular reference to the theory of curvature in which the Ricci lemma does not hold. J. H. Taylor and J. L. Synge introduced a special parallelism. In 1928 Taylor gave the name ‘Finsler Space’ to the manifold equipped with this generalized metric.

2. 1934 – 1950

The second period began in 1934, when E. Cartan [6] published his thesis on Finsler geometry. He showed that it was indeed possible to define connection coefficients and hence covariant derivatives such that the Ricci lemma is satisfied. On this basis Cartan developed the theory of curvature and torsion. All subsequent investigations considering the geometry of Finsler spaces were dominated by this approach. Several mathematicians such as E. T. Devise [7], S. Golab [10], H. Hombu [12], O. Varga [33], V. V. Wagner [34] have studied Finsler geometry along Cartan’s approach. They expressed the opinion that the theory attained his final form. This theory made certain devise, which basically involves the consideration of a space, whose elements are not the points of the underlying manifold, but the line element of latter, which forms a (2n-1) dimensional variety. This facilitates the Cartan called “Euclidean connection” which by means of certain postulates may be derived uniquely from the fundamental metric function.

3. 1951 – 1963

The third period of the history of Finsler geometry began in 1951, when H. Rund [27] introduced a new process of parallelism from the stand point of Minkowskian geometry. Cartan introduced parallelism from the stand point of locally Euclidean geometry. Latter on E. T. Devies and A. Deicke have indicated that Rund's and Cartan's parallelism were the same. Several Mathematicians such as W. Barthel [3], A. Deicke [8], D. Laugwitz [16], R. Sulanke [30] have studied Finsler spaces on Rund's approach.

4. 1963 – up to now

The fourth period of history of development of Finsler geometry began in 1963, when H. Akabar Zadeh [2] developed the modern theory of Finsler spaces based on the geometry of connections of fibre bundles. The reason of modernization is to establish a global definition of connections in Finsler spaces and to reexamine the Cartan's system of axioms; Mathematicians and Physicists began to study special Finsler spaces from the symposium organized by M. Matsumoto on the development of Finsler geometry since 1970. The aim of this symposium was to find real models of Finsler spaces. The contribution of Prof. M. Matsumoto on the development of Finsler geometry is worth records. He correlate Cartan's connections, Berwald connection and Rund's connections by the process called C-process and P-process. His various research papers (1992-1996) on the theory of Finsler space with (α, β)-metric has great contribution in the development of special Finsler spaces.

The study of Finsler spaces in India was started around 1960 under the leadership of Prof. R. S. Mishra, Prof. R. N. Sen and Prof. K. S. Amur. Some important mathematicians in this fields are as follows: - Prof. U. P. Singh, Prof. H. D. Pandey, Prof. R. B. Mishra, Prof. M. D. Upadhaya, Prof. R. S. Sinha, Prof. B. B. Sinha, Prof. Ram Hit, Dr. B. N. Prasad, Prof. T. N. Pandey, Prof. H. S. Shukla, Prof. P. N. Pandey, Prof. S. C. Rastogi, Prof. C. S. Bagewadi, S.K.Narasimhamurthi and some Finslerian are Prof. Z. Shen, H. S Park,. I. Y Lee, Alkou Tadashi, P.L Atonally , Xiaohuan Mo,R.Miran ,H. Akbar-Zadeh etc.

Now, we I will discuss some preliminary concepts of Finsler geometry which have been used in the present thesis.

1.2. Homogeneous function, Curve , Line Element & Tangent bundle

Homogeneous function is a function with multiplicative scaling behavior if the argument is multiplied by a factor, then the result is multiplied by some power of this factor. More precisely, if $f : TM \to W$ is a function between two vector spaces over a field F, and k is an integer, then f is said to be homogeneous of degree k if $f(x,cy) = c^k f(x, y)$ for all nonzero $c \in F$ and $y \in V$.

Let R be a region of n-dimensional differentiable manifold M^n which is covered completely by a co-ordinate system, such that any point P of R is represented by a set of n-real independent variables x^i $(i = 1, 2, 3,\ldots\ldots n)$, called the co-ordinates of the point. A transformation of co-ordinates is represented by a set of n-equations,

(1.2.1) $$x^{i'} = x^{i'}(x^1, x^2, \ldots\ldots, x^n) \qquad (i' = 1,2,\ldots\ldots n)$$

which shows that the co-ordinates x^i of a point P of M^n are represented in the new co-ordinate system by new variables$x^{i'}$. We assume that the functions $x^{i'}$ of (1.2.1) are at least of class C^2 and,

(1.2.2) $$\det\left(\frac{\partial x^{i'}}{\partial x^i}\right) \neq 0$$

A set of points of R, whose co-ordinates may be expressed as functions of a single parameter't' is regarded as a curve of M^n. Thus the equations,

(1.2.3) $$x^i = x^i(t)$$

defines a curve C of M^n. If the functions (1.2.3) are class C^1, we shall regard the entity whose components are given by,

(1.2.4) $$y^i = \frac{dx^i}{dt}$$

as, the tangent vector to *C*. We called the combination (x^i, y^i) a line element of *C*.

Tangent bundle: The tangent bundle [3] of a differentiable manifold M^n is the union of the tangent spaces of M^n, that is $TM = \bigcup_{x\in M} T_xM = \bigcup_{x\in M} \{x\} \times T_xM$ where T_xM denotes the tangent space to M^n at the point *x*. So, an element of *TM* can be thought of as a pair *(x, y)*, where *x* is a point in M^n and *y* is a tangent vector to M^n at *x*.The set of coordinates $\left(,\frac{\partial}{\partial x_k}\right)$ define a basis of the tangent space.

The infinitesimal distance between two points $P(x^i)$ and $Q(x^i, dx^i)$ of curve (1.2.3) lies on Manifold M^n is define by ds = $L(x^i, dx^i) = \sqrt{g_{ij}(x,y)dx^i dx^j}$. The arc PQ become tangent at x on Manifold M^n .

1.3. *Finsler space*

Let M^n be n- dimensional Manifold ,TM tangent bundle of M^n , { $\frac{\partial}{\partial x^i}$ } is basis of tangent spaces at (x) and $y = (y^i) = \frac{dx^i}{dt}$. A function $L: TM \rightarrow [0, \infty)$ of the line elements $(x^i,\ y^i)$ defined on M^n is called fundamental function if it satisfies

(a) The function $L\ (x^i,\ y^i)$ is positively homogeneous of degree one in y^i i. e.,

(1.3.1) $$L(x^i,\ ky^i) = kL(x^i,\ y^i), \qquad k > 0$$

That is, the arc length of curve is independent of the choice of parameter t.

(b) The function $L(x^i,\ y^i)$ is positive if not all y^i vanish simultaneously, i.e.,

(1.3.2) $$L\ (x^i,\ y^i) > 0 \quad \text{with} \quad \Sigma_i (y^i)^2 \neq 0$$

That is, the distance between two distinct points is positive.

(c) The quadratic form,

(1.3.3) $$\dot{\partial}_i \dot{\partial}_j L^2 (x^i,\ y^i) \xi^i \xi^j = \frac{\partial^2 L^2 (x^i, y^i)}{\partial y^i \partial y^j} \xi^i \xi^j$$

is assumed to be positive definite for any variable ξ^i.

That is, $L\ (x^i,\ y^i)$ is a convex function in y^i.

The manifold M^n equipped with the fundamental function L is called a Finsler space [3] .It is denoted by F^n or (M^n, L).

Some example of Finsler spaces are Normed vector spaces, Euclidean spaces, Riemannian spaces, Randers spaces,...

From Euler's theorem on homogeneous functions, we have

(1.3.4) $\dot{\partial}_i L(x,\ y) y^i = L(x,\ y)$

and

(1.3.5) $\dot{\partial}_i \dot{\partial}_j L(x,\ y) y^i = 0$

We put,

(1.3.6) $g_{ij}(x,\ y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2(x,\ y)$

Using the theory of quadratic form and the Condition-c, we deduce from (1.3.4) that

(1.3.7) $g(x,\ y) = |g_{ij}(x,\ y)| > 0$

for all line elements $(x^i,\ y^i)$.

If the function L is of particular form

(1.3.8) $L\,(x^i,\ dx^i) = \sqrt{g_{ij}(x^k) dx^i dx^j}$

where, the coefficients $g_{ij}(x^k)$ are independent of dx^i, the metric defined by this function is called Riemannian metric and the manifold M^n is called a Riemannian space. Throughout the present thesis, the n-dimensional Finsler space will be denoted by F^n or $(M^n,\ L)$, where as n-dimensional Riemannian space will be denoted by R^n.

1.4. Physical motivation

In a perfectly homogeneous and isotropic medium, geometry is Euclidean, and shortest paths are straight lines. In an inhomogeneous space, geometry is Riemannian and the shortest paths are geodesics. If a medium is not only inhomogeneous, but also anisotropic 1, i.e. has innate directional structure, the appropriate geometry is Finslerian [13] [14] and the shortest paths are correspondingly Finsler-geodesics. As a consequence the fundamental metric tensor depends on both position and direction. This is also a natural model for high angular resolution diffusion images.

Finsler geometry has its genesis in integral of the form $\int_a^b L($ x, y) dt, where

$x = (x^i), y = (y^i) = \frac{dx^i}{dt}$ .Let us find out some contexts in which this integral arises.

(a). Suppose x stands for position, y for velocity. Then $L(x, y)$ would have the meaning of speed and t would play the role of time, in this case the integral $\int_a^b L($ x, y) dt measures distance traveled.

(b). In an anisotropic medium (rays and wave fronts are not orthogonal to each other) the speed of light depends on its direction of travail. At each location x, visualize y as an arrow that emanates from x .We denote the time that light takes to trivial from x to the top of y call the result $L(x , y)$. The integral $\int_a^b L($ x, dx) dt represents total time that light takes to traverse is given path in this medium.

(c). It is well-known that the time taken by man in climbing up and going down on same length of the slope of mountain are distincts. It means time measures function $L(x (t), y (t))$ also depends on direction .This fundamental function L together with slope of mountain TM (Tangent bundle) is Finsler space.

(d). Cost of transportation function not only depend on distance but also on direction, except some other physical perturbation such as friction, air resistance e.t.c . This function can be regarded as fundamental function of Finsler space.

(e) . (Mathematical ecology) Suppose x stands for the state of coral reef , and y displacement vector from the state x to new state x+dx , L (x ,dx) represents the energy one needs in order to develop from the state x to the neighboring state $x+dx$. Hence the integral $\int_a^b L(x, dx)$ dt represents the total energy cost of a given path of evolution.

So from above we see that the world is Finslerian and it has wide application in theory of relativity, control theory, thermodynamics, optics, ecology, and mathematical biology .

1.5. Tangent space, Indicatrix and Cotangent spaces

We consider a change of local co-ordinates as represented by the equation (1.2.1), along the curve (1.2.3) referred to an invariant parameter t; the new components of the tangent vector $y^{i'} = \frac{dx^{i'}}{dt}$ are obtained by differentiating the relation,

(1.5.1) $$x^{i'} = x^{i'}(x^i(t))$$

With respect to t, which gives,

(1.5.2) $$y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i$$

Or in terms of differentials,

(1.5.3) $$dx^{i'} = \frac{\partial x^{i'}}{\partial x^i} dx^i$$

Here dx^i is interpreted as the components of a displacement in M^n from a point P (x^i) to a point Q($x^i + dx^i$).

If the point P(x^i) is fixed, i. e. the coefficients $\frac{\partial x^{i'}}{\partial x^i}$ of the transformation (1.4.2)' are fixed, this relation represents a linear transformation of the dx^i onto the $dx^{i'}$. The same is true for the variables y^i and $y^{i'}$ in the transformation (1.4.2). Therefore,, the n-entities of this kind may be taken to define the elements of an n-dimensional linear vector space.

A system of n-quantities X^i whose transformation law under (1.2.1) is equivalent to that of the y^i is called a contravariant vector attached to the point P(x^i) of M^n. Such contravariant vectors constitute the element of vector space. The totality of all contravariant vectors attached to P(x^i) of M^n is the *tangent space* denoted by T_n(P) or $T_n(x^i)$.

Indicatrix

We consider the function $L(x^i, y^i)$ defined for all line elements (x^i, y^i) over the region R of M^n. The equation,

$$L(x^i, y^i) = 1, \qquad (x^i \text{ fixed, } y^i \text{ variable})$$

Represent an (n - 1)-dimensional locus in T_n (P) i. e., a hypersurface. This hypersurface plays the role of unit sphere in geometry of the vector space $T_n(P)$ and is called Indicatrix [28].

Cotangent space

Let M^n be a smooth manifold and let x be a point in M^n. Let T_xM be the tangent space at x. Then cotangent space at x is defined as the dual space of T_xM denoted by T_x^*M or $(T_xM)^*$. Concretely, elements of the cotangent space are linear functional on T_xM. That is, every element $f \in T_x^*M$ is a linear map f: $T_xM \to R^+$,where R^+ set of positive real numbers. The elements of T_x^*M are called cotangent vectors

1.6. Pull- back tangent bundle, Non-linear connection, Decomposition of *T(TM – 0) and T^*(TM – 0).*

***Pull- back tangent bundle* (π*T M)**: Let M^n be an n-dimensional manifold. Suppose $T_x M$ the tangent space at $x \in M$, and $TM = \bigcup_{x\in M} T_xM = \bigcup_{x\in M} \{x\} \times T_xM$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in TxM$. Let $TM_0 = TM \setminus \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$.

The pull-back tangent bundle π^*TM is a vector bundle over TM_0 whose fiber

$\pi_v{}^*TM$ at $v \in TM_0$ is T_xM , where $\pi(v) = x$. Then

$$\pi^* TM = \{(x, y, v) \mid y \in T_x M_0 , v \in T_x M \}.$$

The natural basis for $\pi_v{}^*TM$ is $\{ \partial_i|_v = (v, \frac{\partial}{\partial x^i})|_x\}$ for all $i = 1,2,\ldots\ldots n$.

Non-linear connection

A non- linear connection on a manifold M^n is a collection of locally defined 1-homogeneous function N_j^i on *(TM -0)* satisfying transformation rules

(1.5.4) $$\frac{\partial \bar{x}^j}{\partial x^i} \bar{N}^i{}_j = \frac{\partial \bar{x}^h}{\partial x^j} N^j{}_i - \frac{1}{2} \frac{\partial^2 \bar{x}^h}{\partial x^i \partial x^j} y^j$$ and

(1.5.5) $$N_j^i = \frac{\partial G^i}{\partial y^j}.$$

Decomposition of *T(TM – 0)*

The vector spaces span { $\frac{\partial}{\partial x^i}|_y$: i = 1,2,……n} depend on local coordinates. Therefore, we can not say about " $\frac{\partial}{\partial x^i}$ '' direction in *T(TM – 0)*. However, when M^n is equipped with a non – linear connection N_j^i , let

(1.5.6) $$\frac{\delta}{\delta x^i}|_y = \frac{\partial}{\partial x^i}|_y - N_i^k(x,y)\frac{\partial}{\partial y^j}|_y \in T(TM - 0),$$

whence $\frac{\delta}{\delta x^i}|_y = \frac{\partial \bar{x}^r}{\partial x^i}\frac{\delta}{\delta \bar{x}^i}|_y$. Thus 2n – dimentional vector spaces $T_p(TM - 0)$ has 2n- dimensional subspaces, V_pTM = span { $\frac{\partial}{\partial y^j}|_y$} and

H_pTM = span { $\frac{\delta}{\delta x^i}|_y$} and these are independ of local coordinates. Let us define $VTM = \bigcup_{p\in TM-0} V_pTM$ and $HTM = \bigcup_{p\in TM-0} H_pTM$ hence $T(TM - 0) = VTM \oplus HTM$.

The vectors in *VTM* are called vertical vectors and victors in *HTM* are called horizontal vectors. The tangent of a geodesics is always a horizontal vectors geodesic spray G(x, y) is horizontal for all $(x, y) \in (TM - 0)$.

Decomposition of $T^*(TM - 0)$

On TM the 1- forms dx^i and dy^i satisfying law of transformation

(1.5.7) $$dx^i|_y = \frac{\partial x^i}{\partial \bar{x}^r} d\bar{x}^r|_y$$

(1.5.6) $$dy^i|_y = \frac{\partial x^i}{\partial \bar{x}^r} d\bar{y}^r|_y + \frac{\partial^2 xi}{\partial \bar{x}^r \partial \bar{x}^s} \bar{y}^r d\bar{x}^s|_y .$$

Let $\delta y^i|_y = dy^i|_y + N_i^k(x,y)dx^j|_y$, in whence $\delta y^i|_y = \frac{\partial x^i}{\partial \bar{x}^r} \delta \bar{y}^r|_y$. The 2n – dimensional vector spaces, $T^*(TM - 0)$ has two n- dimensional subspaces, $V_p{}^*TM$ = span { $\delta y^i|_p$ } and, $H_p{}^*TM$ = span { $dx^j|_p$ } and these are independent of local coordinates. Then point wise $T^*(TM- 0) = V^*TM \oplus H^*TM$. Co-vectors in V^*TM are called vertical co-vector and co-vectors in H^*TM are called horizontal co-vectors.

1.7. Metric Tensor & Cartan torsion

From equation (1.4.2)´ we can easily see that the quantities g_{ij} defined by the equation (1.3.7) from the components of the covariant tensor of rank 2, also $g_{ij}(x, y)$ are positively homogeneous of degree zero in y^i and symmetric in their indices. Due to homogeneity condition – (a) of section 3 for the function $L(x, y)$, we have,

(1.7.1) $$L^2(x, y) = g_{ij}(x, y)y^i y^j$$

By condition- (c) of section 3 it follows that inverse of matrix g_{ij} exist. Thus, if g^{ij} denote the inverse of g_{ij}, then

(1.7.2) $$g_{ij}(x,\ y)g^{jk}(x,\ y) = \delta_i^k$$

where, δ_i^k is well known kronecker delta. Therefore,, the tensor whose covariant and contravariant components are $g_{ij}(x,\ y)$ and $g^{ij}(x,\ y)$ is called the metric tensor or the first fundamental tensor of the Finsler space F^n.

Cartan torsion tensor:

Let $x \in M$, $y \in T_xM$ and L fundamental function on Manifold M^n, define

$c_y :\ T_xM \times\ T_xM \times T_xM \rightarrow R$ by $c_y(u,v,w) = c_{ijk}\, u^i\, v^j\, w^k$. The family $c = \{\ c_{ijk}\ \}$ for all $y \in T_xM$, is called Cartan torsion. The tensor $C_{ijk}(x,\ y)$ defined by

(1.7.3) $$C_{ijk}(x,\ y) = \frac{1}{2}\dot{\partial}_k g_{ij} = \frac{1}{4}\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L^2$$

is positively homogeneous of degree -1 in y^i and is symmetric in all their indices. This tensor is called Cartan's C-tensor and satisfies

(1.7.4) $$C_{ijk}(x,\ y)y^i = C_{ijk}(x,\ y)y^j = C_{ijk}(x,\ y)y^k = 0$$

(1.7.5) $$(\partial_h C_{ijk})y^i = (\partial_h C_{ijk})y^j = (\partial_h C_{ijk})y^k = 0$$

1.8. Magnitude of a vector. The Notion of Orthogonality

The metric tensor $g_{ij}(x,\ y)$ may be used in two different ways, in defining the magnitude of a vector and also the angle between two vectors.

Let X^i be a vector, then the scalar X given by

(1.8.1) $$X^2 = g_{ij}(x,\ X)X^iX^j$$

is called the magnitude of this vector.

If Y^i is another vector, then the ratio,

(1.8.2) $$\cos(X,\ Y) = \frac{g_{ij}(x^i,\ X^i)X^iY^j}{L(x^i,\ X^i)L(x^i,\ Y^i)}$$

is called the 'Minkowskian cosine' corresponding to the (ordered) pair of directions $X^i,\ Y^i$(Rund [27]). It is obvious from (1.7.2) that Minkowskian cosine is non-symmetric in X^i and Y^i.

Let X^i be a vector and y^i an arbitrary fixed direction, then the scalar

(1.8.3) $$g_{ij}(x,\ y)X^iX^j$$

is called the square of magnitude of the vector X^i for the preassigned direction y^i.If Y^i is another vector, then the ratio,

(1.8.4) $$\cos(X,\ Y) = \frac{g_{ij}(x,\ y)X^iY^j}{\sqrt{g_{ij}(x,\ y)X^iX^j}\sqrt{g_{ij}(x,\ y)Y^iY^j}}$$

is called the cosine of $X^i,\ Y^i$ for the direction y^i.

It is to be noted that the concepts of magnitude of vector and the cosine between two vectors given by (1.8.3) and (1.8.4) stands at each point of the space in a pre-assigned direction y^i which has been called the element of support. Also the cosine given by (1.8.4) is symmetric in X^i and Y^i (Berwald [4], Synge [31])

To distinguish between the two magnitudes we call the magnitude given by (1.8.1) as the Minkowskian magnitude of X^i and that given by (1.8.3) the magnitude ofX^i.

The equations (1.8.2) and (1.8.4) are used to define the Orthogonality in F^n.

The vector Y^i is said to be orthogonal with respect to X^i if

(1.8.5) $$g_{ij}(x, X)X^iY^j = 0$$

Thus according to this definition if Y^i is orthogonal with respect to X^i then it is not necessary that X^i is also orthogonal with respect to y^i.

The vector X^i and Y^i are said to be orthogonal (for a pre-assigned y^i) if

(1.8.6) $$g_{ij}(x, y)X^iY^j = 0$$

This definition of Orthogonality is symmetric in X^i and Y^i.

1.9. Connections and Covariant Differentiations

Any quantities in a Finsler space is function of line element *(x, y)*. If *S(x, y)* is a scalar field in a Finsler space then $\frac{\partial s}{\partial x^i}$ are not components of a covariant vector. If we have a non-linear connection $N_j^i(x, y)$, we can obtain the covariant vector field of the components

$$S_{|i} = \frac{\delta s}{\delta x^i}, \quad \text{where} \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}.$$

Further, if we have quantities $F_{jk}^i(x, y)$ which obey the transformation rule similar to Christoffel symbol the covariant derivatives $K_{j|k}^i$ of a Finsler tensor field of (1, 1)-type is defined by

(1.9.1) $$K_{j|k}^i = \frac{\delta K_j^i}{\delta x^k} + K_j^r F_{rk}^i - K_r^i F_{jk}^r$$

On the other hand, the partial derivatives of components of a tensor field K_j^i with respect to y^i gives a new tensor field, but we shall modify them as,

(1.9.2) $$K_j^i|_k = \frac{\partial K_j^i}{\partial y^k} + K_j^r C_{rk}^i - K_r^i C_{jk}^r$$

Where, $C_{jk}^i(x, y)$ are components of a tensor field of (1, 2)-type. The collection $(F_{jk}^i, N_j^i, C_{jk}^i)$ constitute a Finsler connection, and covariant derivatives given by (1.9.1) and (1.9.2) are called h- and v-covariant derivatives of K_j^i respectively.

Finsler connection : Suppose N_j^i is is a non-linear connection on M^n and F_{jk}^i, C_{jk}^i are 0 & -1 degree homogeneous function respectivelyin y^i from (TM – 0) to R^+, $\aleph$ (M) the set of vector field on manifold M^n . A Finsler connection is a mapping

$$\nabla(F_{jk}^i, N_j^i, C_{jk}^i): T_p(TM - 0) \times \aleph(M) \rightarrow T_{\pi(p)}(M), (Y, X) \rightarrow \nabla_Y(X)$$

Satisfying the properties

1 - ∇ is linear over *R* in X and Y (but not necessarily in y).

2 – If f ∈ C^∞ (M) and y ∈ $T_xM - 0$ then in local coordinates

$$\nabla_{\frac{\delta}{\delta x^i}|_y}\left(f \frac{\partial}{\partial x^j}|_y\right) = df\left(\frac{\partial}{\partial x^i}|_y\right) \frac{\partial}{\partial x^j}|_x\Big) + f F_{ij}^m(y) \frac{\partial}{\partial x^m}|_x\Big),$$

$$\text{and } \nabla_{\frac{\partial}{\partial x^i}|_y}\left(f \frac{\partial}{\partial x^j}|_y\right) = f\, C_{jk}^i\,(y) \frac{\partial}{\partial x^m}|_x.$$

For all $X \in \aleph(M)$ and ∇ does not depend on the local coordinates.

For any Finsler connection $(F^i_{jk},\ N^i_j,\ C^i_{jk})$ we have five torsion tensors and three curvature tensors hh, hv and vv-curvatures [Riemannian curvature (R), Beraldian Curvature (B) and third curvature (Q)] which are given by,

(1.9.3) (h)h-torsion: $T^i_{jk} = F^i_{jk} - F^i_{kj}$

(1.9.4) (v)v-torsion: $S^i_{jk} = C^i_{jk} - C^i_{kj}$

(1.9.5) (h)hv-torsion: C^i_{jk} as the vertical connection C^i_{jk}

(1.9.6) (v)h-torsion: $R^i_{jk} = \frac{\delta N^i_j}{\delta x^k} - \frac{\delta N^i_k}{\delta x^j}$

(1.9.7) (v)hv-torsion: $P^i_{jk} = \dot{\partial}_k N^i_j - F^i_{kj}$

(1.9.8) h-curvature: $R^i_{hjk} = \frac{\delta F^i_{hj}}{\delta x^k} - \frac{\delta F^i_{hk}}{\delta x^j} + F^m_{hj} F^i_{mk} - F^m_{hk} F^i_{mj} +$

$C^i_{hm} F^m_{jk}$

(1.9.9) hv- curvature: $P^i_{hjk} = \dot{\partial}_k F^i_{hj} - C^i_{hk|j} + C^i_{hm} P^m_{jk}$

(1.9.10) v- curvature: $S^i_{hjk} = \dot{\partial}_k C^i_{hj} - \dot{\partial}_j C^i_{hk} + C^m_{hj} C^i_{mk} - C^m_{hk} C^i_{mj}$

The deflection tensor field D^i_j of a Finsler connection FΓ is given by

(1.9.11) $D^i_j = y^k F^i_{jk} - N^i_j$

when a Finsler metric is given, various Finsler connections are determined from the metric. The well known examples are Cartan's connection, Rund's connection and Berwald's connection.

Cartan's Connection

We are concerned with a Finsler space $F^n = (M^n, L)$ which is to be endowed with the Cartan's connection $C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ constructed from the fundamental function $L(x, y)$. According to the theory of Finsler connections due to M. Matsumoto ([17], [18]), the $C\Gamma$ is determined from the axiomatic stand point as follows:-

There exists a unique Finsler connection $F\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$ which satisfies the following five conditions:

(C_1) $g_{ij|k} = 0$

(C_2) (h)h-torsion: $T^i_{jk} = 0$

(C_3) Deflection tensor field $D^i_j = 0$

(C_4) $g_{ij}|_k = 0$

(C_5) (v)v-torsion: $S^i_{jk} = 0$

This connection is called the Cartan's connection and is denoted by

$C\Gamma = (\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$.

The last two condition C_4 and C_5 give,

(1.9.12) $$C^i_{jk} = \frac{1}{2} g^{ih} \frac{\partial g_{jk}}{\partial y^h}$$

This shows that vertical connection of $C\Gamma$ and Cartan's C-tensor is identical.

The first three conditions C_1, C_2 and C_3 give, (1.9.13) $F^i_{jk} = \Gamma^{*i}_{jk} = \frac{1}{2} g^{ih}[\frac{\delta g_{jh}}{\delta x^k} + \frac{\delta g_{kh}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^h}]$

(1.9.14) $N^i_j = \Gamma^{*i}_{0k} = \gamma^i_{0k} - 2C^i_{km}G^m$

where,

(1.9.15) $G^i = \frac{1}{2}\gamma^i_{00}$

and

(1.9.16) $\gamma^i_{jk} = \frac{1}{2} g^{ih}[\frac{\partial g_{jh}}{\partial x^k} + \frac{\partial g_{kh}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}]$

is the Christoffel symbol of (M^n, L). Here '0' denote contraction with y^i.

It is easy to verify from the axioms C_1, C_3 and equation (1.5.1), that

(1.9.17) a). $y^i_{|h} = 0$, b). $L_{|h} = 0$, c). $l^i_{|h} = 0$

where, l^i is a unit vector in the direction of element of support y^i, i.e.

$$l^i = \frac{y^i}{L(x,\ y)}$$

Since, C_{ijk} is an indicatory tensor, Therefore, from (1.6.12) we have $y^i_{|h} = \delta^i_h$. Thus in view of (1.5.1) and condition C_1, we have $L|_i = \dot{\partial}_i L = l_i$ where $l_i = g_{ij}l^j$. It may also be verified that,

(1.9.18) $\begin{cases} a).\ l^i|_j = L^{-1}h^i_j, & b).\ l_i|_j = L^{-1}h_{ij}, \quad c).\ l_{i|j} = 0, \\ d).\ h_{ij|k} = 0, & e).\ h_{ij}|_k = L^{-1}(l_i h_{jk} + l_j h_{ki}) \end{cases}$

where, h_{ij} is the angular metric tensor defined by

(1.9.19) $$h_{ij} = g_{ij} - l_i l_j$$

and $h^i_j = g^{ik} h_{jk}$

Round's Connection

The Rund's connection of a Finsler space F^n = (M^n, L) is a Finsler connection which is obtained from Cartan's connection CΓ by the C-process [18]. The C-process is characterized by expelling the torsion tensorC^i_{jk}. Thus the first two connection coefficients of the Rund's connection RΓ are the same with those of the Cartan's connection CΓ, while the third is equal to zero. Thus the Rund's connection RΓ of the Finsler space F^n is given by RΓ = (Γ^{*i}_{jk}, Γ^{*i}_{0k}, 0). The torsion tensors of RΓ are such that,

(1.9.20) $$\begin{cases} T^i_{jk} = 0, \quad R^i_{jk} = \text{the same as that of } C\Gamma, \; C^i_{jk} = 0 \\ P^i_{jk} = \text{the same as that of } C\Gamma, \; S^i_{jk} = 0 \end{cases}$$

The curvature tensors of RΓ are as follows

(1.9.21) $$\begin{cases} (a).\ \text{h}-\text{curvature } K\text{: } K^i_{hjk} = R^i_{hjk} - C^i_{hr} R^r_{jk} \\ (b).\ \text{hv}-\text{curvature } F\text{: } F^i_{hjk} = P^i_{hjk} + C^i_{hk|j} - C^i_{hr} P^r_{jk} \end{cases}$$

While, the v-curvature tensor S^i_{hjk} of RΓ vanishes identically. We note that h-covariant differentiations with respect to CΓ and RΓ coincide with each other. Furthermore C^i_{jk} in (1.9.21) is the Carton's C-tensor $C^i_{jk} = g^{ih} C_{jhk}$ which is not the vertical connection of RΓ as it vanishes for RΓ.

The h-curvature K and hv-curvature F of RΓ may be given in terms of connection coefficients as,

(1.9.22) $$\begin{cases} a). \quad K^i_{hjk} = \dfrac{\delta \Gamma^{*i}_{hj}}{\delta x^k} - \dfrac{\delta \Gamma^{*i}_{hk}}{\delta x^j} + \Gamma^{*m}_{hj}\Gamma^{*i}_{mk} - \Gamma^{*m}_{hk}\Gamma^{*i}_{mi} \\ b). \quad F^i_{hjk} = \dot{\partial}_j \Gamma^{*i}_{hj} \end{cases}$$

Bernard's Connection

The Bernard's connection of a Finsler space $F^n = (M^n, L)$ is a Finsler connection which is obtained from Round's connection RΓ by the P[1]-process [18]. The P[1]-process is characterized by expelling the torsion tensor P^i_{jk}. The Berwald's connection of Finsler space F^n is denoted by BΓ=(G^i_{jk}, G^i_j, 0) where

(1.9.23) $$a). \quad G^i_{jk} = \dot{\partial}_j G^i_k, \qquad b). \quad G^i_j = \Gamma^{*i}_{0j} = \dot{\partial}_j G^i$$

The Berwald's connection BΓ is uniquely determined from metric function $L(x, y)$ of F^n by the following five axioms:-

(B$_1$) $L_{|i} = 0$

(B$_2$) (h) h-torsion: $T^i_{jk} = 0$

(B$_3$) Deflection: $D^i_j = 0$

(B$_4$) (v) hv-torsion: $P^i_{jk} = 0$

(B$_5$) (h)hv-torsion: $C^i_{jk} = 0$

Thus the tensors of BΓ are such that

(1.9.24) $$\begin{cases} T^i_{jk} = 0, \quad R^i_{jk} = \text{the same as that of R}\Gamma, \; C^i_{jk} = 0 \\ \mathrm{P}^i_{jk} = \text{the same as that of R}\Gamma, \; S^i_{jk} = 0 \end{cases}$$

The v-connection coefficients G^i_{jk} of BΓ are related to those of CΓ by

(1.9.25) $$G^i_{jk} = \Gamma^{*i}_{jk} + C^i_{jk|0}$$

The curvature tensors of BΓ are as follows

(1.9.2) $$\begin{cases} a). \; h-curvature \; H: \; H^i_{hjk} = K^i_{hjk} + C^i_{hj|0|k} - C^i_{hk|0|j} \\ \qquad\qquad\qquad\qquad\qquad +C^i_{kr|0}C^r_{jk|0} - C^i_{jr|0}C^r_{kh|0} \\ b). \; hv-curvature \; G: \; G^i_{hjk} = F^i_{hjk} + \dot{\partial}_h G^i_{jk} \end{cases}$$

The v-curvature tensor S^i_{hjk} of BΓ vanishes identically.

The simpler forms of H^i_{hjk} and G^i_{hjk} of BΓ may be given by,

(1.9.27) $$H^i_{hjk} = \dot{\partial}_h R^i_{jk}, \qquad G^i_{hjk} = \dot{\partial}_h G^i_{jk}$$

It is to be noted that BΓ is neither h-metrical nor v-metrical in general:-

$$g_{ij(k)} = -2C_{ijk|0}, \qquad g_{ij.k} = C_{ijk}$$

where h- and v-covariant derivatives with respect to BΓ is denoted here by ( ) and '.' respectively.

1.10. Geodesics and paths in a Finsler space

The geodesic of a Finsler space are the curves of minimum or maximum arc-length between any two points of the space. The differential equations of a geodesic in a Finsler space is given by [18]

(1.10.1) $$\frac{d^2x^i}{ds^2} + 2G^i\left(x,\ \frac{dx}{ds}\right) = 0$$

where, s is the arc length of the curve $x^i = x^i(s)$ and

(1.10.2) $$2G^i = \gamma^i_{jk} y^j y^k$$ or

(1.10.3) $$2G^i = g^{ir}\left(y^j \dot{\partial}_r \dot{\partial}_j F - \partial_r F\right),$$

Here Lagrangian function *L* is defined on TM by $F(x, y) = \frac{1}{2} L^2(x, y)$,

Where $F: TM \rightarrow R$ is the Finsler function

Let M^n be a manifold with a Finsler connection $F\Gamma = (F^i_{jk}, N^i_j, C^i_{jk})$. A curve C of the tangent bundle *T(M)* over M^n is called on h-path, if C is the projection of an integral curve of an h-basic vector field $B^h(v)$, corresponding to a fixed $v \in V^n$ [18].

(1. 10.4) $$\begin{cases} \frac{dy^i}{dt} + N^i_j(x(t),\ y(t))\frac{dx^j}{dt} = 0 \\ \frac{d^2x^i}{dt^2} + F^i_{jk}(x(t),\ y(t))\frac{dx^j}{dt}\frac{dx^k}{dt} = 0 \end{cases}$$

Geodesic spray

Geodesic spray $G \in \aleph$ (TM - 0) the set of vector field on (TM- 0) is locally defined as

(1. 10.5) $$G\,|_y = y^i \frac{\partial}{\partial x^i}\,|_y - 2G^i(x,y)\,\frac{\partial}{\partial y^i}\,|_y$$

Here G does not depend on local coordinate and G^i is defined by (1. 10.3). It is also called path space

1.11. Special Finsler Spaces

In Riemannian geometry we have many interesting theorems such that if a Riemannian space is assumed to have special geometrical properties, or to satisfy special tensor equations, or to admit special tensor fields, then the space reduces to one of well-known space forms, for instance, Euclidean space, spheres, topological spheres, projective spaces and so on.

On the other hand, in Finsler geometry we have special Finsler spaces, namely, Riemannian spaces and Minkowskian spaces, but there are various kinds of Riemannian spaces and Minkowskian spaces. As a consequence we have an important problem to classify all the Minkowskian spaces. It is easy to write down concrete forms of fundamental functions $L(x, y)$ which are interesting as a function, for instance, a Randers metric, Kropina metric, generalized Kropina metric, Matsumoto metric and cubic metric.

It is essential for the progress of Finsler geometry to find Finsler spaces, which are quite similar to Riemannian spaces, but not Riemannian and Minkowskian spaces, which are analogous to flat spaces, but not flat. In the present section, we are mainly concerned with special tensor equations satisfied by torsion, curvature and other important tensors. In the following, we give some definitions of special Finsler spaces and their corresponding result.

(A). **Riemannian space**

A Finsler space $F^n = (M^n, L(x, y))$ is said to be a Riemannian space, if its fundamental function $L(x, y)$ is written as,

$$L(x, y) = g_{ij}(x)y^i y^j$$

Among Finsler spaces, the class of all the Riemannian spaces is characterized by $C_{ijk} = 0$ i.e. vertical connection Γ^v of the Cartan's connection $C\Gamma$ is flat.

(B). **Locally Minkowskian space**

A Finsler space $F^n = (M^n, L(x, y))$ is called locally Minkowskian space, if there exists a co-ordinate system (x^i) in which L is a function of y^i only [18].

A Finsler space is locally Minkowskian if and only if

For $C\Gamma$: $\quad R^h_{ijk} = C^h_{ij|k} = 0$

For $R\Gamma$: $\quad K^h_{ijk} = F^h_{ijk} = 0$

For $B\Gamma$: $\quad H^h_{ijk} = G^h_{ijk} = 0$

(C). **Berwald space**

If the connection coefficient G^i_{jk} of the Berwald's connection $B\Gamma$ given by,

$$G^i_{jk} = \dot{\partial}_j G^i_k$$

are function of position alone, the space is called a Berwald space [18].

A Finsler space is Berwald space if and only if

For CΓ: $C^h_{ij|k} = 0$

For RΓ: $F^h_{ijk} = 0$

For BΓ: $G^h_{ijk} = 0$

(D). **Landsberg space**

A Finsler space is called a Landsberg space [18] if the Berwald connection BΓ is h-metrical i.e. $g_{ij(k)} = 0$.

In terms of the Cartan's connection CΓ, a Landsberg space is characterized by,

(a). $P^i_{jk} = 0$, or (b). $P^h_{ijk} = 0$

(E). **C-reducible Finsler space**

A Finsler space of dimension n, more than two, is called C-reducible if C_{ijk} is written in the form [18]:-

$$C_{ijk} = \frac{1}{n+1}\pi_{(ijk)}(h_{ij}C_k)$$

where, $C_i = C_{ijk}g^{jk}$ is the torsion vector, h_{ij} is the angular metric tensor given by $h_{ij} = g_{ij} - l_i l_j$ and $\pi_{(ijk)}$ is the sum of cyclic permutation of *i*, *j*, *k*.

(F). **Semi C-reducible Finsler space**

A Finsler space of dimension n, more than two, is called semi C-reducible if C_{ijk} is written in the form [18]:-

$$C_{ijk} = \frac{p}{n+1}\pi_{(ijk)}\left(h_{ij}C_k\right) + \frac{q}{C^2}C_iC_jC_k$$

where, $C^2 = g^{ij}C_iC_j$ and *p* + *q* = *1*.

(G). **Quasi C-reducible Finsler space**

A Finsler space of dimension n, more than two, is called quasi C-reducible if there exists a symmetric Finsler tensor field A_{ij}, satisfying $A_{i0} = 0$, in terms of which C_{ijk} is written in the form [18]:-

$$C_{ijk} = \pi_{(ijk)}(A_{ij}C_k)$$

(H). **P-reducible Finsler space**

A Finsler space of dimension n, more than two, is called P-reducible if (v)hv-torsion tensor P_{ijk} of CΓ is written in the form ([12], [22]):-

$$P_{ijk} = \frac{1}{n+1}\pi_{(ijk)}(h_{ij}C_{k|0})$$

(I). **C2-like Finsler space**

A Finsler space is called C2-like Finsler space [23] if

$$C_{ijk} = \frac{1}{C^2}C_iC_jC_k$$

(J). **C3-like Finsler space**

A Finsler space is called C3-like Finsler space [25] if

$$C_{ijk} = S_{ijk}\{h_{ij}a_k + C_i C_j b_k\}$$

where, a_k and b_k are components of arbitrary indicatory tensors.

(K). **S3-like Finsler space**

A Finsler space F^n with fundamental function *L(x, y)* is called S3-like Finsler space [18] if v-curvature tensor S_{hijk} of CΓ is written in the forms:-

$$L^2 S_{hijk} = S\{h_{hj}h_{ik} - h_{hk}h_{ij}\}$$

where, *S* is a scalar and function of position alone.

(L). **S4-like Finsler space**

A Finsler space F^n is called S4-like Finsler space [25] if v-curvature tensor S_{hijk} of CΓ is written in the form:-

$$S_{hijk} = h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk}$$

where, M_{ij} are components of a symmetric covariant tensor of second order and are (-2)p-homogeneous in y^i satisfying $M_{0j} = 0$.

(M). **R3-like Finsler space**

A Finsler space of dimension more than three, is called R3-like Finsler space [20] if h-curvature tensor R_{hijk} of CΓ is written in the

forms:-

$$R_{hijk} = g_{hj}L_{ik} + g_{ik}L_{hj} - g_{hk}L_{ij} - g_{ij}L_{hk}$$

where, L_{ij} are components of a covariant tensor of second order.

(N). **Finsler space of scalar curvature**

A Finsler space of scalar curvature K is characterized by [18]:-

$$R_{i0j} = KL^2 h_{ij}$$

where, R_{ijk} are components of (v)h-torsion tensor of CΓ defined by (1.8.6)

(O) One – form

A one-form on a differentiable manifold is a smooth section of the cotangent bundle. It is a smooth mapping of the total space of the tangent bundle of M to R whose restriction to each fiber is a linear functional on the tangent space. Symbolically,

β : TM → R , $\beta_x = \beta|_{T_xM}$: T_xM → R where β_x is linear.

In a local coordinate system, a one-form is a linear combination of the differentials of the coordinates: $\beta_x = b_i dx^i$ where the b_i are smooth functions

(Fibers over x). It is an order-1 covariant tensor field

Examples

1 - The second element of a three-vector is given by the one-form [0, 1, 0]. That is, the second element of [*x* ,*y* ,*z*] is [0, 1, 0] · [*x*, *y*, *z*] = *y*.

2-The mean element of an *n*-vector is given by the one-form [1/*n*, 1/*n*, ..., 1/*n*]. That is, *mean (v) = [1/n, 1/n, 1/n].v*

(P). **Finsler space with (α, β)-metric**

In the paper [26] concerned with the unified field theory of gravitation and electromagnetism Randers wrote, “Perhaps the most characteristic property of the physical world is the unidirectional of time like interval. Since there is no obvious reason why this asymmetry should disappear in the mathematical description it is of interest to consider the possibility of a metric with asymmetrical property. It is known that many reasons speak for the necessity of a quadratic induction. The only way of introducing an asymmetry while retaining the quadratic indicatrix, is to displace the center of the indicatrix. In other words we adopt as indicatrix an eccentric quadratic hypersurface. This involves the definition of a vector at each point of space determining the displacement of the center of the indicatrix. The formula for the length *ds* of a line element dx^i must necessarily be homogeneous of first degree in dx^i. The simplest “eccentric” line element possessing this property and of course being invariant is

(1.11.1) $$ds = \sqrt{a_{ij}(x)dx^i dx^j} + b_i(x)dx^i$$

where, a_{ij} is the fundamental tensor of the Riemannian affine connection and b_i is a covariant vector determining the displacement of the center of the indicatrix.”

After sixteen years, in the monograph [13] concerned with electron microscope Ingarden wrote:-

“In arbitrary curvilinear co-ordinate systems the Lagrangian function of electron of electron optics may be written in the form

$$L(x,\ x') = imc\sqrt{\gamma_{ij}(x)x'^i x'^j} + e_k A_i(x^i,\ x'^i)$$

where, γ_{ij} is an isotropic tensor reducing in Lorentz systems to the constant unit tensor δ_{ij}. According to their physical interpretation, we shall call γ_{ij} the gravitational tensor and A_i the electromagnetic vector.

The special kind of Finsler space with the metric (1.10.1) we shall call a Randers space, since Randers (1941) seems to have been the first consider this kind of spaces, although he regarded them not as Finsler space but as "affinely connected Riemannian spaces" which is rather confusing notion. Randers could not use, Therefore, the methods of Finsler's geometry and tried to reduce the study of (1.9.1) to a sort of 5-dimensional Kalza-Klein geometry, where Riemannian method plus a method of special projecting of tensors are used. Spaces with metric of the form (1.9.1) were also considered by Stephenson and Kilmister [29] in 1953, but in investigations of these spaces they simply use pure Riemannian methods, which are obviously erroneous."

On the other hand, in 1959-1961 Kropina considered protectively flat Finsler spaces equipped with the metrics

(1.11.2) $$L(x,\ y) = \frac{a_{ij}(x)y^i y^j}{b_k(x)y^k}$$

(1. 11.3) $$L(x,\ y) = \{a_{ijk}(x)y^i y^j y^k\}^{1/3}$$

Generalizing these special Finsler metrics of Randers type (1.10.1) and Kropina type (1.10.2), Matsumoto defined in 1972, the notion of $(\alpha,\ \beta)$-metric as follows:-

Definition-1: A Finsler metric $L(x,\ y)$ [18] is called an $(\alpha,\ \beta)$-metric, when L is positively homogeneous function $L(\alpha,\ \beta)$ of first degree in two variables

$$\alpha(x,\ y) = \sqrt{a_{ij}(x)y^i y^j} \quad \text{and} \quad \beta(x,\ y) = b_i(x)y^i$$

It is usual to suppose that α is a Riemannian metric, i.e. non-degenerate (regular) and positive definite, but there are some cases for applications where these restrictions are relaxed. Further, we shall have to confine our discussions to suitable domain of *(x, y)* on account of special form of the function $L(\alpha,\ \beta)$.

Definition-2: The $(\alpha,\ \beta)$-metric $L = \alpha + \beta$ [18] is called a Randers metric and Finsler space with this metric is called a Randers space.

In 1980 Hashiguchi and Ichijyo gave the following interesting remark on Randers metric.

Proposition-11.1: A Randers metric $L = \alpha + \beta$ [11] is positive valued, if and only if $a_{ij} - b_i b_j$ is positive-definite, provided that a_{ij} is positive-definite.

Definition-3: The $(\alpha,\ \beta)$-metric $L = \frac{\alpha^2}{\beta}$ [18] is called Kropina metric and Finsler space with this metric is called a Kropina space.

Wrona [35] has given the interesting example of Kropina metric. For a Kropina space the direction y^i belonging to the hyperplane $\beta(x,\ y) = b_i(x)y^i = 0$ of the tangent space at any point x must be obviously excluded. The indicatrix is to extend asymptotically along this hyperplane. Therefore, a Kropina metric is never positive definite.

Although Kropina herself seems to have played attention to such a metric from a pure mathematical standpoint, there are close relation between this kind of metric and Lagrangian function of analytic dynamics.

Definition-4: The $(\alpha,\ \beta)$-metric $L = \alpha^{m+1}\beta^{-m}$ $(m \neq 0,\ -1)$ [18] is called a generalized m-Kropina metric.

The Finsler metric given by (1.9.3) is called a cubic metric and was considered by Wegener (1935) and also by Kropina. It is regarded as a direct generalization of Riemannian metric in a sense.

In the astronomy we measure the distance in a time, in particular, in the light year. When we take a second as the unit, the unit surface (indicatrix) is a sphere with radius of 300,000 km. To each point of our space is associated such a sphere, this defines the distance (measured in a time) and the geometry of our space is the simplest one, namely, the Euclidean geometry. Next, when a ray of light is considered as the shortest line in the gravitational field, the geometry of our space is Riemannian geometry. Furthermore, in an isotropic medium the speed of the light depends on its direction, and the unit surface is not any longer a sphere.

Now, on the slope of the earth surface we sometimes measure the distance in a time namely, the time required such as seen on a guidepost. Then the unit curve (indicatrix) taken a minute as the unit, will be general closed curve without center, because we can walk only a shortest distance in an uphill road than in downhill road. This defines a general geometry (Finsler geometry), although it is not exact. The shortest line along which we can reach the goal, for instance, the top of a mountain as soon as possible will be a complicated curve.

The exact formulation given by Matsumoto is as follows:-

Proposition-1.1.2. A slope, the graph of a function $z = f(x,\ y)$, [21] of the earth surface is regarded as a two-dimensional Finsler space with fundamental function,

$$L(x,\ y,\ \dot{x},\ \dot{y}) = \frac{\alpha^2}{v\alpha - w\beta}$$

where, *v* and *w* are non-zero constants and

$$\alpha^2 = \dot{x}^2 + \dot{y}^2 + (\dot{x}f_x + \dot{y}f_y)^2$$

$$\beta = \dot{x}f_x + \dot{y}f_y$$

This α is the usual induced Riemannian metric and β is a derived form

$$\beta(x,\ dx) = \alpha f(x,\ y)$$

The two constants *v* and *w* are such that one can walk *v* meters per minute on the horizontal plane and 2*w* is equal to the acceleration of falling. Aikou, Hashiguchi and Yamauchi generalized and normalized the above metric as follows:-

Definition-5: An n-dimensional $(\alpha,\ \beta)$-metric $L = \frac{\alpha^2}{\alpha - \beta}$ [1] is called a slope metric or Matsumoto metric and a Finsler space equipped with this metric is called a Matsumoto metric.

Definition-6 **projective:** If any geodesic of F^n coincides with a geodesic of $\bar{F}^n$ as a set of points and vice versa, then the change $L \rightarrow \bar{L}$ of the metric is called *projective* and F^n is said to be projective to $\bar{F}^n$

Definition-7 **Conformal change** : Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces on a same underlying manifold Mn . If the angle in F^n is equal to that in $\bar{F}^n$ for any tangent vectors, then F^n is called conformal to $\bar{F}^n$ and the change $L \rightarrow \bar{L}$ of the metric is called a conformal change of Finsler metric.

Definition-8 $\boldsymbol{\beta}$ **- change** : Let R^n be associated Riemannian spaces with a Finsler spaces F^n with (α, β) – metric ,the β - change is a change from R^n to F^n . Randers change ,Kropina change .Matsumoto change are β – change .

Definition—9 **Mean Cartan torsion** : Mean Cartan torsion $I_y : T_xM \to R$ defined by $I_y(u) = I_i(y)\, u^i$,

where $I_i = g^{kj} c_{ijk}$ and $u = u^i \frac{\partial}{\partial x^i}|_x$.

Definition—10 **Landsberg Curvature**: The h- covariant derivative of cartan tensor along geodesics gives rise to the landsberg curvature $L_y : T_xM \times T_xM \times T_xM \to R$ by $L_y(u,v,w) = L_{ijk}(y)\, u^i v^j w^k$, where

$L_{ijk} = c_{ijk|0} = c_{ijk|s}\, y^s$ .The family $L = \{ L_y \}$ for all $y \in T_xM$ are called Landsberg curvature .

Definition-11 **Mean Landsberg Curvature**: The h- covariant derivative of Mean cartan tensor along geodesics gives rise to the Mean landsberg curvature $J_y : T_xM \to R$,defined by $J_y(u) = J_i(y)\, u^i$ where $J_i = I_{i|s}\, y^s$, $u = u^i \frac{\partial}{\partial x^i}|_x$.

Definition-12 **Landsberg Metric and Weakly Landsberg Metric**: A metric Landsberg Curvature and Mean Landsberg Curvature are veins *i.e.* $L = 0$ and $J = 0$ are respectively called Landsberg Metric and Weakly Landsberg Metric.

Berwald curvature : For $y \in T_xM_0$,define $B_y : T_xM \times T_xM \times T_xM \to T_xM$ by $B_y(u,v,w) = B^i_{jkl}(y)\, u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ where $B^i_{jkl}(y) = \frac{\partial^3 G^i}{\partial y^j y^k y^l}(y)$. B is called Berwald curvature ,a Finsler metric is called Berwald metric if $B = 0$, that is $\frac{\partial^3 G^i}{\partial y^j y^k y^l} = 0$.

1.12. **Intrinsic fields of Orthonormal frames**

Berwald theory of two-dimensional Finsler space is developed based on the intrinsic field of orthonormal frame which consists of the normalized supporting element l^i and unit vector orthonormal to l^i. Following idea Moor introduced, in a three-dimensional Finsler space, the intrinsic field of orthonormal frame which consists of the normalized supporting element l^i, the normalized torsion vector C^i/C and a unit vector orthogonal to them and developed a theory of three-dimensional Finsler spaces. Generalizing the Berwald's and Moo's ideas, Miron and Matsumoto ([18], [20], [24]) developed a theory of intrinsic orthonormal frame fields on n-dimensional Finsler space as follows.

Let *L(x, y)* be the fundamental function of an n-dimensional Finsler space and introduce Finsler tensor fields of (0, 2α-1) type, α = 1,2,…..,n by

$$L_{i_1 i_2 \dots\dots i_{2\alpha-1}} = \frac{1}{2^\alpha}\dot{\partial}_{i_1}\dot{\partial}_{i_2}\dots\dots\dot{\partial}_{i_{2\alpha-1}}L^2$$

Then we get a sequence of covariant vectors

$$L_{\alpha)i} = L_{i j_1 j_2 \dots\dots j_{2\alpha-3} j_{2\alpha-2}}g^{j_1}g^{j_1}\dots\dots g^{j_{2\alpha-3}j_{2\alpha-2}}$$

Definition-1: If (n-1) covariant vectors $L_{\alpha)i}$, α = 1, 2,….., n-1 are linearly independent, the Finsler space is called strongly non-Riemannian.

Assuming above n-covectors $L_{\alpha)i}$ are linearly independent and put $e^i_{1)} = L^i_{1)}/L = l^i$. Here and in following we use raising and lowering of indices as $L^i_{1)} = g^{ij}L_{1)j}$.

Further putting $N_{1)ij} = g_{ij} - e_{1)i}e_{1)j}$ and matrix $N_{1)} = N_{1)ij}$ is of rank (n-1). Second vector $e_{2)}$ is introduced by

$$e^i_{2)} = L^i_{2)}/L_2,$$

where, L_2 is the length of $L^i_{2)}$ relative to y^i. Next we put $N_{2)ij} = N_{1)ij} - e_{2)i}e_{2)j}$, $E^i_{3)}$=$N^i_{2)j}L^j_{3)}$ and so third vector $e_{3)}$ is defined by,

$$e^i_{3)} = E^i_{3)}/E_3\ ,$$

where, E_2 is the length of $E^i_{2)}$ relative to y^i. The repetition of above process gives a vector $e_{r+1)}$, r = 1, 2,……, n-1 defined by

$$e^i_{r+1)} = E^i_{r+1)}/E_{r+1}$$

where, $E^i_{r+1)}$=$N^i_{r)j}L^j_{r+1)}$ E_{r+1} is the length of $E^i_{r+1)}$ relative to y^i and $N_{r)ij} = N_{r-1)ij} - e_{r)i}e_{2)j}$.

Definition-2: The orthonormal frame $\{e_\alpha\}$, α = 1, 2,….., n as above defined in every in every co-ordinate neighborhood of a strongly non-Riemannian Finsler space is called the 'Miron Frame'.

Consider the Miron frame $\{e_\alpha\}$, If a tensor T^i_j of (1, 1)-type, for instance, is given then

$$T^i_j = T_{\alpha\beta}\, e^i_{\alpha)} e_{\beta)j}$$

where, the scalars $T_{\alpha\beta}$ are defined as

$$T_{\alpha\beta} = T^i_j e_{\alpha)i} e^j_{\alpha)}$$

These scalars $T_{\alpha\beta}$ are called the scalar components of T^i_j with respect to Miron frame.

Let $H_{\alpha)\beta\gamma}$ be scalar components of the h-covariant derivatives $e^i_{\alpha)|j}$ and $V_{\alpha)\beta\gamma}/L$ be scalar components of the v-covariant derivatives $e^i_{\alpha)}|_j$ with respect to CΓ of the vector $e^i_{\alpha)}$ belonging to the Miron frame. Then

$$e^i_{\alpha)|j} = H_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j},$$

$$e^i_{\alpha)}|_j = V_{\alpha)\beta\gamma} e^i_{\beta)} e_{\gamma)j},$$

where, the scalars $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ satisfying the following relations [18].

$$H_{1)\beta\gamma} = 0, \quad H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma},$$

$$V_{\alpha)\beta\gamma} = \delta_{\beta\gamma} - \delta^1_\beta \delta^1_\gamma, \quad V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}$$

Definition-3: The scalars $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called connection scalars.

If $C_{\alpha\beta\gamma}/L$ be the scalar components of the (h)hv-torsion tensor C^i_{jk} i.e.,

$$LC^i_{jk} = C_{\alpha\beta\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}$$

then [13], we have

Proposition-1:[18]

$$C_{1\beta\gamma} = 0$$

$C_{2\mu\mu} = LC,\ \ C_{3\mu\mu} = \cdots \ldots \ldots = C_{n\mu\mu} = 0$ for $n \geq 3$, where C is the length of C^i.

Now, we consider scalar components of covariant derivatives of a tensor field, for instance, T_j^i. Let $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}/L$ be the scalar components of h-and v-covariant derivatives with respect to CΓ respectively of a tensor T_j^i *i*.e.,

(1.12.1) $\qquad T^i_{j|k} = T_{\alpha\beta,\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}$ and

(1.12.2) $\qquad LT^i_j|_k = T_{\alpha\beta;\gamma} e^i_{\alpha)} e_{\beta)j} e_{\gamma)k}$, then we have [34]

(1.12.3) $\qquad T_{\alpha\beta,\gamma} = \left(\delta_k T_{\alpha\beta}\right) e^k_{\gamma)} + T_{\mu\beta} H_{\mu)\alpha\gamma} + T_{\alpha\mu} H_{\mu)\beta\gamma}$ and

(1.12.4) $\qquad T_{\alpha\beta;\gamma} = L\left(\dot{\partial}_k T_{\alpha\beta}\right) e^k_{\gamma)} + T_{\mu\beta} V_{\mu)\alpha\gamma} + T_{\alpha\mu} V_{\mu)\beta\gamma}$.

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ are called h-and v-scalar derivative of $T_{\alpha\beta}$ respectively.

Two-dimensional Finsler space

The Miron frame $\{e_{1)},\ e_{2)}\}$ is called the Berwald frame. The first vector $e^i_{1)}$ is the normalized supporting element $l^i = y^i/L$ and the second vector $e^i_{2)} = m^i$ is the unit vector orthogonal to l^i. If C^i has non-zero length C, the $m^i = \pm C^i/C$. The connection scalars $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ of a two-dimensional Finsler space are such that [18],

$$H_{\alpha)\beta\gamma} = 0,\ \ V_{\alpha)\beta 1} = 0,\ \ V_{\alpha)\beta 2} = \delta^{12}_{\alpha\beta}, \text{ which implies}$$

(1.12.5) $\qquad l^i_{|j} = 0,\ \ m^i_{|j} = 0,\ \ Ll^i|_j = m^i m_j,\ \ Lm^i|_j = -l^i m_j$

There is only one surviving scalar components of LC_{ijk} namely C_{222}. If we put $I = C_{222}$. Then

$$LC_{ijk} = Im_i m_j m_k$$

The scalar *I* is called the main scalar of a two-dimensional Finsler space.

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Chapter 2

H-RANDERS CHANGE OF FINSLER METRIC

2.1 Introduction

Let $F^n = (M^n, L)$ be n-dimensional Finsler space, where M^n is n-dimensional differentiable manifold and $L(x, y)$ is the Finsler fundamental function. In 1974, M. Matsumoto ([1], [7]) introduced the transformation of Finsler metric

(2.1.1). $$L^*(x, y) = L(x, y) + b_i(x)\, y^i,$$

and obtained the relation between the imbedding class numbers of tangent Riemannian spaces to (M^n, L) and (M^n, L^*). In 1980 he [8] obtained the relation between the Cartan's connections of (M^n, L) and (M^n, L^*). Assuming $L(x, y)$ as a Riemannian metric, he obtained various important tensors of Finsler space

(M^n, L^*). The Finsler space equipped with metric

$$L^*(x, y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$$

is called Randers metric. This metric has been introduced by G. Randers in 1941 [11], from the stand-point of general theory of relativity and applied to the theory of electron microscope. R. S. Ingarden [4] named it as Randers metric. The geometrical properties of this space have been studied by various authors ([3], [9], [10], [12]). In all these theories b_i has been considered as functions of coordinate's x^i only.

In 1980, H. Izumi [5], while studying the conformal transformation of Finsler spaces introduced the h-vector b_i which is v-covariantly constant with respect to Cartan's connection $C\Gamma$ (i.e. $b_i|_j = 0$) and satisfies the relations $LC_i{}^h{}_j\, b_h = \rho\, h_{ij}$, where $C_i{}^h{}_j$ are the components of (h)hv- torsion tensor and h_{ij} are components of angular metric tensor. Thus the h-vector is not only a function of coordinates x^i, but also a function of directional arguments satisfying $L\dot{\partial}_j b_i = \rho h_{ij}$.

In this chapter we consider the transformation of Finsler metric given by

(2.1.2) $$\bar{L}(x, y) = L(x, y) + \beta(x, y)$$

where $\beta = b_i(x, y)y^i$ and $b_i(x, y)$ are components of h-vector in (M^n, L). The Finsler space with metric $\bar{L}(x, y)$ will be denoted by $\bar{F}^n$ whereas the metric

$L(x, y)$ will be denoted by F^n. The quantities corresponding to $\bar{F}^n$ will be written by putting bar (i.e. –) on the top of that quantities.

2.2 Cartan's Connection of $\bar{F}^n$.

Let the Cartan's connection of Finsler space F^n be denoted by

$C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$. Since $b_i(x, y)$ are components of h-vector, we have

(2.2.1) (a) $b_i|_j = \dot{\partial}_j b_i - b_h C^h_{ij} = 0$

(b) $L C_i{}^h{}_j\, b_h = \rho\, h_{ij}$.

Hence we obtain,

(2.2.2) $$\dot{\partial}_j b_i = L^{-1} \rho\, h_{ij}.$$

Since h_{ij} are components of an indicatrx tensor i.e. $h_{ij}\, y^j = h_{ij}\, y^i = 0$, we have $\dot{\partial}_i \beta = b_i$. Therefore differentiating (2.1.2) with respect to y^i, we get

(2.2.3) $\bar{L}_i = L_i + b_i$,

where $L_i = \dot{\partial}_i L$. Since the normalized element of support l_i of F^n is given by $l_i = \dot{\partial}_i L$, equation (2.2.3) may be written as $\bar{l}_i = l_i + b_i$.

Differentiating (2.2.3) with respect to y^j, using (2.2.2) and the fact that $\dot{\partial}_j l_i = L^{-1} h_{ij}$, we get

(2.2.4) $\quad \bar{h}_{ij} = \sigma\, h_{ij},$

where $\sigma = L^{-1}\, \bar{L}(1 + \rho)$. Since $h_{ij} = g_{ij} - l_i\, l_j$, from (2.2.4) and (2.2.5), we get

(2.2.5) $\quad \bar{g}_{ij} = \sigma\, g_{ij} + (1 - \sigma)\, l_i\, l_j + (l_i\, b_j + l_j\, b_i) + b_i\, b_j.$

The contravariant components $\bar{g}^{ij}$ of the fundamental tensor $\bar{g}_{ij}$ of $\bar{F}n$, are obtained from

$\bar{g}^{ij}\ \bar{g}_{jk} = \delta^i_k$, and is given by

(2.2.6) $\quad \bar{g}^{ij} = \sigma^{-1}\, g^{ij} - \sigma^{-3}(1+\rho)^2(1 - b^2 - \sigma) l^i\, l^j - \sigma^{-2}(1 + \rho)(l^i\, b^j + l^j\, b^i),$

where b is the magnitude of the vector b_i in F^n and $b^i = g^{ij}\, b_j$. Now we establish the following:

Lemma 2.2.1. For a Finsler Space of dimention n (n > 2) the scalar ρ in h-vector is a function of coordinate's x^i only.

Proof. Since

(2.2.7) $\quad \dot{\partial}_k h_{ij} = 2C_{ijk} - L^{-1}(l_i h_{jk} + l_j h_{ik}),$

Differentiating (2.2.1) (b) with respect to y^k and using (2.2.2) we get

$$l_k C^h_{ij} b_h + L(\dot\partial_k C^h_{ij}) b_h + \rho C^h_{ij} h_{hk} = (\dot\partial_k \rho) h_{ij} + \rho[2C_{ijk} - L^{-1}(l_i h_{jk} + l_j h_{ik})],$$

which after using (2.2.1) (b) and the fact that $\dot\partial_k C^h_{ij} = -2C^m_{ij} C^h_{mk} + g^{hm} \dot\partial_k C_{mij}$, we obtain

(2.2.8) $$L^{-1}\rho l_k h_{ij} - \rho C_{ijk} + L(\dot\partial_k C_{mij} b^n = (\dot\partial_k \rho) h_{ij} + 2\rho C_{ijk} - L^{-1}\rho(l_i h_{jk} + l_j h_{ik}).$$

Taking skew symmetric part in j and k in the above equation, we get

$$L^{-1}\rho(l_k h_{ij} - l_j h_{ik}) = (\dot\partial_k \rho) h_{ij} - (\dot\partial_j \rho) h_{ik} + L^{-1}\rho(l_k h_{ij} - l_j h_{ik}).$$

Hence $(\dot\partial_k \rho) h_{ij} - (\dot\partial_j \rho) h_{ik} = 0$, which after contraction with g^{ij} gives (n – 2) $(\dot\partial_k \rho) = 0$. Hence ρ is independent of y^k for n≠2 and we have the lemma (2.2.1).

From equations (2.1.1), (2.2.4) and lemma (2.2.1) we get

(2.2.9) $$\dot\partial_i \sigma = L^{-1}(1+\rho) m_i,$$

where

(2.2.10) $$m_i = b_i - (L^{-1}\beta) l_i$$

Differentiating (2.2.5) with respect to y^k and using (2.2.4), (2.2.5), (2.2.8) and (2.2.9) we get

(2.2.11) $$\overline{C}_{ijk} = \sigma\, C_{ijk} + \frac{(1+\rho)}{2L}(h_{ij}\, m_k + h_{jk}\, m_i + h_{ki}\, m_j).$$

From the definition of m_i, it is evident that

(2.2.12) (a) $m_i l^i = 0$, (b) $m_i\, b^i = b^2 - \beta^2/L^2 = m_i\, m^i$.

(c) $h_{ij}\, m^i = h_{ij}\, b^i = m_j$, (d) $C_{ihj}\, m^h = L^{-1}\, \rho\, h_{ij}$.

From (2.2.1), (2.2.7), (2.2.11) and (2.2..12), we get

(2.2.13) $$\bar{C}^h_{ij} = C^h_{ij} + \frac{1}{2\bar{L}}(h_{ij} m^h + h^h_j m_i + h^h_i m_j)$$

$$-\frac{1}{\bar{L}}\left[\left\{\rho + \frac{L}{2\bar{L}}\left(b^2 - \frac{\beta^2}{L^2}\right)\right\} h_{ij} + \frac{L}{\bar{L}} m_i m_j\right] l^h .$$

From equations (2.2.2) and (2.2.3) we obtained

(2.2.14) $\bar{L}_{ij} = (1 + \rho)\, L_{ij}$,

(2.2.15) $\bar{L}_{ijk} = (1 + \rho)\, L_{ijk}$,

where $L_{ij} = \dot{\partial}_i \dot{\partial}_j L$, $L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L$. From equation (2.2.14) we obtained

(2.2.16) $\partial_k \bar{L}_{ij} = (1 + \rho)\, \partial_k L_{ij} + L_{ij}\, \rho_k$,

where $\rho_k = \partial\rho/\partial x^k$. Since h-covariant derivative of L_{ij} and $\bar{L}_{ij}$ with respect to $C\Gamma$ vanishes, we have

(2.2.17) $$\partial_k L_{ij} = L_{ijr} G^r_k + L_{rj} F^r_{ik} + L_{ir} F^r_{jk} ,$$

(2.2.18) $$\partial_k \bar{L}_{ij} = \bar{L}_{ijr} \bar{G}^r_k + \bar{L}_{rj} \bar{F}^r_{ik} + \bar{L}_{ir} \bar{F}^r_{jk} .$$

The equation (2.2.14) and lemma (2.2.1) serve the purpose to find the relation between v-connection components of F^n and $\bar{F}^n$ with respect to $C\Gamma$. For this purpose we put

(2.2.19) $$D^i_{jk} = \bar{F}^i_{jk} - F^i_{jk}, \qquad D^i_{0k} = \bar{G}^i_k - G^i_k.$$

Here 0 in index, denote the contraction with y^j, for instance $D^i_{0k} = D^i_{jk} y^j$.

The difference D^i_{jk} is obviously a tensor of type (1,2). Substituting the values of $\partial_k L_{ij}$ and $\partial_k \bar{L}_{ij}$ from (2.2.17) and (2.2.18) in (2.2.16) and using (2.2.14), (2.2.15), (2.2.19), we get

(2.2.20) $$(1+\rho)(L_{ijr} D^r_{0k} + L_{rj} D^r_{ik} + L_{ir} D^r_{jk} = L_{ij} \rho_k.$$

In order to find the difference tensor D^i_{jk}, we had to constructed supplementary equation to (2.2.20). From (2.2.3), we obtain $\partial_j \bar{L}_i = \partial_j L_i + \partial_j b_i$. Since the h-covariant derivative of L_i with respect to CΓ vanishes and $b_{i|j} = \partial_j b_i - G^r_j \dot{\partial}_r b_i - b_r F^r_{ij}$, this equation may be written as

(2.2.21) $$\bar{L}_{ir} \bar{G}^r_j + \bar{L}_r \bar{F}^r_{ij} = L_{ir} G^r_j + L_r F^r_{ij} + b_{i|j} + G^r_j \dot{\partial}_r b_i + b_r F^r_{ij}.$$

Since $\dot{\partial}_r b_i = \rho L_{ir}$, in the light of equations (2.2.14) and (2.2.19), we have

(2.2.22) $$(1+\rho) L_{ir} D^r_{0j} + (l_r + b_r) D^r_{ij} = b_{i|j}.$$

The difference tensor D^i_{jk} is now found from equation (2.2.20) , (2.2.22) and the lemma given below.

Lemma [7] 2.2.2. The system of algebraic equations

(i) $\mathbf{L_{ir} A^r = B_i}$ **(ii)** $\mathbf{(l_r + b_r) A^r = B}$

has a unique solution A^r for given B and B_i such that $B_i\, l^i = 0$. The solution is given by

(2.2.23) $$A^i = L\,B^i + \frac{1}{L}(B - L\,B_r\,b^r)y^i.$$

It is obvious that (2.2.22) is equivalent to the two equations

(2.2.24) $$(1+\rho)(L_{ir}D^r_{0j} + L_{jr}D^r_{0i}) + 2(l_r + b_r)D^r_{ij} = 2E_{ij}$$

(2.2.25) $$(1+\rho)(L_{ir}D^r_{0j} - L_{jr}D^r_{0i}) = 2F_{ij}$$

where, we put

(2.2.26) $$2E_{ij} = b_{i|j} + b_{j|i}, \qquad \text{and} \qquad 2F_{ij} = b_{i|j} - b_{j|i}.$$

Applying Christoffel process to (2.2.20), we get

(2.2.27)

$$(1+\rho)(2L_{rj}D^r_{ik} + L_{ijr}D^r_{0k} + L_{jkr}D^r_{0i} - L_{kir}D^r_{0j}) = L_{ij}\rho_k + L_{jk}\rho_i - L_{ki}\rho_j.$$

Contraction of (2.2.24), (2.2.25) and (2.2.27) by y^j gives

(2.2.28) $$(1+\rho)L_{ir}D^r_{00} + 2(l_r + b_r)D^r_{0i} = 2E_{i0}$$

(2.2.29) $$(1+\rho)L_{ir}D^r_{00} = 2F_{i0}$$

(2.2.30) $$(1+\rho)(L_{ir}D^r_{0k} + L_{kr}D^r_{0i} + L_{kir}D^r_{00}) = L_{ki}\rho_r y^r.$$

Moreover contraction of (2.2.28) by y^i gives

(2.2.31) $$(l_r + b_r)D^r_{00} = E_{00}.$$

Applying lemma (2.2.2) to the equations (2.2.29) and (2.2.31), we obtain

(2.2.32) $$D^i_{00} = \frac{2L}{(1+\rho)}F^i_0 + \frac{1}{L}[E_{oo} - \frac{2L}{(1+\rho)}F_{r0}b^r]y^i$$

where we put $F^i_0 = g^{ij}F_{j0}$. After replacing k by j in equation (2.2.30) and adding it in equation (2.2.25), we get

(2.2.33) $$L_{ir}D^r_{0j} = A_{ij}$$

where

(2.2.34) $$A_{ij} = (1+\rho)^{-1}[F_{ij} + \frac{1}{2}\rho_r y^r L_{ij}] - \frac{1}{2}L_{ijr}D^r_{00}.$$

The equation (2.2.28) can be written as

(2.2.35) $$(l_r + b_r)D^r_{0i} = A_i$$

where

(2.2.36) $$A_i = E_{i0} - \frac{(1+\rho)}{2}L_{ir}D^r_{00}.$$

Substituting the value of D^r_{00} from (2.2.32) in (2.2.34), we obtain

(2.2.37) $$A_{ij} = (1+\rho)^{-1}\Big[F_{ij} + \frac{1}{2}\rho_r y^r L_{ij} - LL_{ijr}F^r_0$$

$$+\frac{1}{L}\{\frac{1}{2}(1+\rho)E_{oo} - LF_{r0}b^r\}L_{ij}\Big].$$

Substituting the value of F_{i0} from (2.2.29) in (2.2.36), we get

(2.2.38) $$A_i = E_{i0} - F_{i0}.$$

Applying lemma (2.2.2) to equations (2.2.33) and (2.2.35) , we get

(2.2.39) $$D^i_{0j} = LA^i_j + \frac{1}{L}(A_j - LA_{rj}b^r)y^i .$$

Finally we deal with (2.2.27) and (2.2.24), we obtain

(2.2.40) (a) $$L_{jr}D^r_{ik} = H_{ijk}$$ (b) $$(l_r + b_r)D^r_{ik} = H_{ik}$$

where

(2.2.41)
$$2H_{ijk} = L_{kir}D^r_{0j} - L_{ijr}D^r_{0k} - L_{jkr}D^r_{0i} + (1+\rho)^{-1}(L_{ij}\rho_k + L_{jk}\rho_i - L_{ki}\rho_j)$$

and

(2.2.42) $$H_{ik} = E_{ik} - \frac{(1+\rho)}{2}(L_{ir}D^r_{0j} + L_{jr}D^r_{0i}) .$$

Hence H_{ijk} and H_{ij} are written in terms of known quantities. Applying lemma (2.2.2) to the equations (2.2.40), we can find the concrete value of

(2.2.43) $$D^i_{jk} = LH^i_{jk} + \frac{1}{L}(H_{jk} - LH_{jrk}b^r)y^i ,$$

where $H^i_{jk} = g^{ir}H_{jrk}$.

Theorem 2.2.1. The cartan's connection $C\Gamma$ = $(\bar{F}^i_{jk}, \bar{G}^i_j, \bar{C}^i_{jk})$ of the Finsler space $\bar{F}^n$ are completely determined by the equations (2.2.13), (2.2.39) and (2.2.43) in terms of the Cartan's connection of F^n.

2.3. Relationship of Randers change with projective change:

We consider the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ which is given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \text{where } F=L^2/2, \quad G^i_j = \dot{\partial}_j G^i \text{ and } G^i_{jk} = \dot{\partial}_k G^i_j.$$

Since $G^i_j y^j = 2G^i$, therefore from (2.2.19) we get $D^i_{00} = 2\overline{G}^i - 2G^i$. Hence from (2.2.32), we obtain

$$2\overline{G}^i = 2G^i + \frac{2L}{(1+\rho)} F^i_0 + \frac{1}{L}[E_{oo} - \frac{2L}{(1+\rho)} F_{r0} b^r] y^i. \tag{2.3.1}$$

From this equation ,we have

$$\overline{G}^i y^j - \overline{G}^j y^i = G^i y^j - G^j y^i + \frac{L}{(1+\rho)} (F^i_0 y^j - F^j_0 y^i). \tag{2.3.2}$$

Now let F^n be a Douglas space. Then $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in y^i [8]. From (2.3.2) we may state the following:

Theorem 2.3.1. Let F^n be a Douglas space and $\overline{F}^n$ is obtained from F^n by h-Randers change of its metric. The $\overline{F}^n$ is a Douglas space if and only if

$L(F^i_0 y^j - F^j_0 y^i)$ is homogeneous polynomial of degree three in y^i.

Now let us suppose that the h-Randers change of Finsler metric given by (1.1.2) is projective. Then [8]

(2.3.3) $\bar{G}^i = G^i + P(x, y)\, y^i,$

where P(x, y) a scalar function and is called the projective factor. Comparing equation (2.3.3) with (2.3.1) we get

$$\frac{L}{(1+\rho)} F^i_0 + \frac{1}{2L}[E_{oo} - \frac{2L}{(1+\rho)} F_{r0} b^r] y^i = P(x, y)\, y^i$$

which may be written as

(2.3.4) $F^i_o = \lambda y^i,$

where $\lambda = \frac{1+\rho}{L}\left[\{P(x,y) - \frac{1}{2L} E_{00}\}\right] + 2F_{r0} b^r$. Since $F^i_0 g_{ih} y^h = 0$, equation (2.3.4) gives $\lambda = 0$. Consequently, we have $F^i_0 = 0$.

Theorem 2.3.2. The h-Randers change of Finsler metric is projective if and only if F^i_0 = 0.

In view if theorem (2.3.1) and (2.3.2) we have the following:

Theorem 2.3.3. Let h-Randers change of Finsler metric is projective. Then a Douglas space is transformed to a Douglas space.

We are concerned with the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ which is given by $2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F)$, where $F = L^2/2$, $G^i_j = \dot{\partial}_j G^i$

and $G^i_{jk} = \dot\partial_k G^i_j$.

The Douglas tensor of Finsler space is defined by

(2.3.5) $$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1}(G_{ijk}y^h + G_{ij}\delta^h_k + G_{jk}\delta^h_i + G_{ki}\delta^h_j)$$

where $G^h_{ijk} = \dot\partial_k G^h_{ij}$ is the hv-curvature tensor of Berwald connection BΓ,

G_{ij} = G^r_{ijr} and $G_{ijk} = \dot\partial_k G_{ij}$ [9]. If the Finsler space F^n ,is projective to $\overline{F}^n$ then $D^h_{ijk} = \overline{D}^h_{ijk}$ [9]. Thus from equation (2.2.5), we obtain

(2.3.6) $$\frac{(1+\rho)}{L} h_{\alpha h} D^\alpha_{ijk} = \frac{1}{\overline{L}} \overline{h}_{\alpha h} \overline{D}^\alpha_{ijk}.$$

Since the hv-torsion tensor of Cartan connection is given by $P_{ijk} = -\frac{1}{2} y_h G^h_{ijk}$, therefore from equation (2.3.5), we have

(2.3.6)' $$h_{\alpha h} D^\alpha_{ijk} = G_{hijk} + \frac{2}{L} P_{ijk} l_h - \frac{1}{(n+1)}(h_{kh} G_{ij} + h_{ih} G_{jk} + h_{jh} G_{ki})$$

where $G_{hijk} = g_{\alpha h} G^\alpha_{ijk}$. Using equation (2.3.6)′ to the equation (2.3.6), we have

(2.3.7) $$(1+\rho)\left[\frac{1}{L} G_{hijk} + \frac{2}{L^2} P_{ijk} l_h - \frac{1}{L(n+1)}(h_{kh} G_{ij} + h_{ih} G_{jk} + h_{jh} G_{ki})\right]$$

$$= \frac{1}{\overline{L}} \overline{G}_{hijk} + \frac{2}{\overline{L}^2} \overline{P}_{ijk} \overline{l}_h - \frac{1}{\overline{L}(n+1)}(\overline{h}_{kh} \overline{G}_{ij} + \overline{h}_{ih} \overline{G}_{jk} + \overline{h}_{jh} \overline{G}_{ki}).$$

Now suppose that the Finsler spaces F^n and $\overline{F}^n$ are Landsberg spaces. Then

P_{ijk} = $\overline{P}_{ijk}$ = 0 [1-721p.]. Under this condition (2.3.7) becomes

$$(2.3.8) \qquad (1+\rho)\left[\frac{1}{L}G_{hijk}-\frac{1}{L(n+1)}(h_{kh}G_{ij}+h_{ih}G_{jk}+h_{jh}G_{ki})\right]$$

$$= \frac{1}{\overline{L}}\overline{G}_{hijk}-\frac{1}{\overline{L}(n+1)}(\overline{h}_{kh}\overline{G}_{ij}+\overline{h}_{ih}\overline{G}_{jk}+\overline{h}_{jh}\overline{G}_{ki}).$$

Moreover, for Landsberg space [1- 721p] $G_{hijk} - G_{ihjk} = 0$ and $\overline{G}_{hijk} - \overline{G}_{ihjk} = 0$. This leads to

$$(2.3.9) \qquad h_{ij}(G_{hk} - \overline{G}_{hk}) + h_{ik}(G_{hj} - \overline{G}_{hj}) - h_{jh}(G_{ki} - \overline{G}_{ki}) - h_{kh}(G_{ij} - \overline{G}_{ij}) = 0$$

Contracting (2.3.9) with g^{ij}, we get

$$(2.3.10) \qquad G_{hk} - \overline{G}_{hk} = \frac{1}{n+1}h_{hk}s(x,y)$$

where $s(x, y) = g^{ij}(G_{ij} - \overline{G}_{ij})$. Therefore we have

Theorem 2.3.3. Let F^n and $\overline{F}^n$ be Landsberg spaces and $\overline{F}^n$ is obtained by projective h-Randers change, then $G_{hk} - \overline{G}_{hk} = \frac{1}{n+1}h_{hk}s(x,y)$, where

$s(x, y) = g^{ij}(G_{ij} - \overline{G}_{ij})$.

If a Finsler space satisfies the condition G_{ij} = o, we call it a Weakly-Berwald space [6]. If F^n and $\overline{F}^n$ are Weakly-Berwald spaces then equation (2.3.7) becomes

$$(2.3.11) \qquad (1+\rho)\left[\frac{1}{L}G_{hijk}+\frac{2}{L^2}P_{ijk}l_h\right]= \frac{1}{\overline{L}}\overline{G}_{hijk}+\frac{2}{\overline{L}^2}\overline{P}_{ijk}\overline{l}_h.$$

But $\frac{1}{L}G_{hijk} + \frac{2}{L^2}P_{ijk}l_h = \frac{1}{L}h_{\alpha h}G^{\alpha}_{ijk}$, therefore in view of equation (2.2.5), above equation reduces to

(2.3.12) $$h_{\alpha h}(G^{\alpha}_{ijk} - \overline{G}^{\alpha}_{ijk}) = 0.$$

Now if $\overline{F}^n$ is projective to F^n, then from (2.3.3) ,we get

(2.3.13) $$\overline{G}^{\alpha}_{ijk} = G^{\alpha}_{ijk} + P_{ijk}y^{\alpha} + P_{ij}\delta^{\alpha}_{k} + P_{jk}\delta^{\alpha}_{i} + P_{ki}\delta^{\alpha}_{j},$$

where $P_{ij} = \dot{\partial}_i\dot{\partial}_j P$, $P_{ijk} = \dot{\partial}_i\dot{\partial}_j\dot{\partial}_k P$. Substituting (2.3.13) in (2.3.12) we get

(2.3.14) $$h_{hk}P_{ij} + h_{ih}P_{jk} + h_{hj}P_{ki} = 0.$$

Contraction of (2.3.14) with g^{hk} gives $(n + 1) P_{ij} = 0$. This equation shows that P_i ($= \dot{\partial}_i P$) does not depend on y^i. Thus we have

Theorem 2.3.4. Let F^n and $\overline{F}^n$ be Weakly Berwald spaces and $\overline{F}^n$ is obtained by projective h-Randers change. Suppose P(x, y) denote the projective factor of this change. Then P_i does not depend on y^i.

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Chapter 3

RANDERS CONFORMAL CHANGE OF A FINSLER SPACE

OF DOUGLAS TYPE

3.1 Introduction

In 1941 ,G.Randers ([1], [7], [10]) introduced a special metric

ds = $\sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$ in a view point of General theory of Relativity. Since then many Physicist had developed the General theory of Relativity. By this time Finsler space has already been coined. This metric was first recognized by , as a kind of Finsler metric in 1957 by R.S.Ingarden , M .Matsumoto introduced the (α,β) – metric by generalizing Randers Metric [2]. The theory of Finslar space with (α, β) - metric has been developed into fruitful branch of Finsler Geometry.

From stand point of Finsler Geometry itself Randers metric is very interesting because its form is simple and properties of Finsler spaces equipped this metric can be looked as Riemannian spaces equipped with the metric L(x,y) = $\sqrt{a_{ij}(x)y^i y^j}$ together with one form β = $b_i(x)y^i$. The curvature tensors of Randers metric R^h_{ijk} , P^h_{ijk} , S^h_{ijk} , of Finsler spaces can be written in terms of Riemannian curvature tensors of metric α , b_i and its covariant derivative with respect to Riemannian connection RΓ.A change of Finsler metric $L(x,y)$ goes to $\bar{L}(x,y)$ is called Randers change of $L(x,y)$ if $\bar{L}(x,y) = L(x,y) + \beta$ .The concept of Douglas space ([1], [8], and [10]) has been introduced by M.

Matsumoto and S. Bacso as a generalization of Berward space from stand point of view of geodesic equation. Finsler space is said to be of Douglas space if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomial in y^i of degree 3. It has been shown by M . Matsomoto in papers ([1], [7], [8], [9], [10]) that

$F^n = (M^n, L)$ is a Douglas type iff the Douglas tensor

$D^h_{ijk} = G^h_{ijk} - \frac{1}{n-1}\left(G_{ijk} y^h + \delta^h_i G_{jk} + \delta^h_j G_{ik} + \delta^h_k G_{ij}\right)$ vanishes identically, where G^h_{ijk} is hv – curvature tensor of Berward connection BΓ. Douglas curvature is a non-Riemannian projective invariant constructed from the Berwald curvature. The said transformation is generalization of conformal as well as Randers change because writing β = 0 it reduces to Conformal change and when σ(x) = 0 it reduces to Randers change .It is compositions of conformal change and Randers change. The conformal theory of Finsler metrics based on the theory of Finsler spaces by M . Matsomoto ([3], [7]).Hashiguchi [3] in 1976 studied the conformal change of a Finsler Metric namely $\bar{L}(x,y) = e^{\sigma(x)} L(x,y)$

In the present paper we shall investigate the condition under which a change of Finsler metric $L(x, y)$ goes to $\bar{L}(x,y) = e^{\sigma(x)} L(x, y)+\beta(x,y)$, where σ is a function of position x^i only, and β a differentiable one-form ([1],[6]), is the Randers Conformal change of Finsler spaces of Douglas type. We have also worked out the condition under which the said changes are Projective also.

3.2. Preliminaries.

The geodesic of an n-dimensional Finsler space $F^n = (M^n, L)$ are given by the system of the differential equation [1]

$$\frac{d^2x^i}{dt^2} y^j - \frac{d^2x^j}{dt^2} y^i + 2\{G^i(x,y)y^j - G^j(x,y)y^i\} = 0$$

where $y^i = \frac{dx^i}{dt}$ in a parameter t . The function $G^i(x,y)$ are given by

$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F)$, where $\dot{\partial}_i =$, $\partial_i =$, $F = L^2/2$,

and $g^{ij}(x,y)$ are inverse of Finsler metric tensor g_{ij} [8] . Let $L_i = \dot{\partial}_i L$,

$L_{ij} = \dot{\partial}_j \dot{\partial}_i L$, $L_{ijk} = \dot{\partial}_k \dot{\partial}_j \dot{\partial}_i L$. Then we have

$$\bar{L}_i = e^{\sigma(x)} L_i + b_i ,$$

$$\bar{L}_{ij} = e^{\sigma(x)} L_{ij} \tag{3.2.1}$$

$$\bar{L}_{ijk} = e^{\sigma(x)} L_{ijk}$$

where $L_i = l_i$, $L L_{ij} = h_{ij}$ and we shall use the notation $2E_{ij} = b_{i/j} + b_{j/i}$ and $2F_{ij} = b_{i/j} - b_{j/i}$, where (/) denote the h- covariant derivative with respect to the Cartan connection $C\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$. Throughout the paper we say homogeneous polynomials in y^i of degree r as hp(r) for brevity. The Finsler spaces with metric = will be denoted by $\bar{F}^n$, where as with metric $L(x,y)$ will be denoted by F^n . The quantities corresponding to $\bar{F}^n$ will be written by putting bar i.e. (-) on the top of that quantities.

3.3 - RANDERS CONFORMAL CHANGE OF DOUGLAS TYPE

For Randers conformal change $L \rightarrow \bar{L} = e^{\sigma(x)} L + \beta$

We put $\bar{G}^i = G^i + D^i$

(3.3.1) $\bar{G}^i_j = G^i_j + D^i_j$,

$\bar{G}^i_{jk} = G^i_{jk} + D^i_{jk}$

Where $D^i_j = \dot{\partial}_j D^i$ & $D^i_{jk} = \dot{\partial}_k D^i_j$. The tensor D^i, D^i_j , D^i_{jk}, are positively homogeneous in y^i of degree 2, 1 & 0 respectively. To find the value of D^i, we deals with the equation

$L_{i/j} = 0$ which implies $\partial_j L_i = L_{ir} G^r_j + L_r F^r_{ij}$ where $L_{ij/k}$ is h – covariant derivative of $L_{ij} = h_{ij}/L$ in $C\Gamma$. Then

(3.3.2) $\partial_k L_{ij} = L_{ijr} G^r_k + L_{rj} F^r_{ik} + L_{ir} F^r_{jk}$.

(3.3.3) $\partial_k \bar{L}_{IJ} = \bar{L}_{ijr} \bar{G}^r_k + \bar{L}_{rj} \bar{F}^r_{jk} + \bar{L}_{ir} \bar{F}^r_{jk}$.

also (3.3.4) $\partial_k \bar{L}_{IJ} = e^{\sigma(x)} (\sigma_k(x) L_{ij} + \partial_k L_{ij})$.

In view of (3.3.3), the equation (3.3 .4) can be written as

(3.3.5) $L_{ijr} D^r_k + L_{rj} D^r_{ik} + L_{ir} D^r_{jk} = \sigma_k(x) L_{ij}$.

Next we deal with $L_{i/j} = 0$ which implies

(3.3.6) $$\partial_j L_i = L_{ir} G^r_j + L_r F^r_{ij}$$

This shows that (3.3.7) $$\partial_j \bar{L}_i = \bar{L}_{ir} \bar{G}^r_j + \bar{L}_r \bar{F}^r_{ij} .$$

Also $b_{i/j} = 0$ implies

(3.3.8) $$\partial_j b_i = b_{i/j} + b_r F^r_{ij}$$

In view of (3.2.1) , (3.3.1) & (3.3.8), the equation (3.3.7) gives

(3.3.9) $$b_{i/j} = e^{\sigma(x)} [L_{ir} D^r_j + l_r D^r_{ij} – L_i \sigma j(x) + b_r D^r_{ij}$$

Therefore we have,

(3.3.10) $$2 E_{ij} = e^{\sigma(x)} [L_{ir} D^r_j + L_{jr} D^r_i + 2 l_r D^r_{ij} - L_i\sigma_j - L_j\sigma_i] +2 b_r D^r ij$$

(3.3.11) $$2 F_{ij} = e^{\sigma(x)} [L_{ir} D^r_j – L_{jr} D^r_i - L_i \sigma j(x) + L_j\sigma_i(x)]$$

Proposition 3.3.1: The tensors E_{ij} & F_{ij} of Randers Conformal change are given by (3.3.10) and (3.3.11).

Applying Christoffel process to (3.3.5), we get

(3.3.12) $$L_{ijr} D^r_k + 2 L_{rj} D^r_{ik} + L_{jkr} D^r_{0i} - L_{kir} D^r_{0j} = L_{ij} \sigma_k + L_{jk} \sigma_i - {}_{ki}\sigma_j .$$

Transvection of (3.3.10) , (3.3.11) & (3.3.12) by y^j gives

(3.3.13) $$2 E_{i0} = e^{\sigma(x)} [2L_{ir} D^r +2 l_r D^r_i - L_i\sigma_j y^j - L\sigma_i] +2 b_r D^r i$$

(3.3.14) $$2 F_{i0} = e^{\sigma(x)} [2L_{ir} D^r - L_i \sigma_j y^j + L\sigma_i(x)]$$

(3.3.15) $L_{ir} D^r_k + L_{kr} D^r_i + 2 L_{kir} D^r = L_{ki} \sigma_0$.

Again transvection of (3.3.13) by y^i gives

(3.3.16) $E_{00} = e^{\sigma(x)} [2 l_r D^r - L\sigma_0] + 2 b_r D^r i$.

Lemma 3.3.1 **The system of algebraic equations**

(3.3.17) **(1)** $\mathbf{L_{ir} A^r = B_i}$

(2) $\mathbf{P\, l_r A^r + Q\, b_r A^r = B}$

has a unique solution $\mathbf{A^i = L\, B^i + l^i\, \tau^{-1} (B - QL\, B_\beta)}$ **, for given B and** $\mathbf{B_i}$ **such that** $\mathbf{B_i\, l^i = 0}$ **.**

It follows from (3.2.1) that 1-(3.3.17) is written in the form $h_{ir} A^r = L\, B_i$, where h_{ij} is angular metric tensor .This implies

(3.3.18-a) $A^i = L\, B^i + l^i (l_r A^r)$.

Contraction by b_r gives

(3.3.18) $b_r A^r = L\, B_\beta + \frac{\beta}{L} (l_r A^r)$

where $B^i b_i = B_\beta$ and $b_i\, l^i = \frac{\beta}{L}$.

In view of (3.3.18) , Lemma-(3.3.1) - 2 is written as

(3.3.19) $l_r A^r = \tau^{-1} (B - QL\, B_\beta)$, where $\tau = P + Q \frac{\beta}{L}$.

In view of (3.3.12) & (3.3.19), the equation (3.3.18-a) is written in the form

$$(3.3.20) \qquad A^i = L\,B^i + l^i\,\tau^{-1}\,(B - QL\,B_\beta)$$

which is the solution of Lemma (3.3.1) . Comparing (3.3.14) & (3.3.17) to Lemma 3.3.1 (2) & (1) respectively,we get

$$A^r = D^r, \quad P = 2, \qquad Q = 2e^{-\sigma(x)}, \qquad B = E_{00}\,e^{-\sigma(x)} + L\,\sigma_0,$$

$$B^i = e^{-\sigma(x)} F^i_o + \frac{1}{2} L^i\,\sigma_j\,y^j - \frac{1}{2} g^{ij}\,\sigma_j\,.$$

Using these results together with equation (3.3.20) we get

$$(3.3.21) \qquad D^i = L[e^{-\sigma(x)} F^i_o + \frac{1}{2}\,l^i\sigma_0 - \frac{1}{2}\sigma_r g^{ri}] + \tau^{-1}(E_{00}\,e^{-\sigma(x)} + L\,\sigma_0$$

$$- 2\,L\,e^{-\sigma(x)} B_\beta)\,l^i\,.$$

where $\tau = 2(1 + \frac{\beta}{L} e^{-\sigma(x)})$, $B_\beta = B_i\,b^i$.

Propositions 3.3.2 The tensor D^i of the equation $\bar{G}^i = G^i + D^i$ arising from Randers Conformal change in Finsler spaces are given by equation (3.3 .21).

From equation (3.3 .21) we have

$$\bar{G}^i y^j - \bar{G}^j y^i = G^i y^j - G^j y^i + L\,e^{-\sigma(x)}\,(F^i_0 y^j - F^j_0 y^i) + \frac{1}{2}(L^i y^j - L^j y^i)\,\sigma_r y^r$$

$$- \delta \frac{1}{2} L \sigma_r (g^{ri} y^j - g^{rj} y^i) .$$

This equation is rewritten in the form

$$\bar{G}^i \delta^j_k - \bar{G}^j \delta^i_k = G^i \delta^j_k - G^j \delta^i_k + L\, e^{-\sigma(x)} (F^i_0 \delta^j_k - F^j_0 \delta^i_k)$$

$$+ \frac{1}{2} (L^i \delta^j_k - L^j y \delta^i_k) \sigma_r y^r - \frac{1}{2} L \sigma_r (g^{ri} y \delta^j_k - g^{rj} \delta^i_k) .$$

Thus we have

Theorem 3.3.1 **Let F^n be a Douglas space and $\bar{F}^n$ a Finsler space obtained by Randers Conformal change. Then $\bar{F}^n$ is Douglas spaces if and only if**

$$L\, e^{-\sigma(x)} (F^i_0 \delta^j_k - F^j_0 \delta^i_k) + \frac{1}{2} (L^i \delta^j_k - L^j \delta^i_k) \sigma_r y^r - \frac{1}{2} L \sigma_r (g^{ri} \delta^j_k - g^{rj} \delta^i_k)$$

is homogeneous polynomial in y^i of degree 2 .

The change is Projective if every geodesics of (M^n, L) is also a geodesics of $(M^n, \bar{L})$ and vice versa. We are going to find out a condition for a Conformal Randers Change to be Projective. The Euler –Lagrange equation for the metric $\bar{L}$ in terms of arc length s is given by [2]

$$\frac{d}{ds}\left(\frac{\partial \bar{L}}{\partial y^i}\right) - \frac{\partial \bar{L}}{\partial x^i} = 0 \qquad \text{which gives}$$

(3.3.22) $$\frac{d\sigma(x)}{ds} L_i + \frac{dL_i}{ds} + e^{-\sigma(x)}\left(\frac{db_i}{ds} - \frac{\partial b_i}{\partial x^i}\frac{\partial x^r}{\partial s}\right) - \frac{\partial \sigma(x)}{\partial x^i} - \partial_i L = 0 .$$

The Euler –Lagrange equation for the metric L is

(3.3.23) $$\frac{dL_i}{ds} - \partial_i L = 0 .$$

In view of (3.3.23), equation (3.3.22) becomes

$$\frac{\partial e^{\sigma(x)}}{\partial x^i}\left(\frac{\partial L}{\partial s} - 1\right) + b_{[ij]}\frac{dx^j}{ds} = 0$$

$$\text{where} \quad b_{[ij]} = \frac{\partial b_i}{\partial x^j} - \frac{\partial b_j}{\partial x^i} .$$

Thus we have

Theorem 3.3.2. Randers Conformal change is Projective iff

$$\frac{\partial e^{\sigma(x)}}{\partial x^i}\left(\frac{\partial L}{\partial s} - 1\right) + b_{[ij]}\frac{dx^j}{ds} = 0 .$$

Now let us suppose that the Randers Conformal change of Finsler metric given by (3.1.1) is Projective. Then [10]

(3.3.24) $$\bar{G}^i = G^i + P(x,y)\, y^i$$

where P(x,y) is a scalars function and is called the Projective factor. Comparing equation (3.3.24) and (3.3.21), we get

$p(x,y)\ y^i = L\ [e^{-\sigma(x)} F^i_o + \frac{1}{2}\ l^i \sigma_0 - \frac{1}{2} \sigma_r g^{ri}\ ] + \tau^{-1}\ (E_{00} e^{-\sigma(x)} + L\ \sigma_0 - 2\ L\ e^{-\sigma(x)}\ B_\beta)\ l^i$

,which may be written as

$$L\ e^{-\sigma(x)} F^i_o - \frac{1}{2}\ L^2\ \sigma^i = [\ p\ (x,y)\ -\ \tau^{-1}\ (E_{00} e^{-\sigma(x)}\ + L\ \sigma_0 - 2\ L\ e^{-\sigma(x)}\ B_\beta)\ L^{-1}]\ y^i.$$

Since $F^i_0\ g_{ih}\ y^h = 0$. So we get

$$P(x,y) = \tau^{-1}\ L^{-1}\ e^{-\sigma(x)}\ (\ E_{00} -\ 2L\ B_\beta\ ) + \sigma_0\ (\tau^{-1} - \frac{1}{2} L^{-1} + \frac{1}{2}\ ) . \tag{3.3.25}$$

Theorem 3.3.3. Randers Conformal change of Finsler metric given by (3.1.1) is Projective if

$$\mathbf{P(x,y)} = \tau^{-1}\ \mathbf{L}^{-1}\ e^{-\sigma(x)}\ (\mathbf{E}_{00} -\ \mathbf{2L\ B}_\beta\ ) + \sigma_0\ (\tau^{-1} - \frac{1}{2}\mathbf{L}^{-1} + \frac{1}{2}\ ) .$$

where , $\tau = 2\ (\ 1 + \frac{\beta}{L}\ e^{-\sigma(x)}\ )$,

$\mathbf{B}_\beta = \mathbf{B}_i\ \mathbf{b}^i$,

$\sigma_0 = \sigma_i\ y^i$.

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Chapter 4

CONFORMAL RANDERS CHANGE OF A FINSLER SPACE WITH (α,β) METRIC OF DOUGLAS TYPE

4.1 Introduction

The conformal theory of Finsler metrics based on the theory of Finsler spaces by M. Matsomoto, M. Hashiguchi ([3], [7]) in 1976 studied the conformal change of a Finsler metric namely

$$\bar{L}(x, y) = e^{\sigma(x)} L(x, y) .$$

G.Randers ([6], [7],) in the year 1941 introduced a special metric

$ds = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i$ in a view point of General theory of Relativity. Since then many Physicist had developed the General theory of Relativity. This metric was first recognized by R.S.Ingarden and M .Matsumoto ([1], [4], [6],) in 1957, produced the (α, β) – Metric by generalizing Randers Metric [2]. The theory of Finsler space with (α,β) - metric has been developed into abundant

branch of Finsler Geometry. From stand point of Finsler Geometry itself

Randers metric is very interesting because its form of simple and properties of

Finsler spaces equipped this metric can be looked as Riemannian spaces equipped with the metric

$$L(\alpha,\beta) = \alpha + \beta .$$

The concept of Douglas space ([1], [8], [9], [10], and [11]) has been introducing by M. Matsumoto and S. Bacso as a generalization of Berward spaces from stand point of view of geodesic equation. Finsler space is said to be of Douglas space if

$$D_{ij} = G_i\, y_j - G_j\, y_i$$

are homogeneous polynomial in y^i of degree – 3. It has been shown by M . Matsomoto in papers ([1], [7], [8], [9], [10]) that $F^n = (M^n, L)$ is a Douglas type iff the Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n-1}\left(G_{ijk}\, y^h + \delta^h_i G_{jk} + \delta^h_j G_{ik} + \delta^h_k G_{ij}\right)$$

vanishes identically, where G^h_{ijk} is hv – curvature tensor of Berward connection $B\Gamma$. The Conformal Randers change can be consider as generalization of Conformal as well as Randers change because writing $\beta = 0$ it reduces to Conformal change and when $\sigma(x) = 0$ it reduces to Randers change .It is compositions of Randers change and Conformal change.

In present chapter we shall workout the condition under which a change of Finsler metric $L(\alpha, \beta) \to \bar{L}(\alpha, \beta) = e^{\sigma(x)}\{L(\alpha, \beta) + \beta\}$ is that Conformal Randers change of Finsler spaces of Douglas type remains to be Douglas type.

4.2. Preliminaries.

Let $\alpha(x, y) = \sqrt{a_{ij}(x) y^i y^j}$ be Riemannian metric and

$\beta(x, y) = b_i(x) y^i$ be a differentiable one-form in an n-dimensional differentiable manifold M^n. If a fundamental metric function $L(\alpha,\beta)$ is positively homogeneous of degree one in α and β in M^n, then

$F^n = (M^n, L(\alpha,\beta))$ is called a Finsler space with (α,β)-metric [5]. The space $R^n = (M^n,\alpha)$ is called a Riemannian space associated with F^n [5], Christoffel symbols of R^n are indicated by $\gamma_j{}^i{}_k$, and covariant differentiation with respect to $\gamma_j{}^i{}_k(x)$ by ∇. We shall use the symbols as follows:

(4.2.1) $$r_{ij} = \frac{1}{2}(\nabla_j b_i + \nabla_i b_j) \ , \ s_{ij} = \frac{1}{2}(\nabla_j b_i - \nabla_i b_j) \ ,$$

$$s^i{}_j = a^{ir} s_{rj} \quad , \quad s_j = b_r s^r{}_j .$$

It is to be noted that $s_{ij} = \frac{1}{2}(\partial_j b_i - \partial_i b_j)$. Throughout the paper the symbols ∂_j and $\dot{\partial}_j$ stand for $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ respectively. We are concerned with the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j)$ which is given by

$$2G^i(x,y) = g^{ij}(y^r \dot{\partial}_j \partial_r F - \partial_j F), \text{where} \quad F = L^2/2, \quad G^i_j = \dot{\partial}_j G^i$$

and $G^i_{jk} = \dot{\partial}_k G^i_j$.

The Finsler space F^n is said to be of Douglas type (or Douglas space) [1] if $D^{ij} = G^i(x, y)\, y^j - G^j(x, y)\, y^i$ are homogeneous polynomial in y^i of degree three. we shall denote the "homogeneous polynomials in y^i of degree r" by hp(r).

For a Finsler space F^n with (α, β)-metric ([3], [5]), we have

(4.2.2) $$2G^i = \gamma^i_{00} + 2B^i ,$$

where

(4.2.3) $$B^i = \frac{E}{\alpha} y^i + \frac{\alpha L_\beta}{L_\alpha} s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} C^* \left(\frac{y^i}{\alpha} - \frac{\alpha}{\beta} b^i \right),$$

$E = \dfrac{\beta L_\beta}{L} C^*$, $\quad C^* = \dfrac{\alpha\beta(r_{00} L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}$, $b^i = a^{ij} b_j$, $\quad \gamma^2 = b^2 \alpha^2 - \beta^2$, $\quad b^2 = a^{ij} b_i b_j$

and the subscript α and β in L denote the partial differentiation with respect to α and β respectively. Since $\gamma^i_{00} = \gamma^i_{jk}(x) y^j y^k$ is homogenous polynomial in (y^i) of degree two, we have

Proposition [7].4.2.1. A Finsler space with (α, β)-metric is a Douglas space if and only if

$$\mathbf{B^{ij} = B^i y^j - B^j y^i \text{ are hp(3).}}$$

Equation (4.2.3) gives

(4.2.4) $$B^{ij} = \frac{\alpha L_\beta}{L_\alpha} (s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i)$$

Lemma [8] 4.2.1. If $\alpha^2 \equiv 0$ (mod - β) that is $a_{ij}(x)\, y^i y^j$ contains $b_i(x) y^i$ as a factor then the dimension is two and $b^2 = 0$ .In this case ,we have $\delta = d_i(x) y^i$ satisfying $\alpha^2 = \beta\, \delta$ and $d_i(x) b^i = 2$.

4.3. Conformal Randers change of Finsler spaces with (α, β)-metric of Douglas type

Let $F^n = (M^n, L)$ and $\overline{F}^n = (M^n, \overline{L}(\alpha, \beta) = e^\sigma [\, L(\alpha, \beta) + \beta \,]\,)$be two Finsler spaces on the same underlying manifold M^n. If we have a function σ(x) in each coordinate neighborhoods of M^n such that

$$\overline{L}(\alpha, \beta) = e^\sigma \; [\, L(\alpha, \beta) + \beta \,]$$

then F^n is called conformal Randers to $\overline{F}^n$, and change $L \to \overline{L}$ of metric is called conformal Randers change of (α, β) metric . For (α,β) metric

$$\overline{L}\,(\alpha,\beta) = e^\sigma [L(\alpha,\beta) + \beta\,] = L\,(\,\overline{\alpha},\; \overline{\beta})\,,$$

(by homogeneity). Therefore, a Conformal Randers change of (α,β) metric is expressed as $(\alpha,\beta) \to (\,\overline{\alpha},\, \overline{\beta})$ where $\overline{\alpha} = e^\sigma\alpha$, $\overline{\beta} = e^\sigma \beta$,

Therefore, we have $\overline{y}^i = y^i$, $\overline{y}_i = e^{2\sigma} y_i$, $\overline{a}_{ij} = e^{2\sigma} a_{ij}$, $\overline{b}_i = e^\sigma b_i$,

$$\overline{a}^{ij} = e^{-2\sigma} a^{ij}\,, \quad \overline{b}^{\,i} = e^{-\sigma} b^i \text{ and } \bar{b}^2 = b^2 .$$

Proposition 4.3.1: In a Finsler spaces with (α,β) – metric the length b of b_i with respect to the Riemannian α is invariant under conformal Randers change.

The Conformal Randers change $(\alpha, \beta) \to (\,\overline{\alpha},\, \overline{\beta})$ gives rise to the conformal change of $R^n : \alpha \to \overline{\alpha} = e^\sigma \alpha$ and hence we get the

Conformal Randers change of Christoffel symbols γ^i_{jk} are same as Conformal change of Christoffel symbols γ^i_{jk} . So it follows [1] as

(4.3.3) $$\bar{\gamma}^i_{jk} = \gamma^i_{jk} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk}$$

where $\sigma_j = \partial_j \sigma$ and $\sigma^i = a^{ij}\sigma_j$.

From (4.3.2) and (4.3.3) we have the following conformal Randers change

(4.3.4) (a) $$\bar{\nabla}_j \bar{b}_i = e^{\sigma}(\nabla_j b_i - b_j \sigma_i + \rho a_{ij})$$

(b) $$\bar{r}_{ij} = e^{\sigma}[r_{ij} - \frac{1}{2}(b_i \sigma_j + b_j \sigma_i) + \rho a_{ij}]$$

(c) $$\bar{s}_{ij} = e^{\sigma}[s_{ij} + \frac{1}{2}(b_i \sigma_j - b_j \sigma_i)]$$

(d) $$\bar{s}^i_j = e^{-\sigma}[s^i_j + \frac{1}{2}(b^i \sigma_j - b_j \sigma^i)]$$

(e) $$\bar{s}_j = s_j + \frac{1}{2}(b^2 \sigma_i - \rho b_i).$$ where $\rho = b_r \sigma^r$,

From (4.3.3) and (4.3.4) we can easily obtain the following:

(4.3.5) (a) $$\bar{\gamma}_0{}^i{}_0 = \gamma_0{}^i{}_0 + 2\sigma_0 y^i - \alpha^2 \sigma^i$$

(b) $$\bar{r}_{00} = e^{\sigma}(r_{00} + \rho\, \alpha^2 - \sigma_0 \beta)$$

(c) $$\bar{s}_0^i = e^{-\sigma}[s_0^i + \frac{1}{2}(b^i\sigma_0 - \beta\sigma^i)]$$

(d) $$\bar{s}_0 = s_0 + \frac{1}{2}(b^2\sigma_0 - \rho\beta)$$

To find the Conformal Randers change of B^{ij} given in (4.2.3), we first find the Conformal Randers change of C* given in (4.2.3). $\bar{\beta}$

Since $\bar{L}(\alpha, \beta) = e^{\sigma}\ [L(\alpha, \beta) + \beta]$, we have (4.3.6)

$$\bar{L}_{\bar{\alpha}} = L_\alpha, \qquad \bar{L}_{\bar{\alpha}\bar{\alpha}} = e^{-\sigma} L_{\alpha\alpha},$$

$$\bar{L}_{\bar{\beta}} = L_\beta + 1 \qquad \bar{\gamma}^2 = e^{2\sigma}\gamma^2.$$

From (4.2.3), (4.3.4) and (4.3.5), we have

(4.3.7) $$\bar{C}^* = e^{\sigma}(C^* + D^*),$$

where

(4.3.8) $$D^* = \frac{\alpha\beta[(\rho\alpha^2 - \sigma_0\beta)L_\alpha - \alpha\{2s_0 + (b^2\sigma_0 - \rho\beta)(L_\beta + 1\}]}{2(\beta^2 L_\alpha + \gamma^2\alpha L_{\alpha\alpha})}$$

Hence Conformal Randers change of B^{ij} is written in the form

(4.3.9) $$\bar{B}^{ij} = B^{ij} + C^{ij},$$

where

(4.3.10)

$$2C^{ij} = \frac{1}{2L_\alpha}[2\alpha e^\sigma(s^i{}_0 y^j - s^j{}_0 y^i) + (L_\beta + e^\sigma)\alpha\{\sigma_0(b^i y^j - b^j y^i) -$$

$$\beta(\sigma^i y^j - \sigma^j y^i)\}] + \frac{D * \alpha^2 L_{\alpha\alpha}(b^i y^j - b^j y^i)}{\beta L_\alpha}$$

Theorem 4.3.1. A Douglas space with (α,β) –metric transformed to a Douglas space with (α,β) –metric under Conformal Randers change if and only If C^{ij} defined in equation (4.3.10) is homogeneous polynomial in y^i of 3.

4 .4. Conformal Randers Change of some particular (α,β) metric

For a **Randers metric**

we have , ***L* = α+β**

so that $L_\alpha = 1$, $L_\beta = 1$ and $L_{\alpha\alpha} = 0$ .Then we have

(4.4.1)

$$2C^{ij} = \alpha[2e^\sigma(s^i{}_0 y^j - s^j{}_0 y^i) + (1 + e^\sigma)\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\}].$$

We know that [6] Finsler spaces with Randers metric is Douglas space iff $s_{ij} = 0$. Under this condition equation (4.4.1) becomes

$$2C^{ij} = \alpha(1+e^{\sigma})\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\}.$$

Since α is irrational function in y^i , from above it follows that C^{ij} are hp(3) if and only if

$$\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i) = 0 \tag{4.4.2}$$

The equation (4.4.2) may also be written as ,

$$(\sigma_k\delta_h^j + \sigma_h\delta_k^j)b^i - (b_k\delta_h^j + b_h\delta_k^j)\sigma^i - (\sigma_k\delta_h^i + \sigma_h\delta_k^i)b^j + (b_k\delta_h^i + b_h\delta_k^i)\sigma^j = 0 \tag{4.4.3}$$

Contracting (4.4.3) by j and h we get $b_i\sigma_j - b_j\sigma_i = 0$ which gives

$$\sigma_i = \frac{\rho}{b^2} b_i .$$

Conversely, if $S_{ij} = 0$ and $\sigma_i = \frac{\rho}{b^2} b_i$ then(4.4.1) gives $C^{ij} = 0$ .Hence equation (4.3.9) gives $\overline{B}^{ij} = B^{ij}$.

Thus we have

Theorem - 4.4.1. The Douglas space with Randers metric transformed to a Douglas space under Conformal Randers change if and only if $S_{ij} = 0$ and $\sigma_i = \frac{\rho}{b^2} b_i$, where $\rho = b_r\sigma^r$.

For a **Kropina metric**,

we have $\boldsymbol{L} = \dfrac{\alpha^2}{\beta}$,

so that $L_\alpha = \dfrac{2\alpha}{\beta}$, $L_{\alpha\alpha} = \dfrac{2}{\beta}$, $L_\beta = -\dfrac{\alpha^2}{\beta^2}$. Hence the value of D* given by (4.3.7) reduces to

(4.4.4)

$$D^* = \frac{1}{4b^2\alpha^2}[\beta^2(\rho\alpha^2 - \sigma_0\beta) - \alpha e^{\sigma}\{2s_0\beta^2 + (b^2\sigma_0 - \rho\beta)(\beta^2 - \alpha^2 e^{-\sigma})\}]$$

Therefore, the value of C^{ij} given by (4.2.10) reduces to,

(4.4.5)

$$C^{ij} = \frac{1}{2}e^{\sigma}\beta(s^i{}_0 y^j - s^j{}_0 y^i) + (\frac{e^{\sigma}\beta}{4} - \frac{\alpha^2}{4\beta})\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\}$$

$$- (b^i y^j - b^j y^i)[(\frac{\beta^2\rho}{2b^2} - \frac{\sigma_0\beta^3}{2b^2\alpha^2}) - \frac{e^{\sigma}\beta^2 s_0}{b^2\alpha} + (b^2\sigma_0 + \rho\beta)\ (\frac{\beta^2\rho}{2b^2} - \frac{\sigma_0\beta^3}{2b^2\alpha^2})].$$

Since

$\frac{1}{2}e^{\sigma}\beta(s^i{}_0 y^j - s^j{}_0 y^i) + \frac{e^{\sigma}\beta}{4}\{\sigma_0(b^i y^j - b^j y^i) - \beta(\sigma^i y^j - \sigma^j y^i)\} - (b^i y^j - b^j y^i)[\frac{\beta^2\rho}{2b^2} - \frac{\alpha}{2b^2}(b^2\sigma_0 + \rho\beta)]$ are hp (3).These terms may be neglected in our future discussion and we treat only

$$H_{ij} = -\frac{\alpha^2}{4\beta}\sigma_0(b^i y^j - b^j y^i) - (b^i y^j - b^j y^i)[\frac{e^{\sigma}\beta^2}{2b^2\alpha}(b^2\sigma_0 + \sigma\beta)]$$

Above equation may be written as

(4.4.6) $4b^2\alpha^2\beta H^{ij} = -(b^i y^j - b^j y^i)\,[\alpha^4 b^2 \sigma_0 - 2\sigma_0 \beta^4 - 4\, e^{\sigma} s_0 \alpha\beta^3 + 2\, e^{\sigma} b^2 \sigma_0 \beta^3 - 2\rho\, e^{\sigma}\alpha\beta^4]$

Equating rational and irrational terms we get

(4.4.7) $4b^2\alpha^2\beta H^{ij} = -(b^i y^j - b^j y^i)\,(\alpha^4 b^2 \sigma_0 - 2\sigma_0 \beta^4)$ and

(4.4.8) $2e^{\sigma}\beta^3(b^i y^j - b^j y^i)\,(2s_0 + b^2\sigma_0 + \rho\,\beta) = 0$.

Take n > 2 , $\alpha^2 \neq 0 \pmod{-\beta}$ [8] . If $(b^i y^j - b^j y^i) = 0$,by transvection of $b_i y_j$ we get $b^2\alpha^2 - \beta^2 = 0$ which gives rise to contradiction. So we must have ,

(4.4.9) $(2s_0 + b^2\sigma_0 + \rho\,\beta) = 0$

Also from equation (4.4.7)

$$b^2\alpha^2\,[\,4\beta H^{ij} + (b^i y^j - b^j y^i)\,\alpha^2\sigma_0] + 2\,\sigma_0\,\beta^4(b^i y^j - b^j y^i) = 0$$

which implies $4\beta H^{ij} + (b^i y^j - b^j y^i)\,\alpha^2\sigma_0 = 0$ and $\sigma_0 = 0$

Therefore from equation (4.4.9) $2s_0 = -\rho\,\beta$.

Thus we have ,

Theorem 4.4.2. A Finsler spaces $\bar{F}^n$ (n > 2) which is obtained by conformal Randers change of a Kropina space F^n with $b^2 \neq 0$ is of Douglas type if and only if $\sigma_0 = 0$ and $2s_0 + \rho\,\beta = 0$,

where $\rho = b_r\sigma^r$,.

For **a Finsler spaces with metric**

(4.4.10) $$\mathbf{L} = \alpha + \frac{\beta^2}{\alpha} \, .$$

Under Randers change it become

(4.4.11) $$L^* = \alpha + \beta + \frac{\beta^2}{\alpha} \, .$$

The (α, β) –metric (4.4.11) is called an Approximate Matsumoto metric.

Lemma[10] 4.4.1.– A Finsler spaces with an Approximate Matsumoto metric is a Douglas spaces if and only if $\alpha^2 \neq 0$

(mod - β) , $b^2 \neq 1$, $\Delta_j b_i = k \{ (1+ 2b^2) a_{ij} - 3 b_i b_j \}$ where

$k = \frac{h}{b^2 - 1}$, $h(x)$ is scalar function, that is b_i is gradient vector.

(1) $\alpha^2 \equiv 0$ (mod - β) : $n = 2$,

$$\Delta_j b_i = \frac{1}{2} \{ v_i(d_j + 3b_j) + v_j(d_i + 3b_i) \} \text{where } v_0 = v_i (x) y^i \, .$$

Also Conformal change of an Approximate Matsumoto metric is approximate Matsumoto metric. Take

(4.4.12) $$A_{ij} = \Delta_j b_i - k \{ (1+ 2b^2) a_{ij} - 3 b_i b_j \} = 0$$

Assume (F^n, $\bar{L} = e^{\sigma}(\alpha + \beta + \frac{\beta^2}{\alpha})$) is Douglas Spaces .Then $\bar{A}_{ij} = 0$ This can be expressed as

(4.4.13) $$e^{\sigma}(A_{ij} + \rho a_{ij} - \sigma_i b_j) = 0$$

In view of equation (4.4.10), the equation (4.4.13) become

$$\rho a_{ij} = \sigma_i b_j,$$

contracting by y^j gives

(4.4.14) $$\rho y_i = \sigma_i \beta$$

Again if n = 2, $\alpha^2 \equiv 0$ (mod - β) assume

(4.4.15) $$w_{ij} = \Delta_j b_i - \frac{1}{2}\{ v_i(d_j + 3b_j) + v_j(d_i + 3b_i)\} = 0$$

The $\bar{w}_{ij} = 0$ implies

(4.4.16) $$e^{\sigma}(W_{ij} + \rho a_{ij} - \sigma_i b_j) = 0 ,$$

we note that $\bar{v}_i = e^{\sigma} v_i$.

In view of (4.4. 14), the equation (4.4.16) becomes $(\rho a_{ij} - \sigma_i b_j) = 0$.

After contacting by y^j , $\rho Y_i = \sigma_i \beta$.

Thus in both cases we see that ,

Theorem 4.4.3. **A Finsler space $\bar{F}^n$ (n > 2) which is obtained by conformal Randers change of F^n = (M^n , L= $\alpha + \frac{\beta^2}{\alpha}$) with $b^2 \neq 0$ is of Douglas type , remains to be Douglas type if and only if**

$$\rho Y_i = \sigma_i \beta \quad \text{where } \rho = b_r \sigma^r .$$

Lemma [11] 4.4.2. Let F^n be a Dauglas space with (α,β) metric

L = ($c_1\alpha + c_2\beta + \frac{\alpha^2}{\beta}$) for which $b^2 \neq 0$ and if $\alpha^2 \neq 0$ (mod - β), then there exists a scalar function u(x) and a tensor function V_{ij}(x) such that $\nabla_j b_i = (r_{ij} + S_{ij})$ is given by

$$S_{ij} = \frac{1}{b^2} (b_i S_j - b_j S_i) - \frac{1}{(n-1)} V_{ij}$$

$$r_{ij} = \frac{c_2}{2c_1} (b_i S_j + b_j S_i) - 4a_{ij}.$$

For a Finsler space with metric $L = (\alpha + \frac{\alpha^2}{\beta})$.

Under Randers change above metric becomes

$$L^* = (\alpha + \beta + \frac{\alpha^2}{\beta}) . \tag{4.4.17}$$

The conformal change of fundamental metric (4.4.17) is a metric of same type . Take

(4.4.18) $\qquad A_{ij} = S_{ij} - \frac{1}{b^2}\,(b_i S_j - b_j S_i) + \frac{4}{(n-1)}\, V_{ij} = 0$ $\qquad$ and

(4.4.19) $\qquad W_{ij} = \boldsymbol{r}_{ij} - \frac{1}{2}(b_i S_j + b_j S_i) + 4\boldsymbol{a}_{ij} = 0$.

Assume (M^n, $\bar{L} = e^{\sigma}(\alpha + \beta + \frac{\alpha^2}{\beta})$) is Douglas type. Then $\bar{A}_{ij} = 0$ and $\bar{W}_{ij} = 0$ .But $\bar{A}_{ij} = e^{\sigma} A_{ij}$, $\bar{V}_{ij} = e^{\sigma} V_{ij}$ so we get $\bar{A}_{ij} = 0$ if $A_{ij} = 0$. Also

(4.4.19) $\qquad \bar{W}_{ij} = W_{ij} + e^{\sigma}(\rho\, a_{ij} + \frac{1}{2}\, b_i b_j - \frac{2+b^2}{4}\,(b_i \sigma_j + b_j \sigma_i)$)

In view of (4.4.19) , $\bar{W}_{ij} = 0$ implies

(4.4. 20) $\qquad \rho\, a_{ij} + \frac{1}{2}\, b_i b_j = \frac{2+b^2}{4}\,(b_i \sigma_j + b_j \sigma_i)$

Contracting by b^j we get $\rho\, b_i = \sigma_i b^2$.

Thus we have,

Theorem 4.4.4. A Finsler space $\bar{F}^n$ (n > 2) which is obtained by Conformal Randers change of a (M^n , $L = \alpha + \frac{\alpha^2}{\beta}$) of Douglas type remains to be Douglas type if and only if $\rho\, b_i = \sigma_i b^2$, where

$$\boldsymbol{\rho = b_r \sigma^r}\ .$$

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Chapter 5

EXPONENTIAL (α,β)-METRIC

5.1. Introduction

The theory of Finsler space with (α, β) - metric has been developed into faithful branch of Finsler Geometry by M. Matsumoto([3],[4]) .A Finsler metric L(x,y) defined on differentiable manifold M^n is called an (α, β) - metric, if L is a positively homogeneous function of degree one of a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and a one form $\beta = b_i(x)y^i$. A Finsler space is a Berwald space [7], if the Berwald connection is linear.

The concept of Douglas space ([2], [8], and [9]) has been introduced by M. Matsumoto and S. Bacso as a generalization of Berward spaces from stand point of view of geodesic equation. Finale space is said to be of Douglas space if $D_{ij} = G_i y_j - G_j y_i$ are homogeneous polynomial in y^i of degree 3. It has been shown by M. Matsomoto that $F^n = (M^n, L)$ is a Douglas type iff the Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n-1}\left(G_{ijk}y^h + \delta^h_i G_{jk} + \delta^h_j G_{ik} + \delta^h_k G_{ij}\right)$$

vanishes identically, where G^h_{ijk} is hv – curvature tensor of Berward connection $B\Gamma$. A Finsler space $F^n = (M^n, L)$ is a locally Minkowski space ([5], [6]) if M^n is covered by a coordinate neighbourhood system (x^i) in each of -which L is a function of y^i only. A Finsler space is projectively flat if it is projective to a locally Minkowski spaces . The Exponential (α, β) metric

(5.1.1) $$L = \alpha\, e^{\frac{\beta}{\alpha}} + \beta$$

was firstly introduced by Yu Yao Yong ([11], [12]). It is expressed as

$$L = \alpha + 2\beta + \frac{\beta^2}{2\alpha} + \frac{\beta^3}{6\alpha^2} + \ldots..$$

for $|\beta| < |\alpha|$.If we regard $b_i(x)$ as very small numerically and neglect all the power greater than two of β , then L(α, β) is called 2nd approximated to exponential (α, β) metric .So we use

$$(5.1.2) \qquad L = \alpha + 2\beta + \frac{\beta^2}{2\alpha} \ .$$

In this chapter, we have worked out the Finsler space with second approximate exponential (α, β) metric in which all powers greater than 2 of β are neglected,to be a Berwald space ,Douglas space &Protectively flat space . We also investigate the Berwald connection of this metric.

Throughout the chapter we shall effectively use the following expressions. The derivatives of approximate exponential (α,β) metric L with respect to α & β are :-

$$(5.1.3) \qquad L_\alpha = 1 - \frac{\beta^2}{2\alpha^2}, \qquad L_\beta = 2 + \beta/\alpha$$

$$L_{\alpha\alpha} = \frac{\beta^2}{\alpha^3} \qquad L_{\beta\beta} = 1/\alpha, \qquad L_{\alpha\beta} = -\beta/\alpha^2$$

5.2 Berwald Connection of F^n

Throughout the chapter the symbols ∂_j and $\dot\partial_j$ stand for $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ respectively. We are concerned with the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ which is given by

$$2G^i(x,y) = g^{ij}(y^r\dot{\partial}_j\partial_r F - \partial_j F),$$

where $F = L^2/2$, $G^i_j = \dot{\partial}_j G^i$ and $G^i_{jk} = \dot{\partial}_k G^i_j$.

we shall denote the "homogeneous polynomials in y^i of degree r" by hp(r).

For a Finsler space F^n with (α, β)-metric ([3], [5]), we have

(5.2.1) $$2G^i = \gamma^i_{00} + 2B^i,$$

where

(5.2.2) $$B^i = \frac{E}{\alpha}y^i + \frac{\alpha L_\beta}{L_\alpha}s^i_0 - \frac{\alpha L_{\alpha\alpha}}{L_\alpha}C^*\left(\frac{y^i}{\alpha} - \frac{\alpha}{\beta}b^i\right),$$

$$E = \frac{\beta L_\beta}{L}C^*, \qquad C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})},$$

$$b^i = a^{ij}b_j, \quad \gamma^2 = b^2\alpha^2 - \beta^2, \quad b^2 = a^{ij}b_i b_j$$

and the subscript α and β in L denote the partial differentiation with respect to α and β respectively. Since $\gamma^i_{00} = \gamma^i_{jk}(x)y^j y^k$ is homogenous polynomial of degree two in (y^i).

Proposition 5.2.1. The Geodesic coefficient G^i of exponential (α, β) metric (5.1.1) is given by

$$\textbf{(5.2.3)} \qquad 2G^i(x,y) = \gamma^i_{00} + 2\,\frac{(1+e^{\frac{\beta}{\alpha}})y^i}{L\delta^2}\,[r_{00}\,\alpha(\alpha-\beta) - \alpha^3\delta^2(1+e^{-\frac{\beta}{\alpha}})s_0] +$$

$$\frac{r_{00}(b^i\alpha^2-\beta y^i)}{\delta^2} - \frac{2s_0\alpha^2(1+e^{-\frac{\beta}{\alpha}})}{\delta^2} + \frac{2s^i{}_0\alpha^2(1+e^{-\frac{\beta}{\alpha}})}{\alpha-\beta},$$

where $\delta^2 = \alpha^2(1+b^2) - \alpha\beta - \beta^2$.

Berwald connection [4] satisfying the following axiomatic system:

B_1 - L-metrical: $L_{|i} = 0$

B_2 - (h) h- torsion tensor $T^i{}_{jk} = G^i{}_{jk} - G^i{}_{kj} = 0$,

B_3 - deflection tensor $D^i{}_j = y^k G_{kj}{}^i - G^i{}_j = 0$

B_4 – (v) hv – torsion tensor $P^i{}_{jk} = \dot\partial_k G^i{}_j - G^i{}_{kj} = 0$

B_5 - (h) hv – torsion tensor $C^i{}_{jk} = 0$,

Where the symbol (|) in B_1 – denotes the h – covariant differentiation with respect to the Finsler connection. We have from (5.2.1) in view of B_2, B_3 & B_4 , we get

$$\text{(5.2.4)} \qquad G^i{}_j = \dot\partial_j G^i = \gamma^i_{0j} + B^i{}_J$$

$$\text{and} \qquad G^i{}_{jk} = \dot\partial_j G^i{}_k = \gamma^i_{jk} + B^i{}_{Jk}$$

where we put $B^i{}_J = \dot\partial_j B^i$ & $B^i{}_{Jk} = \dot\partial_j B^i{}_j$.

The axiom $B_1 : L_{|i} = \partial_i L - G^r{}_i\, \dot\partial_r L = 0$ is written as

$$\text{(5.2.5)} \qquad L_\alpha B^k{}_{ji}\, y^j y_k + \alpha L_\beta (B^r{}_{ji}\, b_r - \nabla_j b_i)\, y^j = 0 .$$

Where $y_k = a_{ki}y^i$ and ∇_j is the differentiation with respect to γ^i_{jk} .

In view of (5.1.3) ,the (5.2.5) becomes

(5.2.4) $\qquad (2\alpha^2 - \beta^2)B^k_{ji}\, y^j y_k + 2\alpha^2(2\alpha+\beta)\,(B^r_{ji}\, b_r - \nabla_j b_i)\, y^j = 0$

Equating rational and irrational terms we get

(5.2.5) $\qquad (2\alpha^2 - \beta^2)B^k_{ji}\, y^j y_k + 2\alpha^2\beta\,(B^r_{ji}\, b_r - \nabla_j b_i)\, y^j = 0$

(5.2.6) $\qquad 4\alpha^3\,(B^r_{ji}\, b_r - \nabla_j b_i)\, y^j = 0$

In view of (5.2.6), equation (5.2.5) becomes $(2\alpha^2 - \beta^2)B^k_{ji}\, y^j y_k = 0$.

If $2\alpha^2 - \beta^2 = 0$ implies $2\alpha^2 = \beta^2$ which contradicts to our assumption $|\beta| < |\alpha|$

So $2\alpha^2 - \beta^2 \neq 0$. Therefore, $B^k_{ji}\, y^j y_k = 0$ & $(B^r_{ji}\, b_r - \nabla_j b_i)\, y^j = 0$.

Which implies $B^k_{ji}\, a_{kh} + B^k_{hi} a_{kj} = 0$ and $B^r_{ji}\, b_r - \nabla_j b_i = 0$.

Thus we get , $B^k_{ji} = 0$ & $\nabla_j b_i = 0$ .Therefore,the Finsler space [3] with metric (5.1.2) is Berwald space.

Theorem 5.2.1. Approximated exponential (α, β) metric (5.1.2) is a Berwald space iff $\nabla_j b_i = 0$ and Berwald connection is (γ^i_{jk} , γ^i_{0j} , 0).

The h-curvature tensor G^i_{jkh} of Berwald connection concides with the curvature tensor R^i_{jkh} of Riemannian connection .So we have $R^i_{jkh} = 0$. Therefore according to Kikuchi theorem if $R^i_{jkh} = 0 = \nabla_j b_i$,

(M^n ,L) is locally Minkowski.

Theorem 5.2.2. Approximated second order exponential (α, β) metric (5.1.2) Space is a locally Minkowski space iff $R^i_{jkh} = 0$ & $\nabla_j b_i = 0$.

5.3. Projective flat Finsler space with Approximated second order exponential (α,β) metric

A Finsler space (M^n ,L) is called Projectively flat [5] with rectangular geodesic if for any point p of M^n there exists a local coordinate neighborhood

(U,x) of *P* in which the geodesics can be represented by (n-1) linear equations of x^i .Here we shall find that Approximated second order exponential (α, β) metric (5.1.2) be Projectively flat .

We define $r_{ij} = \frac{1}{2}(\nabla_j b_i + \nabla_i b_j)$ $\quad s_{ij} = \frac{1}{2}(\nabla_j b_i - \nabla_i b_j)$,

$$s^i{}_j = a^{ir} s_{rj}, \quad s_j = b_r s^r{}_j,$$

$$\gamma_{ijk} = a_{jr}\, \gamma^r_{ik} \ .$$

A Finsler spaces F^n with an (α, β) metric is projectively flat ([5],[6]) if

(**5.**3.1) $$\frac{1+L_{\beta\beta}(\alpha^2b^2-\beta^2)}{\alpha L_\alpha}\{(\gamma^i_{00}-\gamma_{000}y^i/\alpha^2)/2+\frac{\alpha L_\beta}{L_\alpha}s^i_0+$$

$$\frac{L_{\alpha\alpha}}{L_\alpha}\frac{\alpha}{2\beta}(r_{00}-2\frac{\alpha L_\beta}{L_\alpha}s_0)\}.(\alpha^2b^i/\beta-y^i)=0,$$

Provided $\dfrac{1+L_{\beta\beta}(\alpha^2b^2-\beta^2)}{\alpha L_\alpha}\neq 0.$

In view of (5**.**1.2), equation (5**.**3.1) becomes,

(5**.**3.2) $$(4\alpha^4c-2\alpha^2\beta^2(c-1)-\beta^4)(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)+(16\alpha^7-8\alpha^5\beta^2+8\alpha^6\beta-4\alpha^4\beta^3)s^i_0+\{r_{00}(4\alpha^4-2\alpha^2\beta^2)-2\alpha^2(8\alpha^3+4\alpha^2\beta)s_0\}(\alpha^2b^i-\beta y^i)=0.$$

Equating rational and irrational terms,we get

(5**.**3.3) $$(2\alpha^2-\beta^2)s^i_0-2s_0(\alpha^2b^i-\beta y^i)=0$$

(5**.**3.4) $$(4\alpha^4c-2\alpha^2\beta^2(c-1)-\beta^4)(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)+r_{00}(4\alpha^4-2\alpha^2\beta^2)(\alpha^2b^i-\beta y^i)+4\alpha^4\beta[(2\alpha^2-\beta^2)s^i_0-2s_0(\alpha^2b^i-\beta y^i)]=0$$

In view of (5**.**3.3), (5**.**3.4) becomes

(5**.**3.5) $$(2\alpha^2c+\beta^2)(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)+2\alpha^2r_{00}(\alpha^2b^i-\beta y^i)=0.$$

From (5.3.3)

(5.3.6) $$2\alpha^2(s^i_0-s_0b^i)-\beta(s^i_0\beta-2s_0y^i)=0.$$

The terms ($s^i_0-s_0b^i$) must be factor of β ,so there exists $\lambda^i_0(x)$ such that

(5.3.7) $$s^i_0-s_0b^i=\beta\lambda^i$$

Transvecting (5.3.6) by y^i we get $\lambda_i = s_i$.

Thus (5.3.7) becomes

(5.3.8) $$S_{ij} = s_i b_j + s_j b_i$$

Again from (5.3.5) we see that γ_{000} must have a factor α^2 , so there exists 1-form $V_0 = V_i(x)y^i$ such that

(5.3.9) $$\gamma_{000} = V_0\,\alpha^2$$

In view of (5.3.10), equation (5.3.5) becomes

(5.3.10) $$2\alpha^2[(\gamma^i_{00} - V_0 y^i)c + r_{00}b^i] = \beta[2r_{00}\, y^i - (\gamma^i_{00} - V_0 y^i)\,\beta]$$

So there exists V^i_0 of hp (1) in y^i such that

(5.3.11) $$(\gamma^i_{00} - V_0 y^i)c + r_{00}b^i = \beta\, V^i_0$$

In view of (5.3.11) , equation (5.3.10) gives

(5.3.12) $$2\alpha^2\, V^i_0 = 2r_{00}\, y^i - (\gamma^i_{00} - V_0 y^i)\,\beta$$

Transvecting it by y_i and using (5.3.9) we get

(5.3.13) $$2r_{00} = V^i_0\, y_i\,.$$

In view of (5.3.13) , equation (5.3.11) gives

(5.3.14) $$(\gamma^i_{00} - V_0 y^i)c = \beta\, V^i_0 - V_{00}b^i$$

Eliminating $(\gamma^i_{00} - V_0 y^i)$ from (5.3.14) &(5.3.11) we get

(5.3.15) $$V_{i0}(2\alpha^2 c + \beta^2) = V_{00}\,(b_i\beta + 2c\, y_i)\,.$$

We define $E_{ij} = 2a_{ij}\, c + b_i b_j$

Then (5.3.15) becomes $V_{i0}\,E_{ij} = V_{00}\,E_{i0}$.

Which implies

(5.3.16) $E_{hj}\,V_{ik} + E_{jk}\,V_{ih} + E_{kh}\,V_{ij} = E_{ik}\,V_{hj} + E_{ih}\,V_{jk} + E_{ij}\,V_{kh}$.

The reciprocal of tensor E_{ij} is given by

$$E^{ij} = \frac{1}{2c}\left(a^{ij} - \frac{b_i b_j}{2c + b^2}\right).$$

Transvecting (5.3.16) by E^{hj} we get $V_{ik} = E\,E_{ik}$,

where we put $E = \dfrac{E^{hj} V_{hj}}{n}$.

Therefore we have ,

(5.3.17) $V_{ij} = E\,(2\,a_{ij}\,c + b_i b_j)$

and (5.3.13) becomes

(5.3.18) $r_{ij} = \dfrac{E}{2}(\,2a_{ij}\,c + b_i\,b_j)$

In view of equation (5.3.18) ,the equation (5.3.14) become

(5.3.19) $2\,\gamma^i_{jk} = V_i\,\delta^i{}_k + V_k\,\delta^i{}_j + \dfrac{E}{c}(b_j\,\delta^i{}_k + b_k\,\delta^i{}_j - 2a_{jk}b^i)$

Conversely, it can be easily verified that (5.3.7) is a consequence of (5.3.8) , (5.3.18) & (5.3.19).

Thus we have ,

Theorem 5.3.1. A Finsler space with second approximated exponential

(α, β) metric is Projectively flat iff we have (5.3.8),(5.3.18) , and the spaces is covered by coordinate neighbored in which the Christoffel symbol of associated Riemannian space with the metric α are written in the form (5.3.19).

5.4 - Douglas type

A Finsler space is of Douglas type ([2],[8],[9]) if and only if Douglas tensor vanishes identically. It is generalization of the Berward spaces from the view point of Geodesic equations .A Finsler spaces with (α, β) metric is a Douglas space if and only if ([2],[8])

(5.4.1) $$B^{ij} = \frac{\alpha L_\beta}{L_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^*(b^i y^j - b^j y^i).$$

Here $C^* = \dfrac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}$, $b^i = a^{ij} b_j$,

$$\gamma^2 = b^2 \alpha^2 - \beta^2, \quad b^2 = a^{ij} b_i b_j$$

In view of (5.1.3) , (5.4.1) become

(5.4.2) $$B^{ij} (2\alpha^2 - \beta^2)(2k\alpha^2 - 3\beta^2) = 2\alpha^2 (2\alpha + \beta))(2k\alpha^2 - 3\beta^2)$$
$$(s^i_0 y^j - S^j_0 y^i) + \alpha^2 [(2\alpha^2 - \beta^2) r_{00} - 4\alpha^2 s_0 (2\alpha + \beta)] (b^i y^j - b^j y^i) .$$

Suppose that F^n is a Douglas space, that is B^{ij} are hp(3). Separating (5.4.2), in rational & irrational terms of y^i ,we have

(5.4.3) $$(2k\alpha^2 - 3\beta^2)\,(s^i{}_0 y^j - S^j{}_0\, y^i) - 2\alpha^2 s_0\,(b^i y^j - b^j y^i) = 0$$

& (5.4.4) $$(2k\alpha^2 - 3\beta^2)\; B^{ij} - \alpha^2\, r_{00}\,(b^i y^j - b^j y^i) = 0.$$

The term $3\beta^2 B^{ij}$ of equation (5.4.4) seemingly does not contain α^2. Hence there exists V^{ij} in y^i of degree 3 such that $-3\beta^2 B^{ij} = \alpha^2\, V^{ij}{}_{(3)}$ i.e $B^{ij} = \alpha^2\, V^{ij}{}_{(1)}$. Here we divide the following discussion in the two cases

(1) $\alpha^2 \neq 0 \pmod{-\beta}$ (2) $\alpha^2 \equiv 0 \pmod{-\beta}$

Case – 1 In this case $B^{ij} = \alpha^2\, V^{ij}{}_{(1)}$ & (5.4.4) leads to

$$(2k\alpha^2 - 3\beta^2)\;\alpha^2\, V^{ij}{}_{(1)} - \alpha^2\, r_{00}\,(b^i y^j - b^j y^i) = 0\,.$$

Transvecting it by $b_i y_j$ we get $(2k\alpha^2 - 3\beta^2)\; V^{ij}\, b_i y_j = r_{00}\,(b^2\alpha^2 - \beta^2)$.

If $(2k\alpha^2 - 3\beta^2)$ contains $(b^2\alpha^2 - \beta^2)$ then there exists a scalar function $\lambda(x)$ such that

$$(2k\alpha^2 - 3\beta^2) = \lambda(x)\,(b^2\alpha^2 - \beta^2)\,.$$

This implies $\lambda = 3$ and $b^2 = 2$. Thus for $b^2 \neq 2$, $(2k\alpha^2 - 3\beta^2)$ is a factor of r_{00} .

So there exists a scalar function h(x) such that

$$h(x)(2k\alpha^2 - 3\beta^2) = r_{00}\,, \qquad \text{i.e}$$

(5.4.5) $$r_{ij} = h(x)\,(2k\, a_{ij} - 3b_i\, b_j)$$

Also (5.4.3) can be rewritten as

(5.4.6) $(2k\alpha^2 - 3\beta^2)\,(s^i{}_h\delta^j{}_k + s^i{}_k\delta^j{}_h - s^j{}_h\delta^i{}_k - s^j{}_k\delta^i{}_h)$

$- 2\alpha^2\,[(s_h\delta^j{}_k + s_k\delta^j{}_h)b^i - (s_h\delta^i{}_k + s_k\delta^i{}_h)b^j] = 0.$

Contracting (5.4.3) with a^{hk} ,we get

(5.4.7) $(2k\alpha^2 - 3\beta^2)s^{ij} = 2\alpha^2\,(b^is^j - b^js^i)$

Contracting (5.4.6) by with b^h ,we get

(5.4.8) $(2k\alpha^2 - 3\beta^2)\,(s^i{}_kb^j - s^j\delta^i{}_k + s^j\delta^i{}_k - s^j{}_kb^i) = 0$

Contracting j & k , we get $(2k\alpha^2 - 3\beta^2)s^i = 0.$

But $\alpha^2 \neq 0$ (mod - β) , so $s^i = 0$. Thus from (4) , $S^{ij} = 0$. That is

(5.4.9) $s^i = 0 = S^{ij}$.

From (5.4.5) and (5.4.9) ,we have

(5.4.10) $\nabla_j b_i = r_{ij} = h(x)\,(2k\,a_{ij} - 3b_ib_j)$

Conversely, if (5.4.8) holds,then it follows that

$B^{ij} = h(x)\,(b^iy^j - b^jy^i)\alpha^2$ which is h(p)- 3 .

So F^n is Douglas space . Therefore we have,

Theorem 5.4.1. A Finsler spaces F^n with second Approximated exponential (α, β) metric is Douglas space if and only if there exists a scalar function h(x) such that $\nabla_j b_i$ = h(x) (2k $a_{ij} - 3b_ib_j$) . In particular if

h(x) = 0 ,then F^n is Berwald spaces.

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Chapter 6

ON A WEAKLY-LANDSBERG SPACE OF SOME (α,β) METRIC

6.1. Introduction

Let M^n be n-dimensional differentiable manifold and $F^n = (M^n, L)$ be n-dimensional Finsler space, where L is a fundamental function. Let $g_{ij} = \dot{\partial}_i \dot{\partial}_j \frac{L^2}{2}$ be the fundamental tensor, where the oprator $\dot{\partial}_i$ means $\frac{\partial}{\partial y^i}$. Now we define geodesic coefficient G_i as ([8], [9])

$$G_i = \frac{1}{4} y^r \left(\partial_r \dot{\partial}_i L^2 \,\square\, \dot{\partial}_i L^2\right)$$

and $G^i = g^{ij} G_j$, where the symbol ∂_i means $\frac{\partial}{\partial x^i}$ and g^{ij} the inverse of metric of g_{ij}. The equation of geodesics (Canonically parameterized), of F^n is given by

$$\frac{d^2 x^i}{dt^2} = -2G^i(x, y) \quad \left(\frac{dx^i}{dt} = y^i\right).$$

The Berwald connection of Finsler space is defined by ([5], [8])

$(G^i_{jk}, G^i_j, 0)$, where the connection coefficients are defined by

$G^i_j = \dot{\partial}_j G^i, G^i_{jk} = \dot{\partial}_k (G^i_j)$. Let $x \in M$, $y \in T_x M$ and L is fundamental metric on M, TM = $\bigcup_{x \in M} T_x M$, the non-Riemannian quantity $C_y : T_x M \otimes T_x M \otimes T_x M \to R$ defined by

$$C_y(u, v, w) = C_{ijk}\, u^i v^j w^k, \text{ where } C_{ijk} = \frac{1}{2}\frac{\partial}{\partial k} g_{ij}.$$

The family $\{C_{ijk}\}$ is called Cartan tensor. It is well known that $C_{ijk} = 0$ iff L is Riemannian. In late 20th century, defined Berwald space as given bellow.

For $y \in T_x M_0$ define B_y : $T_x M \otimes T_x M \otimes T_x M \to T_x M$ by

$B_y(u, v, w) = G^i_{jkl} u^j v^k w^l \left.\dfrac{\partial}{\partial x^i}\right|_x$

where $G^i_{jkl} = \dfrac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$ is hv-curvature tensor of BΓ

B is called the Berwald curvature. A Finsler metric is called a Berwald metric[13] if B = 0. In this space G^i (x, y) are quadratic in $y = y^i \left.\dfrac{\partial}{\partial x^i}\right|_x \in T_x M$. It is single norm space [5] i.e., all the tangent space T_xM with induced norm L is linearly isometric to each other.

A Finsler space is called a Weakly Berwald space if $G_{mijk}\, g^{ij} = G_{mk}=0$

i.e.(Ricci curvature tensor) vanishes. It is also known as Mean hv-curvature tensor BΓ.

We are going to investigate the characteris of Finsler Space whose

$P_{hijk} = 0$. The condition $P_{hijk} = 0$ are equivalent to [9] $C_{hij|0} = 0$. Such a space of two dimension was first considered by G. Landsberg. Now, we shall investigate in general Landsberg Space, i.e., a non-Riemannian quantity defined on [13] the slit tangent bundle TM – {0} as $L_y : T_xM \otimes T_xM \otimes T_xM \to R$ by L_y (u, v, w) = $C_{ijk|0}$ $u^i\, v^j\, w^k$ where $u = u^i \left.\dfrac{\partial}{\partial x^i}\right|_x$, $v = v^i \left.\dfrac{\partial}{\partial x^i}\right|_x$, $w = w^i \left.\dfrac{\partial}{\partial x^i}\right|_x$. The family {$C_{ijk|0}$} are called Landsberg curvature. But for Landsberg space in BΓ, $y_m G^m_{jkh} = 0$ implies that $C_{jkh|0}$ = 0. So we conclude that the Finsler space with $y_m G^m_{jkh} = 0$ (BΓ) are called Landsberg space.

Now we define mean Cartan torsion as $I_y(u) = C_i\, u^i$ where

$$C_i = g^{jk} C_{ijk}, \qquad u = u^i \left.\frac{\partial}{\partial x^i}\right|_x, \qquad y \in T_x M - \{0\}.$$

The h-covariant derivative of Mean Cartan torsion along geodesics gives rise to the Mean Landsberg curvature $\tau_y : T_x M \to R$ defined by $\tau_y(u) = C_{i|0} u^i$. The families $\{C_{i|0}\}$ are called Mean Landsberg curvature. A Finsler metric is called Weakly Landsberg metric if $C_{i|0} = 0$. This space was first introduced by Z. Shen ([1], [11], [12], and [13]). But for Landsberg space in $B\Gamma$ $C_{ijk|0} = 0$ implies $y_m G^m_{ijk} = 0$. So for Weakly Landsberg space $y_m G^m_{ijk} g^{ij} = 0$. Also $y_m G^m_{ijk} g^{ij} = 0$ implies $G_{0k} = 0$. Thus a Finsler space is called a Weakly Landsberg space if $G_{0k} = 0$. This space also called Mean Landsberg space. Several Finslerian like M. Matsumoto, M. Hashiguchi, Z. Shen, A. Teyebi, G. S. Asanov have studied about Weakly Landsberg space.

The purpose of the present chapter is to investigate the condition that the Finsler space with some (α,β)-metric like Randers, Kropina and Matsumoto metric to be Weakly-Landsberg type.

6.2. Weakly-Landsberg space of (α,β)-metric

In the present section, we deal with the condition that a Finsler space with an (α,β)-metric be a Weakly Landsberg space.Let (M^n, L) be a Finsler space with an (α,β)-metric, where $\alpha = (a_{ij}(x)\, y^i y^j)^{1/2}$ and $\beta = b_i(x)\, y^i$. Here, the symbol (/) stands for h-□covariant derivative with respect to the Riemannian connection in space

(M,α) and γ^i_{jk} stand for the Christoffel symbols in the space (M,α). Let us list the symbols for the later use

$b^i = a^{ir}\, b_r$, $\quad b^2 = a^{rs}\, b_r\, b_s$, $\quad 2r_{ij} = b_{i|j} + b_{j|i}$, $\quad S_i = b_r\, S^r_i$

$2S_{ij} = b_{i|j} - b_{j|i}$, $\quad r^i_j = a^{ir}\, r_{rj}$, $\quad S^i_j$ = air Srj, $\quad r_i = b_r\, r^r_i$.

In [8], the geodesic coefficient function G^i of a Finsler space with an (α,β)-metric are given by ,

(6.2.1) $\quad 2G^i = \gamma^i_{00} + 2B^i$, where

$$B^i = \left(\frac{E^*}{\alpha}\right) y^i + \left(\frac{\alpha L_\beta}{L_\alpha}\right) S^i_0 - \left(\frac{\alpha L_{\alpha\alpha}}{L_\alpha}\right) C^* \left\{\frac{y^i}{\alpha} - \left(\frac{\alpha}{\beta}\right) b^i\right\}$$

(6.2.2) $$E^* = \left(\frac{\beta L_\beta}{L}\right) C^*$$

$$C^* = \frac{\alpha\beta(r_{00} L_\alpha - 2\alpha S_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}$$

$$\gamma^2 = b^2\alpha^2 - \beta^2, \qquad L_\alpha = \frac{\partial L}{\partial\alpha}, \qquad L_\beta = \frac{\partial L}{\partial\beta}, \qquad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha\partial\alpha},$$

$$L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}, \qquad L_{\alpha\alpha\alpha} = \frac{\partial^3 L}{\partial\alpha\partial\alpha\partial\alpha}.$$

Then, we have ,

$$G^i_j = \gamma^i_{0j} + B^i_j, \qquad G^i_{jk} = \gamma^i_{jk} + B^i_{jk} \text{ and } G^i_{jkl} = B^i_{jkl}$$

where $\quad \dot\partial_j B^i = B^i_j, \quad \dot\partial_k B^i_j = B^i_{jk}$ and $\dot\partial_l B^i_{jk} = B^i_{jkl}$.

Thus $\ y_m G^m_{ijk} g^{ij} = 0$ implies $y^r B^m_{mrk} = 0$ i.e., $B_{0k} = 0$.

Theorem 6.2.1 The necessary and sufficient condition for a Finsler space with an (α,β) - metric to be a Weakly Landsberg Space is that

$B^{m}_{mrk} y^{r} = 0$ **i.e., $B_{0k} = 0$.**

Remark: Throughout the present chapter, we say "homogeneous polynomials of degree r will be denoted as due to brevity.

6.3. Weakly – Landsberg space of Randers and Kropina metric

Consider the metric $L(\alpha,\beta) = \alpha + \beta$. It is well known that in a Randers space the geodesic coefficient G^i are given by

(6.3.1) $$2G^{i} = \left\{ \begin{matrix} i \\ 00 \end{matrix} \right\} \frac{(r_{00} - 2\alpha S_{0})}{\alpha+\beta} y^{i} + 2\alpha S^{i}_{0}.$$

This implies

(6.3.2) $$2G^{i}_{j} = 2\left\{ \begin{matrix} i \\ 0i \end{matrix} \right\} + \frac{1}{\alpha+\beta}(2r_{00} - 4\alpha S_{0}) - \frac{r_{00} - 2\alpha S_{0}}{(\alpha+\beta)}.$$

Contracting (6.3.2) by indices i, j gives

(6.3.3) $$2G^{i}_{i} = 2\left\{ \begin{matrix} i \\ 0i \end{matrix} \right\} + \frac{1}{\alpha+\beta}(r_{00} - 2\alpha S_{0}).$$

Again

(6.3.4) $$2G^{i}_{ik} = 2\left\{ \begin{matrix} i \\ ki \end{matrix} \right\} + \frac{1}{\alpha+\beta}(2r_{0k} - \frac{2a_{k0}}{\alpha} - 2\alpha S_{k}) - \frac{(r_{00} - 2\alpha S_{0})}{(\alpha+\beta)^{2}} . \left(\frac{a_{k0}}{\alpha} + b_{k} \right)$$

(6.3.5) $$2G^{i}_{ikl} = \frac{1}{\alpha+\beta}\left(2r_{kl} - \frac{2a_{kl}S_{0}}{\alpha} - \frac{2a_{k0}S_{l}}{\alpha} + \frac{2a_{k0}S_{0}a_{l0}}{\alpha^{3}} - \frac{2a_{l0}S_{k}}{\alpha} \right)$$

$$-\frac{2\left(r_{0k}-\frac{a_{k0}S_0}{\alpha}-2\alpha S_k\right)}{(\alpha+\beta)^2}\cdot\left(\frac{a_{l0}}{\alpha}+b_l\right)$$

$$-\frac{2\left(r_{00}-2\alpha S_k\right)}{(\alpha+\beta)^3}\cdot\left(\frac{a_{l0}}{\alpha}+b_l\right)\cdot\left(\frac{a_{k0}}{\alpha}+b_k\right)$$

$$-\frac{r_{00}-2\alpha S_0}{(\alpha+\beta)^2}\cdot\left(\frac{a_{kl}}{\alpha}-\frac{a_{k0}}{\alpha^2}\frac{a_{l0}}{\alpha}\right).$$

Contracting equation (6.3.5) by y^k, we get

$$y^k G^i_{ikl}=0 \quad \text{i.e.,} \quad G_{0l}=0.$$

Theorem 6.3.1. Randers space of n-dimensional, is always Weakly Landsberg space.

Remark: B. Li and Shen, Z. [6] in his paper quoted that a Randers space is Weakly Landsberg space iff β is parallel with respect to α. He quoted this result from the paper [7] which read as follows "A Randers space is Landsberg space iff β is parallel with respect to α. But Landsberg space is always Weakly Berwald space, the converse may not be true. So the result quoted in paper [7] is not applicable for Weakly Landsberg space.

Now, we consider Kropina metric $L(\alpha,\beta)=\alpha^2/\beta$. It is well known that in a Kropina space the geodesic coefficient function G^i (x, y) are given by

$$2G^i=\begin{Bmatrix} i \\ 0\,0\end{Bmatrix}-2\left(S_0+\frac{\beta}{\alpha^2}r_{00}\right)\frac{y^i}{b^2}-\frac{\alpha^2 S^i_0}{\beta}+\left(\frac{\alpha^2}{\beta}S_0+r_{00}\right)\frac{b^i}{b^2} \tag{6.3.6}$$

where $\begin{Bmatrix} i \\ 00 \end{Bmatrix}$ denotes the Christoffel symbols of the Riemannian metric α (*x, y*) [9].

Differentiating this equation by y^j, we get

(6.3.7) $$2G^i_j = 2\begin{Bmatrix} i \\ i0 \end{Bmatrix} - 2\left[S_j + \frac{b_j r_{00} + 2\beta r_{j0}}{\alpha^2} - \frac{2\beta r_{00} a_{j0}}{\alpha^4}\right]\frac{y^i}{b^2} - \frac{\alpha^2}{\beta}S^i_j$$

$$- S^i_0\left(\frac{2a_{j0}}{\beta} - \frac{\alpha^2}{\beta^2}b_j\right) + \frac{b^i}{b^2}\left\{S_j\frac{\alpha^2}{\beta} + S_0\left(\frac{2a_{j0}}{\beta} - \frac{\alpha^2}{\beta^2}b_j\right) + 2r_{j0}\right\}.$$

Contracting (6.3.7) by indices i and j, we get

(6.3.8) $$2G^i_j = 2\begin{Bmatrix} i \\ i0 \end{Bmatrix} - \frac{2}{b^2}\left(S_0 + \frac{br_{00}}{\alpha^2}\right) + 2S_0 + 2r_{i0}b^i$$

Differentiating this equation by y^j, we get

(6.3.9) $$G^i_{ij} = 2\begin{Bmatrix} i \\ i\,j \end{Bmatrix} + S_j\left(1 - \frac{1}{b^2}\right) + \frac{2r_{j0}\beta}{b^2\alpha^2} + \frac{1}{b^2}r_{00}\left(\frac{b_j}{\alpha^2} - \frac{2\beta a_{j0}}{\alpha^4}\right) + r_{ij}b^i.$$

Again differentiating equation (6.3.9) by y^k, we get (hv)-Ricci tensor in the following form

(6.3.10) $$G^i_{ijk} = \frac{2r_{jk}\beta}{b^2\alpha^2} + \frac{2r_{j0}}{b^2}\left(\frac{b_k}{\alpha^2} - \frac{2\beta a_{k0}}{\alpha^4}\right) + \frac{2r_{0k}}{b^2}\left(\frac{b_j}{\alpha^2} - \frac{2\beta a_{j0}}{\alpha^4}\right)$$

$$+ \frac{1}{b^2}r_{00}\left\{-\frac{b_j a_{k0}}{\alpha^4} - \frac{2\beta a_{jk}}{\alpha^4} - 2a_{j0}\left(\frac{b_k}{\alpha^4} - \frac{4a_{k0}\beta}{\alpha^6}\right)\right\}.$$

Contracting this equation by y^j gives

$$G^i_{ijk}\,y^j = r_{00}\left(\frac{b_k}{\alpha^2} - \frac{2\beta a_{k0}}{\alpha^4}\right).$$

So G^{i}_{ijk} is vanish only when either $r_{00} = 0$ or $\alpha^2 b_k - 2\beta a_{k0} = 0$. But $\alpha^2 \neq \beta$ (mod – 0), so, $r_{00} = 0$.

Theorem 6.3.2. Kropina Space of dimension-n, is Weakly-Landsberg space iff $r_{00} = 0$, i.e., β is closed with respect to α.

Example of Weakly Landsberg Kropina space, which is not Landsberg Kropina space.

Consider a covariant vector field $b_i(x)$ in an odd-dimensional Euclidean space so that $r_{ij} = 0$ and $S_{ij} \neq 0$ hold good. Let A_{ij} be an $n \times n$ type quadratic skew symmetric matrix and x^i denote coordinates of a point.

We consider the following vector field $b_i = A_{ij} x^j + c_i$, where c_i are constants. In this special case $b_{i|j} + b_{j|i} = 0$ and $b_{i|j} - b_{j|i} \neq 0$. So a Kropina space, which is generated by $b_i = A_{ij} x^j + c_j$ is a Weakly-Landsberg space, and it is not Landsberg space.

6.4. Weakly-Landsberg space of Matsumoto metric

A slope of a mountain is represented as the graph S of a differentiable function $x^3 = L(x^1, x^2)$, where (x^1, x^2, x^3) is a rectangular coordinate system in a three- dimensional Euclidean space. The Riemannian metric α is induced on S by

$$\alpha(x, y) = \{(y^1)^2 + (y^2)^2 + (b_1 y^1 + b_2 y^2)^2\}^{1/2}$$

Where $x = (x^i)$, $y = (y^i)$ and $b_i = \partial_I f$. We put

$$\beta(x,y) = b_1 y^1 + b_2 y^2$$

When a man can walk v meters per minute on horizontal plane, how many minutes does it takes him to walk along a road on S.

Recently, M. Matsumoto [9] shows that the man will walk in

s = $\int_0^1 L(x(t), y(t))dt$ minutes along a road x(t) on S ,by taking L as $L = \frac{\alpha^2}{\alpha-\beta}$,where 2ω is the gravitational constant , and we may regard slope of a mountain as a Finsler space with such time measure L.

Since $L = \frac{\alpha^2}{v\alpha-\omega\,\beta} = (\alpha/v)^2 / \{(\alpha/v) - (\omega\,\beta / v^2)\}$, we shall normalize as

$L = \frac{\alpha^2}{\alpha-\beta}$ and taking a general Riemannian metric α and a general none zero one-form β on a general differentiable manifold M,

An n-dimensional (α, β)-metric $L = \frac{\alpha^2}{\alpha-\beta}$ is called a slope metric or Matsumoto metric[1] and a Finsler space equipped with this metric has been called a Matsumoto metric.

Now we have

$$L_\alpha = \frac{\partial L}{\partial \alpha},\ L_\beta = \frac{\partial L}{\partial \beta},\ L_{\alpha\alpha} = \frac{\partial L_\alpha}{\partial \alpha},\ L_{\beta\beta} = \frac{\partial L_\beta}{\partial \alpha}\ \text{and}\ L_{\alpha\beta} = \frac{\partial L_\alpha}{\partial \beta}.$$

We have from equation (6.2.2)

$$C^* = \frac{\alpha-\beta}{2\beta\gamma}[r_{00}(\alpha-2\beta)-2\alpha^2 S_0],\ \text{where}\quad \gamma = \alpha(1+2b^2)-3\beta$$

$$E^* = \frac{1}{2\gamma}[(\alpha-2\beta)r_{00}-2\alpha^2 S_0]$$

and

(6.4.1) $$B^m = [r_{00}(\alpha-2\beta)-2\alpha^2 S_0]\left(\frac{y^m(\alpha-4\beta)+2\alpha^2 b^m}{2\gamma\alpha(\alpha-2\beta)}\right)+\frac{\alpha^2 S_0^m}{\alpha-2\beta}.$$

In view of equation (6.2.1), we have

(6.4.2) $$2G^m = \begin{Bmatrix} m \\ 0\,0 \end{Bmatrix}[r_{00}(\alpha-2\beta)-2\alpha^2 S_0]\left[\left(1-\frac{2\beta}{\alpha-2\beta}\right)\frac{y^m}{\alpha\gamma}+\frac{2\alpha b^m}{\gamma(\alpha-2\beta)}\right]$$
$$+\frac{2\alpha^2 S_0^m}{\alpha-2\beta}.$$

Equation (6.4.1) gives

(6.4.3) $B_i^m =$

$$[r_{00}(\alpha-2\beta)-2\alpha^2 S_0]\left[\frac{y^m}{\alpha\gamma}\left(\frac{2\beta\left(\frac{a_{i0}}{\alpha}-2b_i\right)}{(\alpha-2\beta)^2}-\frac{2b_i}{\alpha-2\beta}\right)-\left(1-\frac{2\beta}{\alpha-2\beta}\right)\frac{y^m}{(\alpha\gamma)^2}\right.$$
$$\left.\left(-3\frac{a_{i0}\beta}{\alpha}+2a_{i0}c-2\alpha b_i\right)+2b^m\left[\frac{a_{i0}}{\delta^2\alpha}-\frac{\alpha}{(\delta^2)^2}\left(2a_{i0}c-\frac{d\beta a_{i0}}{\alpha}-d\alpha b_i+12\beta b_i\right)\right]\right]$$
$$\left\{\left(1-\frac{2\beta}{\alpha-2\beta}\right)\frac{y^m}{\alpha\gamma}+\frac{2\alpha b^m}{\gamma(\alpha-2\beta)}\right\}\left\{2r_{i0}(\alpha-2\beta)+r_{00}\left(\frac{a_{i0}}{\alpha}-2b_i\right)-4a_{i0}S_0-2\alpha^2 S_i\right\}$$

$$+\frac{2S_i^m\alpha^2}{\alpha-2\beta}+2S_0^m\left(\frac{(\alpha-2\beta)a_{i0}+2\alpha^2 b_i}{(\alpha-2\beta)^2}\right).$$

Contracting by the indices i, j

(6.4.4) $$B^i_i = [r_{00}(\alpha-2\beta) - 2\alpha^2 S_0]\left(\frac{1}{\delta^2} - \frac{2\beta}{\alpha\delta^2} - \frac{2}{(\delta^2)^2}A\right)$$

$$+\frac{2\alpha}{\delta^2}[2r_{i0}b^i(\alpha-2\beta) + r_{00}\left(\frac{\beta}{\alpha} - 2b^2\right) - 4\beta S_0] + \frac{4\alpha^2 S_0}{(\alpha-2\beta)^2}.$$

We note that

$c = 1 + 2b^2$, $d = 5 + 4b^2$, $K = 2 + 16\, b^2$,

$\delta^2 = (\alpha c \;\square\; 3\beta).(\alpha \;\square\; 2\beta)$, $K_1 = 2c + 12\, b^2$

$$A = \frac{\partial}{\partial y^k}(\delta^2) = 2ca_{k0} - 2d\frac{\beta a_{k0}}{\alpha} - d\alpha b_k + 12\beta b_k$$

$B = 2(c\alpha^2 - d\alpha\beta + 6\beta^2)$ and $P = \alpha\beta K_1 - d\beta^2 - db^2\alpha^2$.

Differentiating equation (6.4.4) by y^j, we get

(6.4.5) $$B^i_{ij} = [2r_{j0}(\alpha-2\beta) + r_{00}\left(\frac{a_{j0}}{\alpha} - b_j\right) - 4a_j S_0 - 2\alpha^2 S_j]\left(\frac{1}{\delta^2} - \frac{3\beta}{\alpha\delta^2} - \frac{2}{(\delta^2)^2}P\right)$$

$$[r_{00}(\alpha-2\beta) - 2\alpha^2 S_0\Bigg[-\frac{1}{(\delta^2)^2}A\left(1 - \frac{2\beta}{\alpha}\right) - \frac{2}{\delta^2}\left(\frac{b_j}{\alpha} - \frac{\beta a_{j0}}{\alpha^3}\right)$$
$$-\frac{2}{(\delta^2)^2}\left(\frac{a_{j0}}{\alpha}\beta K_1 + K_1\alpha b_j - 2d\beta b_j - 2db^2 a_{j0}\right) + \frac{4(\alpha\beta K_1 - d\beta^2 - db^2\alpha^2)}{(\delta^2)^3}A\Bigg]$$

$$+\frac{2\alpha}{\delta^2}\Bigg[2r_{ij}b^i(\alpha-2\beta) + 2r_{i0}b^i\left(\frac{a_{j0}}{\alpha} - 2b_j\right) + 2r_{j0}\left(\frac{\beta}{\alpha} - 2b^2\right)$$

$$+r_{00}\left(\frac{b_j}{\alpha} - \frac{\beta a_{j0}}{\alpha^3}\right) - 4b_j S_0 - 4\beta S_j\Bigg] + 4S_j\left(\frac{\alpha}{\alpha-2\beta}\right)^2$$

$$+4S_0\left(\frac{2a_{j0}}{(\alpha-2\beta)^2}-\frac{2\alpha^2}{(\alpha-2\beta)^3}\left(\frac{a_{j0}}{\alpha}-2b_j\right)\right)$$

$$+\left[2r_{i0}b^i(\alpha-2\beta)+r_{00}\left(\frac{\beta}{\alpha}-2b^2\right)-4\beta S_0\right]\left(\frac{2a_{j0}}{\delta^2\alpha}-\frac{2\alpha}{(\delta^2)^2}A\right).$$

Contracting it by y^j gives

$$B^i_{ij}y^j=[r_{00}(\alpha-2\beta)-2\alpha^2S_0]\left[\frac{3}{\delta^2}-\frac{6\beta}{\alpha\delta^2}-\frac{1}{(\delta^2)^2}(10P+B)+\frac{2\beta B}{\alpha(\delta^2)^2}+\frac{4PB}{(\delta^2)^3}\right]$$

(6.4.6) $$+\frac{4\alpha}{\delta^2}\left[r_{i0}b^i(\alpha-2\beta)+r_{00}\left(\frac{\beta}{\alpha}-2b^2\right)-4\beta S_0\right]+4S_0\left(\frac{\alpha}{\alpha-2\beta}\right)^2.$$

Differentiating equation (6.4.6) by y^k

(6.4.7) $$\frac{\partial}{\partial y^k}\left(B^i_{ij}y^j\right)=[2r_{k0}(\alpha-2\beta)+r_{00}\left(\frac{a_{k0}}{\alpha}-2b_k\right)-4a_{k0}S_0-2\alpha^2S_k]\times$$

$$\left[\frac{3}{\delta^2}-\frac{6\beta}{\alpha\delta^2}-\frac{1}{(\delta^2)^2}(10P+B)+\frac{2\beta B}{\alpha(\delta^2)^2}+\frac{4PB}{(\delta^2)^3}\right]$$

$$+(r_{00}(\alpha-2\beta)-2\alpha^2S_0)\left[\frac{-3A}{(\delta^2)^2}-\frac{6}{\delta^2}\left(\frac{b_k}{\alpha}-\frac{\beta a_{k0}}{\alpha^3}\right)+\frac{6\beta A}{\alpha(\delta^2)^2}+\frac{2A}{(\delta^2)^3}(10P+B)\right.$$

$$-\frac{1}{(\delta^2)^2}\left[\left(\frac{a_{k0}\beta}{\alpha}+\alpha b_k\right)K_2+2K_3\beta b_k+K_4a_{k0}\right]+\frac{2B}{(\delta^2)^2}\left(\frac{b_k}{\alpha}-\frac{\beta a_{k0}}{\alpha^3}\right)$$

$$\left.+\frac{4B}{\alpha(\delta^2)^2}\left(2ca_{k0}-d\left(\frac{a_{k0}}{\alpha}\beta+\alpha b_k\right)+12\beta b_k\right)-\frac{4\beta AB}{\alpha(\delta^2)^3}-\frac{12ABP}{(\delta^2)^4}\right]$$

$$+\left(\frac{4a_{k0}}{\alpha\delta^2}-\frac{4\alpha A}{\delta^2}\right)\left(2r_{i0}b^i(\alpha-2\beta)+r_{00}\left(\frac{\beta}{\alpha}-2b^2\right)-4\beta S_0\right)$$

$$+\frac{4\alpha}{\delta^2}\left[2r_{ik}b^i(\alpha-2\beta)+2r_{i0}b^i\left(\frac{a_{k0}}{\alpha}-2b_k\right)+2r_{k0}\left(\frac{\beta}{\alpha}-2b^2\right)+r_{00}\left(\frac{b_k}{\alpha}-\frac{\beta a_{k0}}{\alpha^3}\right)\right.$$

$$\left.-4b_kS_0-4\beta S_k\right]+4S_k\left(\frac{\alpha}{\alpha-2\beta}\right)^2+4S_0\left(\frac{2a_{k0}}{(\alpha-2\beta)^2}-\frac{2\alpha^2}{(\alpha-2\beta)^3}\left(\frac{a_{k0}}{\alpha}-2b_k\right)\right).$$

Now,

$$B^i_{ijk}y^j=\frac{\partial}{\partial y^k}(B^i_{ij}y^j)-B^i_{ik}\,. \tag{6.4.8}$$

In view of equation (6.4.5), (6.4.7) equation (6.4.8) gives

$$B^i_{ijk}y^j=[2r_{k0}(\alpha-2\beta)+r_{00}\left(\frac{a_{k0}}{\alpha}-2b_k\right)-4a_{k0}S_0-2\alpha^2S_k]\times \tag{6.4.9}$$

$$\left[\frac{2}{\delta^2}-\frac{4\beta}{\alpha\delta^2}-\frac{1}{(\delta^2)^2}(8P+B)+\frac{2B\beta}{\alpha(\delta^2)^2}+\frac{4PB}{(\delta^2)^3}\right]$$

$$+(r_{00}(\alpha-2\beta)-2\alpha^2S_0)\left[\frac{-2A}{(\delta^2)^2}+\frac{6\beta A}{\alpha(\delta^2)^2}-\frac{4}{\delta^2}\left(\frac{b_k}{\alpha}-\frac{\beta a_{k0}}{\alpha^3}\right)\right.$$

$$-\frac{1}{(\delta^2)^2}\left\{\frac{a_{k0}\beta}{\alpha}(12K_1-4d)+\alpha b_k(11K_1-3d+4c)+\beta b_k(36-28d)\right.$$

$$\left.+a_{k0}(6c-22db^2)-48\frac{\beta^2}{\alpha}b_k+4d\frac{a_{k0}\beta^2}{\alpha^2}-24\frac{a_{k0}\beta^3}{\alpha^3}+\frac{2Ka_{k0}\beta}{\alpha}\right\}$$

$$\left.+\frac{1}{(\delta^2)^3}\left\{16AP+AB\left(2-\frac{4\beta}{\alpha}\right)\right\}-\frac{12ABP}{(\delta^2)^4}\right]$$

$$+2\left[2r_{i0}b^i(\alpha-2\beta)+r_{00}\left(\frac{\beta}{\alpha}-2b^2\right)-4\beta S_0\right]\left(\frac{a_{k0}}{\delta^2\alpha}-\frac{\alpha A}{(\delta^2)^2}\right)$$

$$+\frac{2\alpha}{\delta^2}\Bigg[2r_{ik}b^i(\alpha-2\beta)+2r_{i0}b^i\left(\frac{a_{k0}}{\alpha}-2b_k\right)+2r_{k0}\left(\frac{\beta}{\alpha}-2b^2\right)+r_{00}\left(\frac{b_k}{\alpha}-\frac{\beta a_{k0}}{\alpha^3}\right)$$

$$-4b_kS_0-4\beta S_k\,].$$

Thus we get,

Theorem 6.4.1. Matsumoto space is Weakly Landsberg space if r_{ij} = 0 and S_0 = 0, i.e., covariant vector field b_i is parallel with respect to the Riemannian connection of the Riemannian space, provided $\delta^2=\alpha^2(1+2b^2)-(5+4b^2)\alpha\beta+6\beta^2\neq 0$.

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CHAPTER 7

Quartic Rander's Change of Finsler Metric

Introduction : Let M^n be an $n-$dimensional differentiable manifold and F^n be a Finsler space equipped with a fundamental function $\alpha(x,y)$, $(y^i = \dot{x}^i)$ of M^n. If a differential 1-form $\beta(x,y) = b_i(x)y^i$ is given on M^n, then M. Matsumoto [4] introduced another Finsler space whose fundamental function is given by

$$L(x,y) = \alpha(x,y) + \beta(x,y) \tag{1.1}$$

This change of Finsler metric has been called $\beta-$change [11], [12]. If $\alpha(x,y)$ is a Riemannian metric, then the Finslerspace with a metric $L = \alpha + \beta$ where $\alpha = \{a_{ij}(x)y^iy^j\}^{1/2}$ is a Riemannian metric. This metric was introduced by G. Rander's [10]. In papers [1], [2], [3], [5] and [7] Randers spaces have been studied from a geometrical view point and various theorem were obtained. In 1978 S. Numata [9] introduced another $\beta-$change of Finsler metric given by $L = \mu + \beta$ where $\mu = \{a_{ij}(y)y^iy^j\}^{1/2}$ is a Minkowski metric and β is as above. This metric is of the similar form of Rander's one, but there are different tensor properties, because the Riemannian space with the metric α is characterized by $C^i_{jk} = 0$ and on the other hand the locally Minkowski space with metric μ by $R_{hijk} = 0$, $C_{hij|k} = 0$.

In 1978 M. Matsumoto and S. Numata [8] introduced the so called cubic metric on a differential manifold with the local coordinate x^i defined by

$$L = \{a_{ijk}(x)y^iy^jy^k\}^{1/3} \qquad (y^i = \dot{x}^i)$$

where $a_{ijk}(x)$ are component of a symmetric tensor field of $(0,3)$ type depending on the position x alone and has been called a cubic Finsler space. This cubic metric is of the similar form to the Riemannian metric α, which is characterized by $\dot\partial_i\dot\partial_j\dot\partial_k\alpha^2 = 0$, where as cubic metric L is characterized by $\dot\partial_i\dot\partial_j\dot\partial_k\dot\partial_p L^3 = 0$.

In the present paper we shall introduced a Finsler space with a metric

$$\overline{L}(x,y) = L(x,y) + \beta(x,y) \tag{1.2}$$

This metric is of the similar form to the Rander's one in the sense that the Riemannian metric is replaced with the Quartic metric, that is, why we will call the cahnge (1.2) as Quartic Randers change of Finsler metric. The relation between v-curvature tensor of Quartic Finsler space and its Quartic Rander's changed Finsler space has been obtained.

The Fundamental tensors of F^n :

We consider an n-dimensional Finsler space F^n with a metic $\overline{L}(x,y)$ given by

$$\overline{L}(x,y) = L(x,y) + b_i(x)y^i \tag{2.1}$$

where

$$L^4 = a_{ijkp}(x)y^iy^jy^ky^p \tag{2.2}$$

By putting

$$a_{ijk} = \frac{a_{ijkh}y^r}{L}, \qquad a_{ij} = \frac{a_{ijkr}y^ky^r}{L^2}, \qquad a_i = \frac{a_{ijkr}y^jy^ky^r}{L^3}. \tag{2.3}$$

We obtained the normalized element of support $\bar{l}_i = \dot{\partial}_i \overline{L}$ and the angular metric tensor $\overline{h}_{ij} = \overline{L}\dot{\partial}_i\dot{\partial}_j\overline{L}$ as

$$\bar{l}_i = a_i + b_i, \tag{2.4}$$

$$\frac{\overline{h}_{ij}}{\overline{L}} = \frac{h_{ij}}{L} \tag{2.5}$$

where h_{ij} is the angular metric tensor of Quartic Finsler space with metric L given by

$$h_{ij} = 3(a_{ij} - a_i a_j). \tag{2.6}$$

The fundamental metric tensor $\overline{g}_{ij} = \dot{\partial}_i\dot{\partial}_j\left(\frac{\overline{L}^2}{2}\right) = \overline{h}_{ij} + \bar{l}_i\bar{l}_j$ of Finsler space F^n are obtained from equations (2.4), (2.5) and (2.6) which is given by

$$\overline{g}_{ij} = 3\tau a_{ij} + (1-3\tau)a_i a_j + (a_i b_j + a_j b_i) + b_i b_j \qquad \text{where } \tau = \frac{\overline{L}}{L} \tag{2.7}$$

It is easy to show that

$$\dot{\partial}_i\tau = \frac{\{(1-\tau)a_i + b_i\}}{L}, \qquad \dot{\partial}_j a_i = \frac{3(a_{ij} - a_i a_j)}{L}, \dot{\partial}_k a_{ij} = \frac{2(a_{ijk} - a_{ij}a_k)}{L}.$$

Therefore from (2.7), it follows (h) hv-torsion tension tensor $\overline{C}_{ijk} = \dot{\partial}_k\left(\frac{\overline{g}_{ij}}{2}\right)$ of the Cartan's connection $C\Gamma$ are given by

$$2L\overline{C}_{ijk} = 6\tau a_{ijk} + 3(1-3\tau)(a_{jk}a_i + a_{ij}a_k + a_{ki}a_j) + 3(a_{ij}b_k + a_{jk}b_i + a_{ki}b_j) \tag{2.8}$$
$$-3(a_i a_j b_k + a_i a_k b_j + a_j a_k b_i) + 3(7\tau - 3)a_i a_j a_k$$

In view of equation (2.6) the equation (2.8) may be written as

$$\overline{C}_{ijk} = \tau C_{ijk} + \frac{(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j)}{2L} \tag{2.9}$$

where $m_i = b_i - \frac{\beta}{L} a_i$ and C_{ijk} is the (h) hv-torsion tensor of the Cartan's connection $C\Gamma$ of Quartic Finsler metric L given by

$$LC_{ijk} = 3[a_{ijk} - (a_{ij}a_k + a_{jk}a_i + a_{ki}a_j) + 2a_i a_j a_k] \tag{2.10}$$

Let us suppose that the intrinsic metric tensor $a_{ij}(x, y)$ of the Quartic metric L has non-vanishing determinant. Then the inverse matrix (a^{ij}) of (a_{ij}) exists.

Therefore the reciprocal metric tensor g^{ij} of F^n is obtain from equation (2.7) which is given by

$$\overline{g}^{ij} = \frac{1}{3\tau} a^{ij} + \frac{(b^2 + 3\tau - 1)}{3\tau(1+q)^2} a^i a^j - \frac{(a^i b^j + a^j b^i)}{3\tau(1+q)} \tag{2.11}$$

where

$$a^i = a^{ij} a_j, \qquad b^i = a^{ij} b_j$$
$$b^2 = b^i b_i, \qquad q = a^i b_i = a_i b^i = \frac{\beta}{L}$$

The v-Curvature tensor of F^n :

From (2.6), (2.10) and definition of m_i and a^i, we get the following identities

$$a_i a^i = 1, \qquad a_{ijk} a^i = a_{jk}, \qquad C_{ijk} a^i = 0, \qquad h_{ij} a^i = 0 \tag{3.1}$$

$$m_i a^i = 0, \qquad h_{ij} b^j = 3m_i, \qquad m_i b^i = (b^2 - q^2)$$

To find the v-curvature tensor of F^n, first we find (h) hv-torsion tensor $\overline{C}^i_{jk} = \overline{g}^{ir} \overline{C}_{jrk}$

$$\overline{C}^i_{jk} = \frac{1}{3} C^i_{jk} + \frac{1}{6\overline{L}} (h^i_j m_k + h^i_k m_j + h_{jk} m^i) \tag{3.2}$$

$$-\frac{a^i}{\overline{L}(1+q)}\{m_j m_k + \frac{1}{6}(b^2-q^2)h_{jk}\} - \frac{1}{3(1+q)}a^i C_{jrk} b^r$$

where

$$LC^i_{jk} = LC_{jrk}a^{ir} = 3\{a^i_{jk} - (\delta^i_j a_k + \delta^i_k a_j + a^i a_{jk}) + 2a^i a_j a_k\}$$

(3.3) $$h^i_j = h_{jr}a^{ir} = 3(\delta^i_j - a^i a_j)$$

$$m^i = m_r a^{ir} = b^i - qa^i$$

and $$a^i_{jk} = a^{ir}a_{jrk}.$$

From (3.1) and (3.3) we have the following identities

$$C_{ijr}h^r_p = C^r_{ij}h_{pr} = 3C_{ijp}, \qquad C_{ijr}m^r = C_{ijr}\, b^r, \qquad m_r h^r_i = 3m_i,$$

$$m_i m^i = (b^2-q^2), \qquad h_{ir}h^r_j = 3h_{ij}, \qquad h_{ir}m^r = 3m_i.$$

From (2.9) and (3.2) we get after applying the identities (3.4)

(3.5) $$\overline{C}_{ijr}\overline{C}^r_{hk} = \frac{\tau}{3}C_{ijr}C^r_{hk} + \frac{1}{2L}(C_{ijh}m_k + C_{ijk}m_h + C_{hjk}m_i + C_{hik}m_j)$$
$$+\frac{1}{6L}(C_{ijr}h_{hk} + C_{hrk}h_{ij})b^r + \frac{1}{12L\overline{L}}(b^2-q^2)h_{ij}h_{hk}$$
$$+\frac{1}{4L\overline{L}}(2h_{ij}m_h m_k + 2h_{hk}m_i m_j + h_{jh}m_i m_k + h_{jk}m_i m_h + h_{ih}m_j m_k + h_{ik}m_j m_h)$$

Now we shall find the $v-$curvature tensor $\overline{S}_{hijk} = \overline{C}_{ijr}\overline{C}^r_{hk} - \overline{C}_{ikr}\overline{C}^r_{hj}$. The tensor is obtained from (3.5) and given by

(3.6) $$\overline{S}_{hijk} = \underset{(jk)}{Q}\left\{\frac{\tau}{3}C_{ijr}C^r_{hk} + h_{ij}m_{hk} + h_{hk}m_{ij}\right\}$$
$$= \frac{\tau}{3}S_{hijk} + \underset{(jk)}{Q}\{h_{ij}m_{hk} + h_{hk}m_{ij}\}$$

where

(3.7) $$m_{ij} = \frac{1}{6L}\left\{C_{ijr}b^r + \frac{(b^2-q^2)}{4\overline{L}}h_{ij} + \frac{3}{2}\overline{L}^{-1}m_i m_j\right\}$$

and the symbol $\underset{(jk)}{Q}\{\cdots\}$ denotes the exchange of j, k and subtraction.

Preposition 1: The $v-$curvature tensor $\overline{S}_{hijk}$ of $\overline{F}^n$ with respect to Carton's connection $C\Gamma$ is of the form (3.6).

Thus (3.6) may be written as

$$\overline{S}_{hijk} = \frac{\tau}{3} S_{hijk} + \underset{(jk)}{Q}\{h_{ij}m_{hk} + h_{hk}m_{ij}\} \tag{3.8}$$

It is well known [6] that the $v-$curvature tensor of any three dimensional Finsler space is of the form

$$L^2 S_{hijk} = S(h_{hj}h_{ik} - h_{hk}h_{ij}) \tag{3.9}$$

Owing to this fact M. Matsumoto [6] defined the S3-like Finsler space F^n $(n \geq 3)$ as such a Finsler space in which $v-$curvature tensor is of the form (3.9). The scalar S in (3.9) is a function of x alone.

The $v-$curvature tensor of any four dimensional Finsler space may be written as [6]

$$L^2 S_{hijk} = \underset{(jk)}{Q}\{h_{hj}K_{ki} + h_{ik}h_{hj}\} \tag{3.10}$$

where K_{ij} is a (0, 2) type symmetric Finsler tensor field which is such that $K_{ij}y^j = 0$. A Finsler space $F^n\,(n \geq 4)$ is called S4-like Finsler space [6] if its $v-$curvature tensor is of the form (3.10).

From (3.8), (3.9), (3.10) and (2.5), we have the following theorems.

Theorem 3.1 : The Quartic Rander's change of S3-like or S4-like Finsler space is S4-like Finsler space.

Theorem 3.2 : If $v-$curvature tensor of Quartic Rander's changed Finsler space $\overline{F}^n$ vanishes identically, then the Quartic Finsler space F^n is S4-like.

If $v-$curvature tensor of Quartic Finsler space F^n vanishes then equation (3.8) reduces to

$$\overline{S}_{hijk} = h_{ij}m_{hk} + h_{hk}m_{ij} - h_{ik}m_{hj} - h_{hj}m_{ik} \tag{3.11}$$

By virtue of (3.11) and (2.11) and the Ricci tensor $\overline{S}_{ik} = \overline{g}^{hk}\overline{S}_{hijk}$ is of the form

$$\overline{S}_{ik} = \left(-\frac{1}{3\tau}\right)[mh_{ik} + 3(n-3)m_{ik}]$$

where $m = m_{ij}a^{ij}$, which in view of (3.7) may be written as

$$\overline{S}_{ik} + H_1 h_{ik} + H_2 C_{ikr} b^r = H_3 m_i m_k \tag{3.12}$$

where $$H_1 = \frac{m}{3\tau} + \frac{(n-3)(b^2-q^2)}{24\,\overline{L}^2}$$

$$H_2 = \frac{(n-3)}{6\overline{L}}$$

$$H_3 = -\frac{(n-3)}{4\overline{L}^2}$$

From (3.12), we have the following:

Theorem 3.3 : If the $v-$curvature tensor of Quartic Finsler space vanishes then there exist scalar H_1 and H_2 in Quartic Rander's changed Finsler space F^n $(n \geq 4)$ such that matrix $||S_{ik} + H_1 h_{ik} + H_2 C_{ikr} b^r||$ is of rank two.

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SUMMARY OF THE PH. D. THESIS

" MODELS OF FINSLER SPACES "

THESIS

SUBMITTED FOR THE DEGREE

OF

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IN

MATHEMATICS

By

Dhirendra Thakur

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS & STATISTICS

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Candidate's Declaration

I hereby declare that the work which is being presented in the thesis entitled "*MODELS OF FINSLER SPACES*" in fulfillment of the requirement for the award of the degree of Doctor of Philosophy in Mathematics of D.D.U. Gorakhpur University, Gorakhpur is an authentic record of my own work carried out during a period from April 2009 to January 2012 under the supervision of Dr. T. N. Pandey, Professor in Mathematics, Department of Mathematics & Statistics, D.D.U. Gorakhpur University, Gorakhpur,India.

The matter presented in this thesis has not been submitted by me for the award of any other degree of this or any other Institute/University.

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This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

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The present thesis is an outcome of my investigations in the Department of Mathematics and Statistics, D.D.U. Gorakhpur University, Gorakhpur under the supervision of Prof. T.N.Pandey. The purpose of the present thesis is to study models of Finsler spaces .

The whole thesis is divided into seven chapters and each chapter is subdivided into various sections. A short and concise presentation of this thesis as follows.

Table of Contents

P. No.

The symbols ∂ and $\dot{\partial}$ denote the partial derivative with respect to x^i and y^i respectively. Small and long vertical lines (I and |) stands for h and v-covariant derivative respectively.

First chapter is an introductory in nature and consists of preliminary details. Some useful results and definitions such as Finsler space, some connections like the Berwald, Carton and Runds have been mentioned therein.

The **second chapter** deals with the relation between Carton's connections of two Finsler spaces (M^n, L) and $(M^n, \overline{L})$ where $\overline{L}$ obtained from L by h-Randers change. It has been obtained the conditions under which this change is projective. It also deals the conditions under which Douglas space, Landsberge space or Weakly Berwald space becomes invariant.

The **third chapter** is devoted to study for Finsler spaces $\overline{F}^n$ obtained by Randers Conformal change of Finsler spaces F^n of Douglas type to be also of Douglas type and vice versa .It has been also worked out the condition under which the said transformation is projective.

In the **fourth chapter** we discuss the Finsler space $\overline{F}^n = (M^n, \bar{L}(\alpha, \beta))$ obtained by Conformal Randers change of Finsler space $F^n = (M^n, L(\alpha, \beta))$ of Douglas type remains to be Douglas type and vice versa .

The **fifth chapter** is devoted to investigate the Berwald connection, condition for projectively flatness of Finsler space with 2nd approximated exponential (α, β) metric $L = \alpha e^{\frac{\beta}{\alpha}} + \beta$ and the conditions under which said space is Douglas type.

In the **sixth chapter**, we investigate condition that the Finsler space with $(\alpha,\beta)-$metric like Randers metric, Kropina metric and Matsumoto metric become Weakly Landsberg space. We also give an example for Weakly Landsberg space which is not Landsberg space.

The **seventh chapter** is the last chapter of my thesis and is devoted to study the S_4- likeness of Quartic Rander's change of a Finsler space and the relation between V-curvature tensor of Quartic Rander's changed Finsler space.

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1. *On Weakly – Landsberg Space of Some* (α,β) – Metric, Bulletin of the Iranian Mathematical Society.

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