ABSTRACT

ON PROBLEMS OF VON NEUMANN AND MAHARAM

A

THESIS SUBMITTED FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN THE FACULTY OF SCIENCES

(MATHEMATICS)

T.M. BHAGALPUR UNIVERSITY

Supervised By:

Dr. Jai Shankar Jha

Professor and Former Head
P. G. Dept. of Mathematics
T.M. Bhagalpur University
Bhagalpur

Submitted By:

Manoj Kumar Raut

Research Scholar
Reg. No. 5405/09

T.M. Bhagalpur University, Bhagalpur-812007

2013

UNIVERSITY DEPARTMENT OF MATHEMATICS
T.M. BHAGALPUR UNIVERSITY
BHAGALPUR-812007

Jai Shankar Jha

M.Sc. (Gold Med.), Ph.D.
Professor & Former Head
Univ. Dept. of Mathematics
T.M. Bhag. Univ., Bhagalpur-812007
And
Vice-President,
Bihar Mathematical Society, Bhagalpur

☎ : 0641-2500372 (O)
0641-2302353 (R)
Mob.: 09430429099

Ref. No………… *Date……………*

CERTIFICATE

This is to certify that the subject matter of the thesis is record of work done by the candidate ***SRI MANOJ KUMAR RAUT*** himself under my guidance and that the contents for his thesis entitled ***"ON PROBLEMS OF VON NEUMANN AND MAHARAM"*** do not form a basis of the award of any previous degree to him or to the best of my knowledge, to any body else and that the thesis has not been submitted by the candidate for any research degree in any other university.

Further certified that the candidate in habit and character is a fit and proper person for the award of the Ph. D. degree.

Date : Signature of the Supervisor

ACKNOWLEDGEMENT

I bow my head before the Almighty God, without whose blessings the research idea could have neither been conceived nor executed.

Gratitude is a noble response of ones' soul in kindness or help generously rendered by another & its acknowledgement is a duty & joyance. So, it is that I express briefly my debt gratitude & thanks to those who have made the eventual completion of my Ph.D. research work possible.

I express my deep sense of gratitude and indebtedness to Dr. Jai Shankar Jha, Professor & Former Head, University Dept. of Mathematics, T.M. Bhagalpur University, Bhagalpur for suggesting me the problems, his constant encouragement, valuable criticism and able guidance during the course of this research. In fact, this study was made possible mainly through his competent supervision and guidance.

I am very much thankful to Dr. Mrityunjay Jha, Professor & Head, University Dept. of Mathematics, T.M. Bhagalpur University, Bhagalpur for his constant encouragement during the tenure of my research work.

I also wish to express my deep respect and eulogy for Dr. Binay Kumar Singh and Dr. P. K. Lal, University Dept. of Mathematics, T.M. Bhagalpur University, Bhagalpur for their valuable criticism, comments and scholarly advice.

I also express my thanks to non-teaching staff of Dept. of Mathematics, T.M. Bhagalpur University, Bhagalpur.

I would also like to thank to Mr. Ballav Prasad Paudel, Campus Chief; Dr. Harihar Bhandari, Former Campus Chief and Mr. Raju Ram Thapa and all other colleagues in the Dept. of Mathematics, P.G. Campus, Tribhuvan University, Biratnagar, Nepal.

I owe my special thanks to my parents Mr. Durga Bahadur Raut and Mrs. Uttara Raut for their grace and blessings which gave me strength for the completion of this Ph.D. work.

I would also like to thanks my wife Mrs. Sunita Raut & my dearest daughter Miss Swechha Raut for their unending assistance and moral support which proved a great source of motivation and encouragement to me that kept my working spirit alive.

Last but not the least I would like to extend my appreciation and thanks to Sri Raj Madhavan, an artist, who finally printed out the entire thesis with great care.

(Manoj Kumar Raut)
Research Scholar

ABSTRACT

Lattices are ubiquitous in mathematics. The beauty and simplicity of the abstraction and the ability to tie together seemingly unrelated pieces of mathematics are certainly appealing to mathematician-as-artist. The introduction of new and nontrivial techniques for the solution of outstanding problems, for example in measure theory, is mathematically rewarding; the discovery of new questions which become natural to ask in the context of lattice flavour in measure theory is undoubtedly vigorously intriguing. Lattice pervades all branches of human knowledge. Because of diagrammatic representation the very simplicity of the basic concepts and the degree of abstraction in its relationship to other branches, lattice theory has an intuitive and aesthetic appeal.

Our effort here is to create relations between different lattices. We have considered some important class of lattices which are necessary understand this dissertation and are useful for applications in other branches of mathematics. The diagram depicts the inclusion relationships among some important subclasses of lattices.

Our endeavour here is to investigate the weak distributivity of Boolean σ-algebras satisfying the countable chain condition (ccc). It addresses primarily the question when such algebras carry a σ-additive measure. We use as a starting point the year old problem of John von Neumann stated in 1937 in the Scottish Book. He posed the problem if the countable chain condition (ccc) and weak distributivity are sufficient for the existence of σ-additive measure.

Subsequent research has shown that the problem has two aspects: one set theoretic and one combinatorial. Recent results provide a complete solution of both the set theoretic and combinatorial problems. Our effort here is to survey the history of von Neumann's Problem and outline the solution of the set theoretic problem. The technique that we describe owes much to the early work of Dorothy Maharam. We investigate von Neumann's question : whether every weakly distributive complete ccc Boolean algebra is a measure algebra. We shall present a number of additional necessary and sufficient conditions for a complete ccc Boolean algebra *B* to carry Maharam submeasure. It turns out that some of the properties are natural generalizations of the conditions of weakly distributive. We remark that the assumption of weak distributivity is not necessary in reformulation of the Housdorffness but is necessary in reformulation of the G_δ property. Finally we shall show that strategic versions of weak distributivity are equivalent in carrying a Maharam submeasure. To make the idea clear we have considered an infinite game of two players. Players I and II take turns to successively produce two infinite sequences of moves. The games in terms of winning strategies that we have considered here are : weak distributivity game, diagonal game and bounding game. We also cite here some examples of weak distributivity complete ccc algebras that are not Maharam.

We investigate the sequential topology τ_s on a complete Boolean algebra *B* determined by algebraically convergent sequences in *B*. We show the role of weak distributivity of *B* in separation axioms for sequential topology. We deal with sequential topologies on complete Boolean algebras from the point of view of separation axioms. Our motivation comes from the still open Control Measure Problem of D. Maharam. Maharam asked whether every σ-complete Boolean algebra that carries a strictly positive continuous submeasure admits a σ-additive measure. The main result is that a necessary and sufficient condition for *B* to carry a strictly positive Maharam submeasure is that *B* is ccc and that the space (B, τ_s) is Hausdroff. We also characterize sequential cardinals. We study sequential topologies on complete Boolean algebras in a more general setting. We review some notations from topology and consider those complete Boolean algebras for which the sequential topology is Fréchet. A necessary and sufficient condition for (B, τ_s) to be a Fréchet space is also one of our important results. We show that for complete ccc Boolean algebra, Hausdorffness of the sequential topology is a strong property : it implies metrizability, and equivalently, the existence of a strictly positive Maharam submeasure.

We continue the investigation of order convergence and related topologies on orthomodular lattices (OMLs). Our main results explain why atomic OMLs behave much better than arbitrary ones, not only from the algebraic, but also from the topological point of view: the complete atomic and meet-continuous OMLs are just the order-topological ones, i.e., those complete OMLs which have topological order convergence and form a topological lattice with respect to the order topology; in this situation, the orthocomplementation automatically becomes continuous, being a dual automorphism. Moreover, we shall find that these OMLs are precisely the algebraic (= compactly generated complete) ones, and that it even suffices to postulate continuity in the sense of Scott in order to ensure atomicity. This will be achieved by applying earlier results of the M. Erné to blocks, i.e., to the maximal Boolean subalgebras of the given OML. Among other characterizations, we shall find that a complete OML is order-topological iff it is a totally separated (and, moreover, a totally order-disconnected) topological lattice in its order topology.

Our theory essentially extends the compact case, because there are interesting order-topological complete OMLs which fails to be compact. In contrast to this fact, an order-topological complete Boolean algebra is always compact. Several purely algebraic characterizations of compact order-topological OMLs will be given. The incomplete case is not yet settled entirely as yet, however, we are able to establish some necessary and sufficient conditions for an OMLs to have a MacNeille completion which is an order-topological atomic OML.

Our research work is divided into four chapters with a number of sections and subsections :

Chapter I : INTRODUCTION AND PRELIMINARIES

It is an introductory chapter. We discuss basic terminologies that are required for smooth understanding of this thesis. This chapter also depicts diagrammatic representation of inclusion relationships among some important subclasses of lattices. This gives us the background needed to begin our exploration "On Problems of von Neumann and Maharam."

Chapter II : CONDITIONS FOR A COMPLETE BOOLEAN ALGEBRA TO CARRY MAHARAM SUBMEASURE

We trace out a number of necessary and sufficient conditions for a Boolean σ-algebra satisfying countable chain condition to carry Maharam submeasure. We remark that weak distributive of Boolean algebra is not necessary in reformulation of Housdorffness and is necessary in reformulation of the G_δ property.

Chapter III : TOPOLOGY DETERMINED BY CONVERGENT SEQUENCES IN A COMPLETE BOOLEAN ALGEBRA

We investigate the sequential topology τ_s on a complete Boolean algebra *B* determined by algebraically convergent sequences in *B*. We have established conditions for (B, τ_s) to a Fréchet space and to be a Hausdroffspace. We show that for complete Boolean algebra satisfying countable chain condition, Hausdorffness of the sequential topology is a strong property.

Chapter IV : COMPACT TOPOLOGICAL ORTHOMODULAR LATTICES

We show that the following properties are equivalent for any complete orthomodular lattices *L*:

(a) *L* is order-topological.
(b) *L* is continuous.
(a) *L* is algebraic.
(b) *L* is compactly atomistic.
(c) *L* is a totally order-disconnected topological lattice in the order topology.

(Manoj Kumar Raut)

CONTENTS

*****

CHAPTER I

INTRODUCTION AND PRELIMINARIES

1.0. INTRODUCTION

Lattice theory, as an independent branch of mathematics has had a somewhat stormy existence for more than hundred years ago. Its origins are to be found in Boole's mid-nineteenth century work in classical logic; and the success of what we now call Boolean algebra in this field led to the late nineteenth century attempts at the formalization of all of mathematical reasoning, and eventually to mathematical logic.

Schroder and Peirce introduced the concept of an abstract lattice as a generalisation of Boolean algebras, while Dedekind's work on algebraic numbers led him to the introduction of lattices outside of logic and to the concept of modular lattices. These late-nineteenth century investigations did not lead to widespread interest in lattice theory-it was not until the thirties that lattice theory truly became an object for independent and systematic study by mathematicians.

Stone's representation theory for Boolean algebras and distributive lattices, Menger's work on the subspace structure of geometries, von Neumann's coordinatisation of continuous geometry and Birkoff's recognition of the lattice as a basic tool in algebra were among the forces which combined in the late thirties to enable Birkhoff successfully to promote the idea that lattice theory is a branch of mathematics worthy of the attention of the community [3].

The very simplicity of the basic concepts in lattice theory and the degree of abstraction in its relationship to other branches of mathematics have proved to be at once both its strongest and weakest points.

Lattices are ubiquitous in mathematics. The beauty and simplicity of the abstraction and the ability to tie together seemingly unrelated pieces of mathematics are certainly appealing to the mathematician-as-artist. The introduction of new and nontrivial techniques for the solution of outstanding problems, for example in universal algebra, is mathematically rewarding; and the discovery of new questions which become natural to ask in the context of lattice theory is undoubtedly intriguing.

Through the vehicle of lattice theory one can hope to contribute to progress in functional analysis by the study of vector lattices or the investigation of von Neumann algebras [5]. Closely related is the work that has been done in quantum mechanics and the special partially ordered sets and lattices that arise there. A different type of work in the field is that being done in combinatorial theory where geometric lattices have been used extensively by Rota and his school in the investigations inspired by Whitney and begun by Dilworth.

These are but a few of the avenues of development of general lattice theory since the surge of interest in it forty years ago. During these ensuing years the sailing was not always smooth. Since lattice theory many times does not provide answers to questions in other fields it is often considered too abstract to be really useful.

Together with category theory it suffers from the fact that abstraction many times leads to oversimplification and loss of the basic properties of the underlying structure. Perhaps even more unfortunate is that the simplicity of the basic definitions leads to the production of an inordinate amount of printed trivia. While the appearance of trivia is not foreign to any branch of mathematics that which arises in lattice theory is quite easily recognizable as such by nonexperts in the field. An obvious result of this phenomenon is that the entire field at times becomes discounted as a serious branch of mathematics.

The role played by distributive lattices in the development of lattice theory is both an important and an interesting one. Boolean algebra and modular lattices have been with us since the last century. Since distributive lattices are weaker than the former and stronger than the later, they have a history as old as that of lattice theory. Distributive lattices have in one sense come full circle in the scheme of things; lattice theory began with Boolean algebras in logic; today there is a great deal of interest in non-Boolean distributive lattices which arise in nonclassical logic [6].

Along with the general development of lattice theory came the parallel development of the study of distributive lattices. Because of their relative age and the central role they have played, they have received more attention than some other types of lattices, and today distributive lattices are among those best understood. Any distributive lattice is isomorphic to a ring of sets, and any distributive lattice can be considered as a sub direct product of two-element chains. Results like these help to explain the structure of distributive lattices and give one somewhat of an intuitive grasp as to their nature.

The analysis of distributive lattices has led to investigations of the distributive parts of other types of lattices and distributive lattices naturally related to arbitrary ones.

The study of distributive lattices has influenced the development of lattice theory in a different way as well. Many of the properties which have been investigated by lattice theorists are weakened forms of distributivity. The obvious example is modularity which in turn has led to the symmetry and semi modularity properties so useful in geometry and combinatorial theory. In another direction the retention of complementation and weakening of distributivity have given rise to some generalizations of Boolean algebras where properties such as orthomodularity have been developed in investigations of functional analysis, mathematical physics and generalized statistics. A thorough knowledge of distributive lattices is clearly indispensable for mathematicians working in lattice theory and related areas.

Any discussion of distributive lattices would be incomplete without mention of universal algebra. Its name goes back at least to Sylvester, Whitehead (1898), Hamilton and De Morgan, these nineteenth century mathematicians recognized and studied universal principles in algebra, but it was not until Birkhoff introduced the modern concept of universal algebra and popularized it in the mid-forties. Today it attracts much attention and the works of Cohn, Gratzer, and Pierce have made its ideas readily accessible. Questions arising in universal algebras have led to quite a bit of renewed interest in lattice in general and distributive lattices in particular.

In this setting, it is clear that a book on distributive lattices can occupy an important place in mathematical literature. Balbes and Dwinger have written such a book and it should be well received and widely used. The book is quite carefully put together which is an absolute necessity because of the author's three-fold approach to distributive lattices-order theoretic, algebraic and categorical. The preliminary sections on universal algebra and category theory are self-contained and enable the reader to progress through the six chapters on general distributive lattices with an appreciation of the new approaches resulting from universal algebra and the relationship of various classes of distributive lattice as best expressed in the language of category theory. In these chapters the main results of distributive lattice theory are gathered together and they provide a foundation for the reading of the last five chapters which deal with special distributive lattices.

The special topics which we have chosen to include pseudocomplemented distributive lattices; Heyting, de Morgan and Lukasiewicz algebras; and lattices satisfying higher degrees of distributivity. The introductory remarks on the special algebras are particularly helpful in that they include some development from a historical perspective, references for further investigation, and an encapsulation of the role of these special lattices and results to be emphasized. Balbes and Dwinger point out that they have selected the special topics out of the many possibilities at least partially on the natural bases of their own research interests and taste. However they present references to topics which they have omitted and have wisely selected some topics which are not extensively discussed in other texts. This book differs considerably, for example, from Gratzer's recent text on distributive lattices both in emphasis and approach.

Balbes and Dwinger should have appeal to three audiences. It is a useful reference work for lattice theorists because of the amalgamation of the results in general distributive lattice theory as well as the special topics [3]. In addition the bibliography is quite valuable - not only for source material on subjects in the text but for the references mentioned as suggested for further reading. The mathematician not working in lattice theory can get from this book a good idea of the history and importance of distributive lattices and their role in logic and universal algebra. The book is also suitable as a text for graduate students and the numerous exercises scattered throughout should be quite helpful especially to independent reading. In two areas the reviewer wishes things might have been different. The authors use + and $\cdot$ instead of $\vee$ and $\wedge$ throughout; one's preference here is somewhat a matter of "creature comfort" and it must be said that the authors are in good company with respect to their notation (see for example von Neumann's Continuous geometry). In a text such as this one, designed to bring the reader to the frontiers of current research, it would have been a natural thing to include in addition to the exercises some specific open problems. Although it is to be hoped that any such collection will soon become out of date, the inclusion of such problems does give a feeling for what the experts are asking and sometimes provides impetus and a challenge.

In summary, Distributive lattices is worth having. It was carefully planned and well written, providing a survey of the general area, special topics, and information on where to find more. It is a useful reference work for lattice theorists and a good source of information for those not conversant with the field, where perhaps it can kindle a spark of interest in the position and development of one of the lattice theory's oldest branches.

1.1. LATTICE APPLICATIONS

Partial order and lattice theory now play an important role in many disciplines of computer science and engineering [7]. For example, they have applications in distributed computing (global predicate detection), concurrency theory, programming language semantics (fixed-point semantics), and data mining (concept analysis). They are also useful in other disciplines of mathematics such as combinatorics, number theory and group theory. In this course the following topics have already been covered up:

Posets : Basics calculating height and width of a poset, Dilworth's theorem, decomposition of a poset.

Lattices : Distributive and modular lattices, lattices as algebraic structures.

Ideals : Ideals and Filters, Birkhoff's theorem, Join and meet Irreducible elements, Application to global predicate Detection.

Fixed Point Theorems : Complete lattices, monotone and continuous functions, Knaster-Tarski theorem, Application to Semantics.

Sperner Property : Ranked Posets, Erdos–Szekeres theorem, Sperner property, Hall's theorem.

Enumeration : Enumerating ideals, Lexical and Gray code; enumerating level sets, enumeration of linear extensions.

Dimension theory : Encoding posets, applications to encoding distributed computations, Hiraguchi's conjure [7].

Special classes of posets : 2-dimensional posets, series - parallel poset interval posets.

Mobius Inversion : Mobius function and application to number theory.

Slicing: Slicing with application to distributed computing and combinatorics.

1.2. LATTICE (ORDER)

In mathematics, a lattice is a partially ordered set (or poset) whose nonempty finite subsets all have a supremum (called join) and an infimum (called meet). Lattices can also be characterized as algebraic structures satisfying certain axiomatic identities. Since the two definitions are equivalent, lattice theory draws on both order theory and universal algebra. Semilattices include lattices, which in turn include Heyting and Boolean algebras. These "lattice-like" structures all admit order-theoretic as well as algebraic descriptions.

Lattices as posets

Consider a poset $(L, \leq)$. L is a lattice if for all elements x and y of L, the set $\{x, y\}$ has both a least upper bound in L (join, or supremum) and a greatest lower bound in L (meet, or infimum).

The join and meet of x and y are denoted by $x \vee y$ and $x \wedge y$, respectively. Because joins and meets are assumed to exist in a lattice, $\vee$ and $\wedge$ are binary operations. Hence this definition is equivalent to requiring L to be both a join-and a meet-semilattice.

A bounded lattice has a greatest and least element, denoted 1 and 0 by convention (also called top and bottom). Any lattice can be converted into a bounded lattice by adding the greatest and least elements.

Using an easy induction argument, one can deduce the existence of suprema (joins) and infima (meets) of all non-empty finite subsets of any lattice. With additional assumptions, further conclusions may be possible; see Completeness (order theory) for more discussion of this subject. That article also discusses how one may rephrase the above definition in terms of the existence of suitable Galois connections between related posets - an approach of special interest for the category theoretic approach to lattices.

1.3. LATTICES AS ALGEBRAIC STRUCTURES

Let L, be set with two binary operations, $\vee$ and $\wedge$. A lattice is an algebraic structure $(L, \vee, \wedge)$ of type (2, 2), such that the following axiomatic identities hold for all members a, b, and c of L:

Commutative laws: $a \vee b = b \vee a$

$$a \wedge b = b \wedge a$$

Associative laws: $a \vee (b \vee c) = (a \vee b) \vee c$

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

Absorption laws: $a \vee (a \wedge b) = a$

$$a \wedge (a \vee b) = a$$

The following important identity follows from the above:

Idempotent laws: $a \vee a = a$

$$a \wedge a = a$$

These axioms assert that $(L, \vee)$ and $(L, \wedge)$ are each semilattices. The absorption laws, the only equations in which both meet and join appear, distinguish a lattice from a pair of semilattices and assure that the two semilattices interact appropriately. In particular, each semilattice is the dual of the other. A bounded lattice requires that meet and join each have a neutral element, called 1 and 0 by convention. See the entry semilattice.

Lattices have some connections to the groupoid family. Because meet and join both commute and associate, a lattice can be viewed as consisting of two commutative semigroups having the same carrier. If a lattice is bounded, these semigroups are also commutative monoids. The absorption law is the only defining identity that is peculiar to lattice theory.

The closure of L under both meet and joint implies, by induction, the existence of the meet and join of any finite subset of L, with one exception: the meet and join of the empty set are the greatest and least elements, respectively. Therefore a lattice contains all finite (including empty) meets and joins only if it is bounded. For this reason, some authors define a lattice so as to require that 0 and 1 be members of L. While defining a lattice in this manner entails no loss of generality, because any lattice can be embedded in a bounded lattice, this definition will not be adopted here.

The algebraic interpretation of lattices plays an essential role in universal algebra.

Connection between the two definitions

The algebraic definition of a lattice implies the order theoretic one, and vice versa. Obviously, an order-theoretic lattice gives rise to two binary operations $\vee$ and $\wedge$. It is easy to see that these operations make $(L, \vee, \wedge)$ into a lattice in the algebraic sense. The converse is true also: Consider an algebraically defined lattice $(M, \vee, \wedge)$. Now define a partial order $\leq$ on M by setting.

$$x \leq y \text{ if and only if } x = x \wedge y$$

or, equivalently,

$$x \leq y \text{ if and only if } y = x \vee y$$

for elements x and y in M. The laws of absorption ensure that both definitions are indeed equivalent. One can now check that the relation $\leq$ introduced in this way defines a partial ordering within which binary meets and joins are given through the original operations $\vee$ and $\wedge$. Conversely, the order induced by the algebraically defined lattice $(L, \vee, \wedge)$ that was derived from the order theoretic formulation above coincides with the original ordering of L.

Since the two definitions of a lattice are equivalent, one may freely invoke aspects of either definition in any way that suits the purpose at hand.

1.4. Examples

- For any set A, the collection of all subsets of A (called the power set of A) can be ordered via subset inclusion to obtain a lattice bounded by A itself and the null set. Set intersection and union interpret meet and join, respectively.

- For any set A, the collection of all finite subsets of A, ordered by inclusion, is also a lattice, and will be bounded if and only if A is finite.

- The natural numbers in their usual order form a lattice, under the operations of "min" and "max". 0 is bottom; there is no top.

- The Cartesian square of the natural numbers, ordered by $\leq$.

  So that, $(a, b) \leq (c, d) \leftrightarrow (a \leq c)$ & $(b \leq d)$. $(0, 0)$ is bottom; there is no top.

- The non-zero natural numbers also form a lattice under the operations of greatest common divisor and least common multiple operations, with divisibility interpreting the usual order relation. Bottom is 1; there is no top.

- Any complete lattice (also see below) is a (rather specific) bounded lattice. This class gives rise to a broad range of practical examples.

- The set of compact elements of an arithmetic complete lattice is a lattice with a least element, where the lattice operations are given by restricting the respective operations of the arithmetic lattice. This is the specific property which distinguish arithmetic lattices from algebraic lattices, for which the compacts do only form a join-semilattice. Both of these classes of complete lattices are studied in domain theory.

Further examples are given for each of the additional properties discussed below.

1.5. MORPHISMS OF LATTICES

The appropriate notion of a morphism between two lattices flows easily from the above algebraic definition. Given two lattices $(L, \vee, \wedge)$ and $(M, \cup, \cap)$, a homomorphism of lattices is a function $f : L \to M$ such that $f(x \vee y) = f(x) \cup f(y)$, and $f(x \wedge y) = f(x) \cap f(y)$.

Thus f is a homomorphism of the two underlying semilattices. If the lattices are bounded, then f should preserve the bounds so that:

$$f(0) = 0,$$

and $$f(1) = 1.$$

In the order-theoretic formulation, these conditions just state that a homomorphism of lattices is a function preserving binary meets and joins. For bounded lattices, preservation of least and greatest elements is just preservation of join and meet of the empty set.

Any homomorphism of lattices is necessarily monotone with respect to the associated ordering relation: see preservation of limits. The converse is of course not true: monotonicity by no means implies the required preservation properties.

Given the standard definition of isomorphisms as invertible morphisms, a lattice isomorphism is just a bijective lattice homomorphism. Lattices and their homomorphisms form a category.

1.6. PROPERTIES OF LATTICES

We now introduce a number of important properties that lead to interesting special classes of lattices. One, boundedness, has already been discussed.

Completeness

A highly relevant class of lattices are the complete lattices. A lattice is complete if all of its subsets have both a join and a meet, which should be constructed to the above definition of a lattice where one only requires the existence of all (non-empty) finite joins and meets. Details can be found within [8].

Distributivity

Since any lattice comes with two binary operations, it is natural to consider whether one distributes over the other. A lattice $(L, \vee, \wedge)$ is distributive, if the following condition is satisfied for every three elements x, y and z of L:

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

This condition is equivalent to the dual statement:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Other characterizations exist, and can be found in the article on distributive lattices. For complete lattices one can formulate various stronger properties, giving rise to the classes of frames and completely distributive lattices. For an overview of these different notions, see distributivity in order theory [4].

Modularity

Distributivity is too strong a condition for certain applications. A strictly weaker property is modularity: a lattice (L, $\vee$, $\wedge$) is modular if, for all elements x, y and z of L, we have

$$x \vee (y \wedge (x \vee z)) = (x \vee y) \wedge (x \vee z)$$

Another equivalent statement of this condition is as follows: if $x \leq z$ then for all y one has

$$x \vee (y \wedge z) = (x \vee y) \wedge z$$

For example, the lattice of submodules of a module, and the lattice of normal subgroups of a group, all have this special property. Moreover, every distributive lattice is modular.

Continuity and algebraicity

In domain theory, it is natural to seek approximating the elements in a partial order by “much simpler” elements. This leads to the class of continuous posets, consisting of posets where any element can be obtained as the supremum of a directed set of elements that are way-below the element. If one can additionally restrict these to the compact elements of a poset for obtaining these directed sets, then the poset is even algebraic. Both concepts can be applied to lattices as follows:

- A continuous lattice is a complete lattice that is continuous as a poset.

- An algebraic lattice is a complete lattice that is algebraic as a poset.

Both of these classes have interesting properties. For example, continuous lattice can be characterized as algebraic structures (with infinitary operations) satisfying certain identities. While such a characterization is not known for algebraic lattices, they can be described "syntactically" via Scott information systems [5].

Complements and pseudo-complements

Let L be a bounded lattice with greatest element 1 and least element 0. Two elements x and y of L are complements of each other if and only if:

$$x \vee y = 1 \text{ and } x \wedge y = 0$$

In this case, we write $x^1 = y$ and equivalently, $y^1 = x$. A bounded lattice for which every element has a complement is called a complemented lattice. The corresponding unary operation over L, called complementation, introduces an analogue of logical negation into lattice theory. The complement is not necessarily unique, nor does it have a special status among all possible unary operations over L. A complemented lattice that is also distributive is a Boolean algebra. For a Boolean algebra, the complement of x is probably unique.

Heyting algebras are the example of distributive lattices having at least some members lacking complements. Every element x of a Heyting algebra has, on the other hand, a pseudo-complement, also denoted $\neg x$. The pseudo-complement is the greatest element y such that $x \wedge y = 0$. If the pseudo-complement of every element of a Heyting algebra is in fact a complement, then the Heyting algebra is in fact a Boolean algebra.

A lattice is called relatively complemented if all its closed intervals are complemented.

Sublattices

A sublattice of a lattice L is a nonempty subset of L which is a lattice with the same meet and join operations as L. That is, if L is lattice and $M \neq \Phi$ is subset of L such that for every pair of elements a, b in M both $a \wedge b$ and $a \vee b$ are in M, then M is a sublattice of L.

Free lattices

Using the standard definition of universal algebra, a free lattice over a generating set S is a lattice L together with a function $i : S \to L$, such that any function f from S to the underlying set of some lattice M can be factored uniquely through a lattice homomorphism f^0 from L to M. Stated differently, for every element s of S we find that $f(s) = f^0(i(s))$ and that f^0 is the only lattice homomorphism with this property.

Important lattice-theoretic notions :

In the following, let *L* be a lattice. We define some order-theoretic notions that are of particular importance in lattice theory.

An element *x* of *L* is called join-irreducible if and only if

- $x = a \vee b$ implies $x = a$ or $x = b$ for any *a*, *b* in *L*,
- if *L* has a 0, *x* is sometimes required to be different from 0.

When the first condition is generalized to arbitrary joins $\vee a_i$, *x* is called completely join-irreducible. The dual notion is called meet-irreducibility. Sometimes one also uses the terms $\vee$-irreducible and $\wedge$-irreducible, respectively.

An element *x* of *L* is called join-prime if and only if

- $x \leq a \vee b$ implies $x \leq a$ or $x \leq b$,
- if *L* has *a* 0, *x* is sometimes required to be different from 0.

Again, this can be generalized to obtain the notion completely join-prime and dualized to yield meet-prime. Any join-prime element is also join-irreducible, and any meet-prime element is also meet-irreducible. If the lattice is distributive the converse is also true.

Other important notions in lattice theory are ideal and its dual notion filter. Both terms describe special subsets of a lattice (or of any partially ordered set in general). Details can be found in the respective articles.

Algebraic lattice

We define a lattice L to be compactly generated or algebraic when every element of L is a join of compact elements of L.

Continuous lattice

Let $\{a_\alpha : \alpha \in D\}$ be a family of elements of a complete lattice L, where D is a directed set. We write $a_\alpha \uparrow a$ when $\alpha_1 \leq \alpha_2 \Rightarrow a\alpha_1 \leq a\alpha_2$ and $a = \vee\, a_\alpha$. A complete lattice L is called upper continuous or ($\wedge$-continuous) lattice if in L $a_\alpha \uparrow a \Rightarrow a_\alpha \wedge b \uparrow a \wedge b$ for every b. Dually a lower continuous lattice can be defined. When a complete lattice is both upper and lower continuous, it is called a continuous lattice.

Atomistic lattice

An element p of a lattice with 0 is called an atom when $0 < p$, i.e., p covers 0. An element h of a lattice with 1 is called dual atom or an anti-atom or a hyperplane when $h < 1$. A lattice L with 0 is called atomic when every non-zero element of it contains an atom. L is called atomistic if every non-zero element a of L is join of atom contained in a.

Compactly atomistic lattice

An atomistic complete lattice L is called a compactly atomistic when L satisfies the following condition:-

if p is an atom and L_0 is a set of atoms in L such that $p \leq \vee\, q$, then there exists a finite subset $\{q_1, q_2, ..., q_n\}$ of L_0 such that $p \leq q_1 \vee .... \vee q_n$.

Symmetric Lattice

Any two elements a and b of a lattice L form a modular pair, denoted by $(a, b)\,M$, when

$$(c \vee a) \wedge b = c \vee (a \wedge b) \text{ for every } c \leq b.$$

Dually a and b form a dual modular pair, denoted by $(a, b)\,M^*$, when

$$(c \wedge a) \vee b = c \wedge (a \vee b) \text{ for every } c \geq b.$$

A lattice L is called modular when $(a, b)M$ for all a, $b \in L$. Evidently in modular lattice $(a, b)M^*$ holds for all $a, b \in L$. A lattice L is said to be M-symmetric (M^*-symmetric) when $(a, b)\,M$

$\Rightarrow (b, a)\,M$ ($(a, b)\,M^*$ in L).

We see that L^* is M^*-symmetric iff its dual L^* is M-symmetric. A lattice L is called semi-modular if to any triplet of elements a, b, c of L satisfying the conditions: a and b are incomparable and $a \wedge b < c < a$, there can be found d such that

(i) $a \wedge b < d \leq b$, and

(ii) $(c \vee d) \wedge a = c$

It is obvious that every modular lattice is semi-modular in which the following covering property holds:

$$a \wedge b < b \Rightarrow a < a \vee b$$

A lattice L with 0 is called weakly modular when in L

$$a \wedge b \neq 0 \Rightarrow (a, b)\, M$$

A lattice with 0 is called ⊥- symmetric when in L

$$(a, b)\, M \text{ and } a \wedge b = 0 \Rightarrow (b, a)\, M.$$

It is evident that M-symmetric lattice with 0 is ⊥-symmetric and a weakly modular ⊥-symmetric lattice is M-symmetric.

An element of a lattice with 0 is called a finite element when it is either zero or the join of a finite number of atoms. In a lattice L an element a is called a modular element when $(x, a)\, M$ for every x in L. A lattice L with 0 is called a finite modular when every finite element of L is modular. A compactly atomistic M-symmetric lattice is called a matriod lattice. "A complete atomic algebraic lattice" defined by Birkhoff (lattice theory, AMS, 1984 (fourth Edition)) coincides with compactly atomistic lattice in our sense. "An atomic matriod lattice" defined by Birkhoff coincide with a matriod lattice in our sense [3].

Centre of a lattice :

An element z of a lattice L with 0 and 1 is called a central element when there exist two lattices L_1, L_2 and an isomorphism between L and the direct product $L_1 \times L_2$ such that z corresponds to the element $(1_1, 0_2) \in L_1 \times L_2$. Evidently 0 and 1 are central elements. The set of all central elements of L is called the centre of L and is denoted by C (L). When C (L) is the set of 0 and 1 alone, L is called irreducible lattice, otherwise reducible. If in a complete lattice L and the following conditions are satisfied:

(i) *C (L)* of L is a complete sublattice L.

(ii) $Z_\alpha \in C(L)$ for each $\alpha \in \mathrm{I} \Rightarrow (\bigvee_{\alpha \in I} z_\alpha) \wedge a = \bigvee_{\alpha \in I} (z_\alpha \wedge a)$

For every $a \in L$. Then there exists a unique least central element z such that $a \leq z$. We call this the central cover of a and is denoted by c (a). When in a lattice L every interval $[a, b] = \{x \in L : a \leq x \leq b\}$ is complemented, it is called relevatively complemented. Since a lattice L with 0 and 1 is complemented under certain conditions, it is always a bounded lattice. A lattice L with 0 is called semicomplemented if for every element $a \in L$ (with $a \neq 1$, if 1 exists), there exists a non-zero element $b \in L$ such that $a \wedge b = 0$. L is called section semicomplement (for brevity SSC lattice) when the interval $[0, a]$ is semicomplemented for every $a > 0$, in other words, when L satisfies the following condition: (1) if $a > b$ in L, then there exists $c \in L$ such that $0 < c \leq a$ and $c \wedge b = 0$

A lattice L with 1 is called a dual section semicomplemented lattice (for brevity, SSC* lattice) evidently a relatively, complemented lattice with 0 and 1 is SSC and SSC*.

Ortholattices :

A lattice L with 0 is called a semiortholattice if there exists a binary relation ($\perp$) which satisfies the following set of axioms.

$(\perp_1)\ a \perp a \Rightarrow a = 0$

$(\perp_2)\ a \perp b \Rightarrow b \perp a$

$(\perp_3)\ a \perp b, a_1 \leq a \Rightarrow a_1 \perp b$

$(\perp_4)\ a \perp b, a \vee b \perp c \Rightarrow a \perp b \vee c$

The elements a, b of a lattice L with 0 are said to be semi-orthogonal if $a \perp$ b. A family F of elements of a semi-ortholattice L is called a semi-orthogonal family and write $(a : a \in F) \perp$, if for any pair disjoint finite subsets F_1 and F_2 of F, we have

$$\bigvee_{\alpha \in F_1} a \perp \bigvee_{\alpha \in F_1} a$$

A semi ortholattice L with 1 is called semi-orthocomplemented lattice if for every element a in L there exists an element $a^{\perp}$ in L such that $(\perp_5) 1 = a \vee a^{\perp}$ and $a \perp a^{\perp}$. $a^{\perp}$ is called semi-orthocomplement of a. A semi ortholattice L is said to be relatively semi-orthocomplemented if for every element $a \leq b$, there exists an element c in L such that $b = a \vee c$ and $a \perp c$. c is called relative semi-orthocomplement of a in b. Clearly a semi-complement of an element a is a complement of a.

A lattice L with 0 and 1 is called orthocomplemented when there is a mapping $a \perp a^{\perp}$ of L onto itself satisfying the following conditions.

$(\perp_6)$ $a^{\perp}$ is a complement of a

$(\perp_7)$ $a \leq b \Rightarrow a^{\perp} \geq b^{\perp}$

$(\perp_8)$ $a^{\perp\perp} = a$ for every a

We call $a^{\perp}$ the orthocomplement of a. When $a \leq b^{\perp}$, we say that a and b are orthogonal and we write $a \perp b$. A lattice L with 0 is said to be left complemented if for every pair of elements a and b in L there exists an element b_1 in L such that $a \vee b = a \vee b_1$, $a \wedge b_1 = 0$, (b_1, a) M and $b_1 \leq b$. b_1 is called a left complement within b of a in $a \vee b$. A lattice L with 0 is said to be pseudo-complemented or Brouwerian complemented if for every element $a \in L$ there exists a largest element $x \in L$ such that $a \wedge x = 0$, x is called pseudo complement of a and is denoted by $\neg a$. An element a of a lattice L is said to be a D-element if the lattice $[0, a]$ is distributive. If a D-element is such that $c(a) = 1$, then it is called regular. A complete complemented modular lattice is called semi-distributive if it contains a regular element and anti-distributive if it does not contain non-zero D-element. L is a complete modular lattice of dimension n if there exists in L an independent system of regular elements $a_1, ..., a_n$ an such that $a_1 \vee ... \vee a_n = 1$.

Logic

A lattice L with 0 is called a Brouwerian lattice or a Brouwerian algebra if it is closed with respect to an operation $\rightarrow$ which satisfies.

$$a \leq b \rightarrow c \Leftrightarrow a \wedge b \leq c, \text{ for all } a, b, c \in L.$$

It is evident that a Brouwerian lattice is distributive. If it is complemented, it is simply a Boolean algebra and every element a has a unique complement a^1. Dually the concept of dual Brouwerian lattice can be introduced. A dual Brouwerian lattice is called a Brouwerian logic.

An orthocomplemented lattice L is called orthomodular when in L

$$a \perp b \Rightarrow (a, b) M$$

An orthomodular poset is called a logic and such a poset is called a σ-orthocomplete orthomodular poset or σ-logic, if the suprememum of every countable family of pairwise orthogonal elements of L exists in L.

A finitely atomistic lattice without infinite chain is called a geometric lattice G provided (G) for every element a and every atom p, $a \wedge p = 0 \Rightarrow a < a \vee p$.

A geometric lattice G is clearly a semimodular lattice. In a geometric lattice every pair of distinct atoms is covered by their supremum the dual of which leads to the introduction of projective lattice. A geometric lattice is called a projective lattice P if it satisfies the dual of (G), i.e., for every element a and any anti-atom h

$$(G^*)\ a \vee h = 1 \Rightarrow a \wedge h < h$$

Evidently a projective lattice is modular.

Po-groupoid

A po-groupoid (or M-poset) is a poset M with binary multiplication which satisfies the isotonicity condition.

$$a \leq b \Rightarrow ax \leq bx, \qquad (1)$$

If the multiplication is commutative or associative, po-groupoid is called commutative po-groupoid or po-semigroup. A po-semigroup with identity (or neutral element) 1 such that

$$x1 = 1x = x \text{ , for all } x$$

is called a po-monoid. A zero of po-groupoid is an element $0 \in M$ such that for all $x \in M$,

$$0 \leq x \text{ and } 0x = x0 = 0.$$

Trivially, a pogroupoid can have at most one zero.

Note that a multiplicative semi lattice is a semilattice M under $\vee$ with a multiplication such that

$$a\,(b\vee c) = ab \vee ac \text{ and } (a \vee b)\, c = ac \vee bc \qquad (2)$$

for all $a, b, c \in M$. If M is a lattice with a multiplication and (2) holds, M is called an m-lattice or ℓ-groupoid. An ℓ-groupoid which is a semigroup (monoid) under multiplication is called an ℓ-semigroup (ℓ-monoid, short for lattice ordered monoid) Note that (1) follows trivially from (2) if $b \leq c$, then $ac = a\,(b \vee c) = ab \vee ac \Rightarrow ab \leq ac$. In otherwords any m-semilattice is trivially a po-groupoid (i.e., m-poset). Note that any pogroup which is a lattice satisfies (2) and its dual i.e., $a\,(b \wedge c) = ab \wedge ac$ and $(a \wedge b)\, c = ac \wedge bc$, it remains an ℓ-group under dualisation. However, in general, the dual of an ℓ-semigroup is not an ℓ-semigroup.

A group which is at the sometime a lattice L in which every group translation is isotone, is called a lattice-ordered group or ℓ-group. If the additive notation will be used for group operation so that group translation will be written $x \to a + x + b$, the group identity as 0, the inverse of x as $-x$ and $a + a + ... + a$ (to n summands) as na. Hence the assumption that group translation are isotone is equivalent to.

$$x \leq y \Rightarrow a + x + b \leq a + y + b, \text{ for all } a, b \in L.$$

Residuated lattice :

Let L be any po-groupoid. The right-residual $a : b$ of a by b is the largest x (if it exists) such that $bx \leq a$; the left residual $a : b$ of a by b is the largest y such that $yb \leq a$. A residuated lattice is an ℓ-groupoid L in which $a : b$ and $b : a$ exist for all $a, b \in L$.

Metric lattices

Most important applications of lattices to mathematics involve limiting process like these of real analysis. Such process can be defined in many ways. The simplest way is in terms of valuation. By a valuation on a lattice L is meant a real-valued function $\upsilon[x]$ on L which satisfies. $[V_1]$ $\upsilon[x] + \upsilon[y] = \upsilon[x \wedge y] + \upsilon[x \vee y]$ A valuation is isotone if and only if $[V_2]$ $x \geq y$ implies $\upsilon[x] \geq \upsilon[y]$ and positive if and only if $x > y \Rightarrow \upsilon[x] > \upsilon[y]$. In any modular lattice of finite length, the height function $h[x]$ is a positive valuations. In any real finite - dimensional vector space R^n, lattice ordered by letting $(x_1, ..., x_n) \leq (y_1, ..., y_n)$ mean that $x_k \leq y_k$ for all k, any linear functional $c[x] = c_1x_1 + ... + c_nx_n$ is a valuation. In any lattice L with an isotone valuation, the distance function.

(1) $d(x, y) = \upsilon[x \vee y] - \upsilon[x \wedge y]$ Satisfies for all $x, y, z, a \in L$:

(2) $d(x, y) \geq 0$, $d(x, x) = 0$, $d(x, y) = d(y, x)$

(3) $d(x, y) + d(y, z) \geq d(x, z)$

(4) $d(a \vee x, a \vee y) + d(a \wedge x, a \wedge y) \leq d(x, y)$

Any set M on which is defined a distance or metric $d(x, y)$ satisfying (2) and (3) is called a pseudo-metric (or quasi-metric) space. Hence a lattice with isotone valuation is called a pseudo-metric (or quasi-metric) lattice. If $d(x, y) = 0$ implies $x = y$ in a pseudo-metric space M, then M is by definition a Metric space. In a pseudo-metric lattice, this condition is equivalent by $[V_1]$ and $[V_2]$ to the condition $x \vee y > x \wedge y$ should imply $\upsilon[x \vee y] > \upsilon[x \wedge y]$. Hence a pseudo-metric lattice yields a metric space under (1) if and only if the valuation which defines it is positive. Lattices with positive valuation are therefore called metric lattices. Every metric lattice is modular. Any pseudo-metric lattice L is a pseudo-metric space in which joins and meets are uniformly continuous. The relation $d(x, y) = 0$ is a congruence relation.

Heyting algebra

In mathematics, Heyting algebras are special partially ordered sets that constitute a generalization of Boolean algebras, named after Arend Heyting. Heyting algebras arise as models of intuitionistic logic, a logic in which the law of excluded middle does not in general hold. Complete Heyting algebras are a central object of study in pointless topology.

Formal definition

A Heyting algebra H is a bounded lattice such that for all a and b in H there is a greatest element x of H such that

$$a \wedge x \leq b.$$

This element is the relative pseudo-complement of $\neg a$ with respect to b, and is denoted by $a \to b$. We write 1 and 0 for the largest and the smallest element of H, respectively.

In any Heyting algebra, one defines the pseudo-complement $\neg x$ of any element x by setting $\neg x = x \to 0$. By definition, $a \wedge \neg a = 0$. However, it is not in general true that $a \vee \neg a = 1$.

A complete Heyting algebra is a Heyting algebra that is a complete lattice.

A subalgebra of a Heyting algebra H is a subset H_1 of H containing 0 and 1 and closed under the operations, $\vee$, $\wedge$ and $\to$. It follows that it is also closed under $\neg$. A subalgebra is made into a Heyting algebra by the induced operations.

1.7. ALTERNATIVE DEFINITIONS

Lattice-theoretic definitions

An equivalent definition of Heyting algebras can be given by considering the mappings.

$$f_a : H \to H \text{ defined by } f_a(x) = a \wedge x.$$

for some fixed a in H. A bounded lattice H is a Heyting algebra if and only if all mappings f_a are the lower adjoint of a monotone Galois connection. In this case the respective upper adjoints g_a are given by $g_a(x) = a \to x$, where $\to$ is defined as above.

Yet another definition is as a residuated lattice whose monoid operation is $\wedge$. The monoid unit must then be the top element 1. Commutativity of this monoid implies that the two residuals coincide as $a \rightarrow b$.

Bounded lattice with an implication operation

Given a bounded lattice A with largest and smallest elements 1 and 0, and a binary operation $\rightarrow$, these together form a Heyting algebra if and only if the following hold:

1. $a \rightarrow a = 1$
2. $a \wedge (a \rightarrow b) = a \wedge b$
3. $b \wedge (a \rightarrow b) = b$
4. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$ (distributive law for $\rightarrow$)

Distributivity

Heyting algebras are always distributive. Specifically, we always have the identities.

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

The distributive law is sometimes stated as an axiom, but in fact it follows from the existence of relative pseudo-complements. The reason is that, being the lower adjoint of a Galois connection, $\wedge$ preserves all existing suprema. Distributivity in turn is just the preservation of binary suprema by $\wedge$.

By a similar argument, the following infinite distributive law holds in any complete Heyting algebra:

$$x \wedge \vee \mathrm{Y} = \vee \{x \wedge y : y \in Y\}$$

for any element x in H and any subset Y of H. Conversely, any complete lattice satisfying the above infinite distributive law is a complete Heyting algebra, with

$$a \rightarrow b = \vee \{c : a \wedge c \leq b\}$$

being its relative pseudo-complement operation.

Regular and complemented elements

An element x of a Heyting algebra H is called regular if either of the following equivalent conditions hold:

1. $x = \neg\neg\, x$.

2. $x = \neg y$ for some y in H.

The equivalence of these conditions can be restated simply as the identity $\neg\neg\neg\, x = \neg\, x$, valid for all x in H.

Elements x and y of a Heyting algebra H are called complements to each other if $x \wedge y = 0$ and $x \vee y = 1$. If it exists, any such y is unique and must in fact be equal to $-x$. We call an element x complemented if it admits a complement. It is true that if x is complemented, then so is $-x$, and then x and $\neg x$ are complements to each other. However, confusingly, even if x is not complemented, $-x$ may nonetheless have a complement (not equal to x). In any Heyting algebra, the elements 0 and 1 are complements to each other.

Any complemented element of a Heyting algebra is regular, though the converse is not true in general. In particular, 0 and 1 are always regular.

For any Heyting algebra H, the following conditions are equivalent:

1. H is a Boolean algebra,
2. Every x in H is regular,
3. Every x in H is complemented.

In this case, the element $a \rightarrow b$ is equal to $\neg a \vee b$.

The De Morgan laws in a Heyting algebra

One of the two De Morgan laws is satisfied in every Heyting algebra, namely

$$\neg(x \vee y) = \neg x \wedge \neg y, \text{ for all } x, y \in H$$

However, the other De Morgan law does not always hold. We have instead a weak de Morgan law:

$$\neg(x \wedge y) = \neg\neg(\neg x \vee \neg y), \text{ for all } x, y \in H$$

The following statements are equivalent for all Heyting algebras H:

1. H satisfies both De Morgan laws;

(i) $\neg(x \wedge y) = \neg x \vee \neg y$, for all $x, y \in H$

(ii) $\neg(x \wedge y) = \neg x \vee \neg y$, for all regular $x, y \in H$;

2. $\neg\neg(x \vee y) = \neg\neg x \vee \neg y$, for all x, y in H;

3. $\neg\neg(x \vee y) = x \vee y$, for all regular x, y in H;

4. $\neg(\neg x \wedge \neg y) = x \vee y$, for all regular x, y in H;

5. $\neg x \vee \neg\neg x = 1$, for all $x \in H$.

Topological lattice:

A topological lattice is lattice L equipped with a topology T such that meet and join operations from $L \times L$ (with the product topology) to L are continuous. Let $(x_i)_{i \in I}$ be a net in L, we say that (x_i) converges to x if (x_i) is eventually in any open neighbourhood (nhd) of x, and we write $x_i \to x$ [6].

If (x_i) and (y_i) are net indexed by I, J respectively, then $(x_i \wedge y_i)$ and $(x_i \vee y_i)$ are nets, both indexed by $I \times J$. This is clear, and is stated in preposition below:

Preposition : If $x_i \to x$ and $y_j \to y$, then

$$x_i \wedge y_j \to x \wedge y \text{ and } x_i \vee y_j \to x \vee y$$

Proof : Let us show the first convergence, and the other follows similarly. The function.

$f : x \to (x, y) \to x \wedge y$ is a continuous function, being composition of two function. If $x \wedge y \in U$ is open, then $x \in f^{-1}(U)$ is open. As $x_i \to x$, there is an $i^0 \in I$ such that $x_i \in f\,(U)$ for all $i \geq i^0$ which means $x_i \wedge y = f(x_i) \in U$. By the same token, for each $i \in I$, the function $g_i : y \to (x_i\, y) \to x_i \wedge y$ is a continuous function. Since $x_i \wedge y \subset U$ is open, $y \in g^{-1}(U)$ is open. As $y_j \to y$, there is an $j^0 \in J$ such that $y_j \in g^{-1}(U)$ for all $j \geq j^0$, or $x_i \wedge y_j = g_i\,(y_i) \in U$ for all $i \geq i^0$ and $j \geq j^0$. Hence $x_i \wedge y_j \to x \wedge y$. For any net (x_i), the set A = $\{a \in L / x_i \to a\}$ is a sublattice. This follows from the fact that if a, $b \in$ A, then $x_i = x_i \wedge x_i \to a \wedge b$. So $a \wedge b \in$ A. Similarly $a \vee b \in A$.

There are two approaches to finding examples of topological lattices. One way is to start with a topological space X such that X is partially ordered, then find two continuous binary operations on X to form the meet and join operation of a lattice [4].

The real numbers with operations $a \wedge b = \inf \{a, b\}$ and $a \vee b = \sup \{a, b\}$ is one such example. This can be easily generalised to the real-valued continuous functions, since, given any two real valued continuous functions *f* and *g*.

$f \vee g = \max (f, g)$ and $f \wedge g = \min (f, g)$ are well defined real valued continuous function as well (it is enough to say that for any continuous function *f*, its absolute value $| f |$ is also continuous, so that

$$\max (f, 0) = 1/2 (f + | f |)$$

and thus $$\max (f, g) = \max (f - g, 0) + g$$

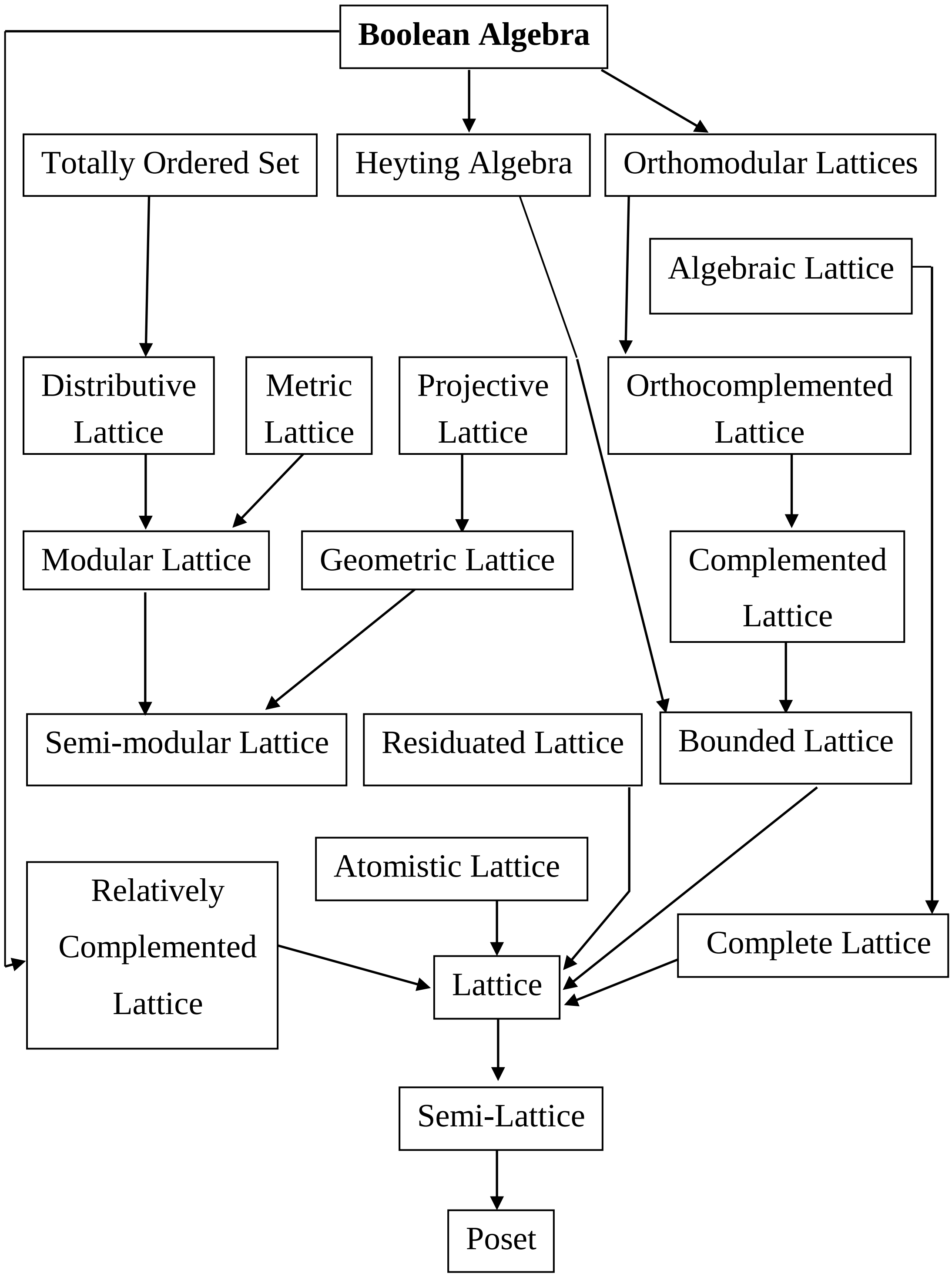
and $$\min (f, g) = f + g - \max (f, g)$$

are both continuous.

The second approach is to start with a general lattice *L* and define a topology *T* on the subset of the underling set *L* with the hope that both $\vee$ and $\wedge$ are continuous under *T*. The obvious example using the second approach is to take the discrete topology of the underlying set. Another way is to impose conditions, such as requiring that the lattice be meet and join continuous. Of course, finding a topology on underlying of a lattice may not guarantee a topological lattice unless and until the lattice operations are continuous.

1.8 MAP OF LATTICES

The concept of a lattices arises in order theory, a branch of Mathematics. The Hasse diagram below depicts the inclusion relationships among some important subclasses of lattices.



Proofs of the relationships in the map are inherited from the book: Introduction to lattice theory By Rutherford, Oliver and Boyd publication, 1965 which are explicity mentioned as below [8].

1. A Boolean algebra is a complemented distributive lattice (definition) (P.77).

2. A Boolean algebra is a Heyting algebra (P. 32-33)

3. A distributive orthocomplemented lattice is orthomodular.

4. A Boolean algebra is orthocomplemented (P22).

5. A Boolean algebra is orthomodular (P-31)

6. An orthomodular lattice is orthocomplemented (definition) (theorem 25.1, P.74).

7. An orthocomplemented lattice is complemented (definition) (theorem 8.1, P22).

8. A complemented lattice is bounded (definition) (P. 87) [8].

9. An algebraic lattice is complete (definition) (P.94)

10. A complete lattice is bounded (theorem 32.1, P.92).

11. A Heyting algebra is bounded (P.89)

12. A bounded lattice is a lattice (definition)

13. A Heyting algebra is residuated.

14. A residuated lattice is a lattice.

15. A distributive lattice is modular

16. A modular complemented lattice is relatively complemented.

17. A Boolean algebra is relatively complemented.

18. A relative complemented lattice is a lattice

19. A Heyting algebra is distributive.

20. A totally ordered set is a distributive lattice.

21. A metric lattice is modular.

22. A modular lattice is semi-modular

23. A projective lattice is modular.

24. A projective lattice is geometric.

25. A geometric lattice is semi-modular.

26. A semi-modular lattice is atomistic.

27. An atomistic lattice is a lattice (def)

28. A lattice is a semi-lattice (def)

29. A semi-lattice is a partially ordered set (def)

30. For weakly atomistic continuous lattice M-symmetry is equivalent to semi-modularity.

31. For algebraic lower continuous lattices M-symmetry is equivalent to semi-modularity.

32. For lattices which are algebraic and dually algebraic. M-symmetry is equivalent to semi-modularity.

The last three results follows from the book. The theory of Symmetric lattice by F. Maeda and S. Maeda, Springer-verlag, New York, 1970 [2] and a reputed paper by Chinthayamma Malliah and S.P. Bhatta, Bull of London Mathematical Society 18 (1986), 338-342 [1].

REFERENCES

[1] Chintayamma and S. P. Bhatta (1986): Bull of London Math. Soc., 18, 338-342.

[2] F. Maeda and S. Maeda (1970): The Theory of Symmetric lattices, Springer Verlag, New York.

[3] G. Brikhoff (1984): Lattice Theory, AMS Publication. (Reprinted)

[4] G. Bezhamishirti and J. Mamulha (2008): Lattices & Topologies: An introductory course. Comb. Univ. Press, London.

[5] G. Geirz, Hofmon, K. Keinel (2003): Continuous lattices & domain, comb. Univ. Press.

[6] P. G. Cataresec (2007): J. of Phil. Logic, vol. 34, No. 4, 363-401.

[7] Q. Zhang; W. Xie and L. Fan (2009): Fuzzy sets and system, 160, (16)

[8] Ruthsford (1965): Inoduation to Lattice theory Oliver & Boyd Publication.

*****

CHAPTER II

CONDITIONS FOR A COMPLETE BOOLEAN ALGEBRA TO CARRY MAHARAM SUBMEASURE

2.0. INTRODUCTION

This chapter investigates the weak distributivity of Boolean σ-algebras satisfying the countable chain condition. It addresses primarily the question when such algebras carry a σ-additive measure. We use as a starting point the problem of John von Neumann stated in 1937 in the Scottish Book. He asked if the countable chain condition and weak distributivity are sufficient for the existence of such a measure.

Subsequent research has shown that the problem has two aspects: one set theoretic and one combinatorial. Recent results provide a complete solution of both the set theoretic and the combinatorial problems. We shall survey the history of von Neumann's Problem and outline the solution of the set theoretic problem.

2.1. COMPLETE BOOLEAN ALGEBRAS AND WEAK DISTRIBUTIVITY

A Boolean algebra is a set B with Boolean operations $a \vee b$ (join), $a \wedge b$ (meet) and $-a$ (complement), partial ordering $a \leq b$

defined by $a \wedge b = a$ and the smallest and greatest element, 0 and 1. By Stone's Representation Theorem, every Boolean algebra is isomorphic to an algebra of subsets of some nonempty set *S*, under operations $a \bigcup b$, $a \bigcap b$, S - a, ordered by inclusion, with $0 = \phi$ and $1 = S$.

If every subset *A* of *B* has a least upper bound $\omega\, A$ and the greatest lower bound $\varpi\, A$ then *B* is a complete Boolean algebra. An antichain in *B* is a set $A \subseteq B$ such that distinct elements $a, b \in A$ are disjoint i.e., $a \wedge b = 0$. *B* satisfies the countable chain condition (ccc) if it has no uncountable antichains.

If *B* is a ccc Boolean σ-algebra, i.e., $\omega_{n\in\omega}\, a_n$ and $\varpi_{n\in\omega}\, a_n$ exist for countable sets, then *B* is a complete Boolean algebra. For this and other basic facts on Boolean algebras, we refer the reader to [24], [30].

The set of all nonzero elements of a Boolean algebra *B* is denoted B^+. A set $D \subseteq B^+$ is dense in *B* if for each $b \in B^+$ there exists some $d \in D$ with $d \leq b$. For every Boolean algebra *A* there exists a unique (up to isomorphism) complete Boolean algebra *B* such that *A* is a subalgebra of *B* and A^+ is dense in *B*. The complete Boolean algebra *B* is called the completion of *A*.

An atom of *B* is a nonzero $a \in B$ that cannot be splitted into two disjoint nonzero elements. *B* is atomic if the set of all atoms is dense in *B*, and atomless if it has no atoms.

Examples.

I. The **power set algebra $\mathcal{P}(\omega)$.** Consider the algebra $\mathcal{P}(\omega)$ of all sets of natural numbers. This algebra is complete, with least upper bounds $\bigcup X$ for $X \subseteq \mathcal{P}(\omega)$, and satisfies ccc. This algebra is atomic, where the atoms are the singletons.

II. The Cohen algebra. Let A be the countable atomless Boolean algebra (this is unique up to isomorphism), and let $\mathcal{C} = \mathcal{C}_\omega$ be the completion of A. The standard representation of A is the algebra of all clopen sets of the Cantor space, the Cantor algebra. Since A has a dense set isomorphic to an infinite binary tree (the set of all finite 0-1 sequences under reverse inclusion), the algebra $\mathcal{C}$ is nowadays called the Cohen algebra, in recognition of its role in forcing.

The open intervals with rational endpoints form a countable dense set. Since $\mathcal{C}$ has a countable dense set, every antichain in $\mathcal{C}$ is necessarily countable, and so $\mathcal{C}$ satisfies ccc. Thus $\mathcal{C}$ is a complete Boolean algebra, and is atomless.

III. A measure algebra. Consider the quotient algebra $\mathcal{M}$ of Borel sets in the interval [0, 1] modulo null sets, i.e., sets of Lebesgue measure 0. $\mathcal{M}$ is an atomless σ-algebra, and carries a (σ-additive strictly positive) measure, a numerical function m with the following properties

$m(0) = 0$, $m(a) > 0$ for $a \neq 0$, and $m(1) = 1$ (2.1)

$m\left(\bigvee_{n\in\omega} a_n\right) = \sum_{n\in\omega} m(a_n)$ whenever the a_n are pairwise disjoint.

An atomless σ-algebra that carries a measure is called a measure algebra.

If A is an antichain in a measure algebra then for every n, only finitely many $a \in A$ have measure greater than $\frac{1}{n}$; and A is necessarily countable. Hence every measure algebra satisfies ccc (and is complete). By a special case of the classification theorem of Maharam [26], the algebra $\mathcal{M}$ is the unique atomless measure algebra with countably many generators.

The complete Boolean algebras $\mathcal{M}$ and $\mathcal{C}$ are different, as $\mathcal{C}$ does not carry a measure (cf. [35]): Assume m is such a measure. Since $\mathcal{C}$ has a countable dense set $\{d_n: n < \omega\}$, choose for each n some nonzero $x_n \leq d_n$ with $m(x_n) < \frac{1}{2^{n+1}}$. If $x = \bigvee_{n\in\omega} x_n$ and $y = -x$, then $y \neq 0$ but no d_n is below y; a contradiction.

We remark that the atomic algebra $\mathcal{P}(\omega)$ does carry a measure, for instance one such that $m(\{n\}) = \frac{1}{2^{n+1}}$ for each $n \in \omega$. Also, the Cohen algebra carries a finitely additive measure, i.e., a function that satisfies $m(a \vee b) = m(a) + m(b)$ for disjoint a, b.

The algebras $\mathcal{M}$ and $\mathcal{C}$ were considered by von Neumann in his 1936-37 lectures [42], [43] on Continuous Geometry at the Institute for Advanced Study. Among others, he introduced the weak distributivity law, an algebraic property that distinguishes these two algebras.

Every complete Boolean algebra satisfies the following generalization of the distributive law

$$\bigvee_{x\in X} a_x \wedge \bigvee_{y\in Y} b_y = \bigvee_{\substack{x\in X\\ y\in Y}} \left(a_x \wedge b_y\right), \tag{2.2}$$

where X and Y are arbitrary indexed sets.

A general distributive law

$$\bigwedge_{x\in X} \bigvee_{y\in Y} a_y^x = \bigvee_f \bigwedge_x a_{f(x)}^x, \tag{2.3}$$

where f ranges over all functions from X to Y, holds only when the algebra is atomic. In fact, both $\mathcal{M}$ and $\mathcal{C}$ fail to satisfy the simplest case of infinitary distributivity, namely

$$\bigwedge_{n\in\omega} \left(a_0^n \vee a_1^n\right) = \bigvee_{f:\omega\to\{0,1\}} \bigwedge_{n\in\omega} a_{f(n)}^n, \tag{2.4}$$

To see this, consider the binary expansions $0.\varepsilon_1\varepsilon_2\varepsilon_3...$ of numbers in [0, 1], and let A_i^n (i = 0, 1) be the set of all reals with $\varepsilon_n = i$. Let a_i^n be A_i^n modulo either null or meager sets. Then the left hand side of (2.4) is 1 while for every f, $\bigwedge_{n\in\omega} a_{f(n)}^n = 0$.

To characterize measure algebras, von Neumann formulated the following weak distributivity law:

If $a_0^n \leq a_1^n \leq \ldots$ for n = 1, 2, ..., then (2.5)

$$\bigwedge_{1}^{k} \bigvee_{k=1}^{\infty} a_k^n = \bigvee_{f:\omega\to\omega} \bigwedge_{n=r}^{\infty} a_{f(n)}^n .$$

(The left hand side of (2.5) is always ≥ the right hand side, in every complete Boolean algebra.) That a measure algebra satisfies (2.5) is proved as follows (this idea appears earlier in [6] where Banach and Kuratowski proved that under the Continuum Hypothesis, there is no σ-additive extension of Lebesgue measure to all sets of reals).

Let m be a measure on B, and let $a_0^n \leq a_1^n \leq \ldots$ for n = 0, 1, 2, … . Without loss of generality we assume that $\bigvee_k a_k^n = 1$ for all n. To verify (2.5) it suffices to find for each $\varepsilon > 0$ some f such that $m\left(\bigwedge_n a_{f(n)}^n\right) \geq 1 - \varepsilon$. And that can be done by choosing $f(n)$ for each n so that $m\left(a_{f(n)}^n\right) \geq 1 - \frac{\varepsilon}{2^{n+2}}$.

Unlike $\mathcal{M}$, the algebra $\mathcal{C}$ is not weakly distributive. Let $\{d_n : n \in \omega\}$ be a countable dense set in $\mathcal{C}$. For each n we can find a strictly increasing sequence $a_0^n \leq a_1^n \leq \ldots$ with $\bigvee_k a_k^n = 1$ such that $d_n \not\leq a_k^n$ for all $k \in \omega$. Now if $f : \omega \to \omega$ is arbitrary, we have $a_f = \bigwedge_n a_{f(n)}^n = 0$ because otherwise there would have to exist some $d_n \leq a_f$, which is impossible.

2.2. THE PROBLEM OF VON NEUMANN.

From 1935 until 1941, a group of mathematicians in the (then) Polish city of Lwów met frequently in the Scottish Coffee House, often with visitors, and recorded a number of problems in a large notebook started by Stefan Banach. After World War II a copy of the notebook found its way to the United States, where Stanislaw Ulam, one of the original participants, published the collection of almost two hundred problems under the title 'The Scottish Book'. An annotated edition, edited by Daniel Mauldin, appeared as [28].

Problem no. 163, dated July 4, 1937, was entered by John von Neumann. It states the weak distributive law (2.5) and asks if ccc and weak distributivity are sufficient for a complete Boolean algebra to carry a measure. (It offers a prize: A bottle of whiskey of measure > 0.)

The rest of this article investigates von Neumann's question, whether every weakly distributive complete ccc Boolean algebra is a measure algebra. With this in mind, we henceforth consider only complete Boolean algebras that satisfy ccc.

2.3. THE WORK OF DOROTHY MAHARAM AND SEQUENTIAL TOPOLOGY.

In order to motivate the technique introduced by Maharam in [27], let B be a measure algebra with measure m. For any $a, b \in B$, let $d(a, b) = m(a \Delta b)$, where $a \Delta b$ is the symmetric difference $(a - b) \vee (b - a)$. Since

$$\left.\begin{array}{l} d(a,b) = d(b,a). \\[1em] d(a,a) = 0 \text{ and } d(a,b) > 0 \text{ if } a \neq b, \\[1em] d(a,b) + d(b,c) \geq d(a,c), \end{array}\right\} \quad (2.6)$$

d is a distance function on *B*, and, as a consequence of σ-additivity of *m*, (*B*, *d*) is a complete metric space.

We remark that if *B* is the atomic algebra $\mathcal{P}(\omega)$, then under the identification of $\mathcal{P}(\omega)$ with 2^{ω}, (*B*, *d*) is homeomorphic to the Cantor space and so is a compact Hausdorff space.

The first observation of Maharam was that in order to prove ccc and weak distributivity one does not need a measure on *B*, but an ostensibly weaker property:

A function *m* on *B* is a continuous submeasure if

$$\left.\begin{array}{ll} \text{(a)} & m(0) = 0,\ m(a) > 0 \text{ for } a \neq 0, \text{ and } m(1) = 1, \\ \text{(b)} & m(a) \leq m(b) \text{ if } a \leq b, \\ \text{(c)} & m(a \vee b) \leq m(a) + m(b), \\ \text{(d)} & \lim_n m(a_n) = 0 \text{ for every decreasing sequence } a_n \\ & \text{with } \bigvee_n a_n = 0. \end{array}\right\} \quad (2.7)$$

We call a continuous submeasure a Maharam submeasure, and complete Boolean algebra *B* a Maharam algebra if it carries a Maharam submeasure. Every measure is Maharam submeasure.

Proposition 2.1. A Maharam algebra satisfies ccc and is weakly distributive.

Proof. For ccc, we claim that for every $\varepsilon > 0$, there exist only finitely many disjoint elements a such that $m(a) \geq \varepsilon$. If there existed an infinite antichain $\{a_n\}$ such that $m(a_n) \geq \varepsilon$ for each n, then letting $b_n = \bigvee_{k \geq n} a_k$, we would get a descending sequence violating the continuity of m.

As for weak distributivity, the proof is the same as for a measure.

The problem of von Neumann splits naturally into the following two problems:

Problem 2.1. Is every Maharam algebra a measure algebra?

Problem 2.2. Is every weakly distributive complete ccc Boolean algebra a Maharam algebra?

The first problem has been studied in functional analysis and is known as the Control Measure Problem, see [21] or [10], vol. 3. We shall address it in the next section. The second problem, the von Neumann-Maharam Problem, is our main object, and we shall outline its solution.

Before we introduce Maharam's method we present another observation from the paper [27].

The ordering of the real line is the unique linear order (up to isomorphism) that is complete, dense, with no endpoints, and has a countable dense subset. As a consequence, it satisfies the countable chain condition (ccc), i.e., every disjoint collection of open intervals is at most countable. A problem of Mikhail Suslin [34] from 1920 asks whether every complete ccc dense linear order without endpoints is isomorphic to the real line. The problem remained unsolved until the 1960's when it was established that it is undecidable: it is both consistent with and independent of the axioms ZFC of set theory. See [17], [37], [33].

A Suslin line is a complete ccc dense linear order that does not have a countable dense subset (a counterexample to Suslin's problem). A Suslin tree is an ω_1-tree with no uncountable chains or antichains. A Suslin algebra is an atomless complete ccc Boolean algebra that satisfies the (ω, ω)-distributive law (i.e., the distributivity law (2.3) with $X = Y = \omega$).

A Suslin line, a Suslin tree and a Suslin algebra can be constructed from each other (see [25], [29] or [19] for details).

Maharam showed that a Suslin algebra does not carry a continuous sub-measure. To see this, let *B* be a Suslin algebra and let *m* be a continuous sub-measure on *B*. First we claim that for every $\varepsilon > 0$, the set $\{a \in B: m(a) < \varepsilon\}$ is dense in *B*. Otherwise, one could find a decreasing sequence *a,,* with $\vee_n a_n = 0$ and $m(a_n) \geq \varepsilon$, contradicting the continuity of *m*.

Thus for each $n \in \omega$ there exists a maximal antichain A_n in B such that $m(a) < 1/n$ for all $a \in A_n$. By the (ω, ω)-distributive law there exists some $b > 0$ such that for every n, $b \leq a$ for some $a \in A_n$. It follows that $m(b) < 1/n$ for every n, a contradiction.

Thus a Suslin algebra is counterexample to the von Neumann-Maharam Problem, and one has to modify the problem as follows

Problem 2.3. Is it consistent that every weakly distributive complete ccc Boolean algebra is a Maharam algebra?

Now we shall introduce Maharam's method. Let B be a Boolean σ-algebra that carries a Maharam submeasure m. Then $d(x, y) = m(x \Delta y)$ is a metric on B, (B, d) is a complete metric space, and for each $a \in B$, the mapping $T^a(x) = a \Delta x$ is an isometry $(d(x \Delta a, y \Delta a) = d(x, y))$. As $(B, \Delta, 0)$ is an abelian group, d is an invariant metric on this group. The metric topology on B is determined by neighborhoods of 0 and is invariant under the translations T^a. Moreover, the Boolean operations $\vee$, $\wedge$ and Δ are continuous and $(B, \Delta, 0)$ is a topological group.

It turns out that this topology τ can be defined algebraically on any Boolean σ-algebra B, and the existence of a Maharam submeasure on B is related to properties of the topological space (B, τ).

Convergence and the sequential topology on *B*.

Let B be a Boolean σ-algebra. An infinite sequence $\{a_n\}_n$ converges to a, $\lim_n a_n = a$, if

$$\limsup_n a_n = \liminf_n a_n = a.$$

where $$\limsup_n a_n = \bigwedge_n \bigvee_{k\geq n} a_k, \liminf_n a_n = \bigvee_n \bigwedge_{k\geq n} a_k.$$

Equivalently, we define $\lim_n a_n = 0$ whenever there exists a decreasing sequence b_n with $\bigwedge_n b_n = 0$ such that $a_n \leq b_n$ for all n. Then we let $\lim_n a_n = a$ if $\lim_n(a_n \Delta a) = 0$.

We summarize the basic properties of convergence:

(a) If $a_n = a$ for all n then $\lim_n a_n = a.$. (2.8)

(b) If $\{a_n\}_n$ converges to a and π is a permutation of ω then $\{a_{\pi(n)}\}_n$ also converges to a.

(c) $\lim_n a_n = 0$ if and only if $\limsup_n a_n = 0$,

(d) if the a_n are pairwise disjoint then $\lim_n a_n = 0$,

(e) $\limsup_n (a_n \vee b_n) = \limsup_n a_n \vee \limsup_n b_n$,

(f) if $\lim_n a_n = a$ and $\lim_n b_n = b$ then $\lim_n - a_n = -a$

$\lim_n (a_n \vee b_n) = a \vee b$ and $\lim_n (a_n \wedge b_n) = a \wedge b$.

For details, see [27] or [41].

A set $F \subseteq B$ is closed if $\lim a_n \in F$ whenever $\{a_n\}_n$ is a sequence in F. Let τ denote the topology on B so obtained; it is the sequential topology on B. The space (B, τ) is T_1 (every singleton is closed). The closure cl(A) of a set $A \subseteq B$ is generally obtained by taking limits of convergent sequences and iterating this ω_1 times. Maharam pointed out that the iteration is not necessary if B is ccc and weakly distributive: in this case cl(A) is the set of all limits of sequences of A. We shall return to this in section 2.5.

The operations $\vee$, $\wedge$ and Δ are not necessarily continuous as functions of two variables, even though (by (2.8)) they are continuous separately in each variable. Since each T^a is a continuous translation, (B, τ) is a homogeneous space and τ is determined by neighborhoods of 0.

If B is a Maharam algebra and (B, d) is the metric space with the distance function $d(a, b) = m(a \Delta b)$, then the metric topology and the sequential topology τ coincide. Maharam showed that conversely, metrizability of τ is itself sufficient for the existence of a Maharam submeasure.

Theorem 2.1 (Maharam [27]). A complete Boolean algebra B is *a* Maharam algebra if and only if the sequential topology on B is metrizable.

We shall outline a proof of the theorem. It has two main ingredients: continuity of Boolean operations and a metrization theorem of Kakutani. A topological space is called first countable if every point a has a countable system of open neighborhoods $\{U_n\}_n$ such that for every open neighborhood V *of* a there is some n with $U_n \subseteq V$.

Theorem 2.2 (Kakutani [20], see also (2.5)). If $(G, +, 0)$ is a topological abelian group and its topology is first countable, then it is metrizable and has an invariant metric.

Proof of Theorem 2.1. Let τ be the sequential topology on B and assume that it is metrizable. Let d be a metric on B such that its topology coincides with τ. It is easy to see that if $\lim_n a_n = 0$ then $\lim_n d(a_n, 0) = 0$ and if $\lim_n d(a_n, 0) = 0$ then some subsequence of a_n converges to 0.

We claim that the function Δ is continuous. Since the translations T^a are continuous, it suffices to prove continuity at (0, 0). If Δ is not continuous at (0, 0), then there exist sequences x_n and y_n and some $\in > 0$ such that $\lim_n d\{x_n, 0) = \lim_n d\{y_n, 0) = 0$ but $d(x_n \Delta y_n, 0) \geq \varepsilon$ for all n. There exists a subsequence $n(1)$, $n(2)$, ... such that $\lim_k x_{n(k)} = \lim_k y_{n(k)} = 0$. It follows that $\lim_k (x_{n(k)} \Delta y_{n(k)}) = 0$, a contradiction.

Thus $(B, \Delta, 0)$ is a topological abelian group, and so by Kakutani's theorem 2.2, τ is metrizable by an invariant metric ρ, i.e., $\rho(x, y) = \rho(x \Delta y, 0)$.

If we define $v(x) = \rho(x, 0)$ then the function v satisfies

$$v(x \Delta y) < v(x) + v(y), \tag{2.9}$$

(by the triangle inequality for ρ) and so, if we let further

$$\mu(x) = \min(\nu(x), 1)$$

and

$$m(a) = \sup\{\mu(x): x \leq a\},$$

we can verify that m is a continuous submeasure on B.

Investigating the sequential topology on B, Maharam was able to formulate a better sufficient condition for the existence of Maharam submeasure.

Theorem 2.3 (Maharam [27]). *A* complete ccc Boolean algebra B is a Maharam algebra if and only if

(i) B is weakly distributive, and

(ii) the space (B, τ) is first countable.

Maharam proved (2.9) by using the assumptions to show that Δ is continuous, and then applied Kakutani's theorem. Later on we shall introduce a condition weaker than first countability (the G_δ property) and prove its sufficiency.

A final result of Maharam was that the following additional requirement on the countable neighborhood base U_n produces not just a submeasure but a measure.

$$\text{if } x, y \notin U_n \text{ and } x \wedge y = 0 \text{ then } x \vee y \notin U_{n+1}. \qquad (2.10)$$

On this, no progress has been made to date.

2.4. FURTHER PROGRESS AND THE EVENTUAL SOLUTION OF THE VON NEUMANN-MAHARAM PROBLEM.

There has been a vast number of publications related to von Neumann's problem since 1947. We shall only mention the ones most relevant to our results.

In [16], Alfred Horn and Alfred Tarski investigated systematically measures on Boolean algebras, both σ-additive and finitely additive. They presented in detail the work of von Neumann on Boolean algebras and introduced the terminology (with some modifications). Among others, they introduced the following two chain conditions:

σ-bounded cc: (2.11)

There is a decomposition $B^+ = \bigcup\{S_n : n \in \omega\}$ such that for every n, S_n contains no antichain.

σ-finite cc: (2.12)

There is a decomposition $B^+ = \bigcup\{S_n : n \in \omega\}$ such that for every n, S_n contains no infinite antichain.

If B carries a finitely additive measure m then it satisfies (2.11): let $S_n = \left\{a \in B : m(a) \geq \frac{1}{n+1}\right\}$. In particular, (2.11) is a necessary condition for B to be a measure algebra. Every Maharam algebra must satisfy the weaker condition (2.12), and it is still an open problem whether (2.11) is equivalent to (2.12). Clearly, this is related to the Control Measure Problem.

In [23], John Kelley investigated Boolean algebras that carry a finitely additive measure, as well as complete Boolean algebras with a σ-additive measure. He showed that these two properties are related, and gave an algebraic characterization of both. Theorem 2.4 had been previously known to A. G. Pinsker, see [22, pp. 428-430].

Theorem 2.4 (Pinsker; Kelley [23]). A complete Boolean algebra *B* carries a σ-additive measure if and only if

(i) *B* is weakly distributive, and
(ii) *B* carries a finitely additive measure.

Kelley's main result is the following characterization of (ii):

For each finite sequence $s = (a_1, ..., a_n)$ of not necessarily distinct elements of B^+, let $k(s)$ be the maximum size of a subset $E \subseteq \{1,..., n\}$ such that $\bigvee_{j\in E} a_j \neq 0$ and let $i(s) = \frac{k(s)}{n}$. For a nonempty $X \subseteq B^+$, the intersection number of X is

$$\inf\{i(s) : s \text{ is a finite sequence in } X\}.$$

If m is a finitely additive measure on B and $S_n = \left\{a \in B : m(a) \geq \frac{1}{n+1}\right\}$ then one can verify that the intersection number of S_n is greater than or equal to $\frac{1}{n+1}$.

Theorem 2.5 (Kelley [23]). A necessary and sufficient condition for a Boolean algebra to carry a finitely additive measure is that there is a decomposition $B^{+} = \bigcup_{n\in\omega} S_n$ such that each S_n has a positive intersection number.

Consequently, a complete Boolean algebra *B* is a measure algebra if and only if *B* is weakly distributive and $B^{+} = \bigcup_{n\in\omega} S_n$ such that each S_n has a positive intersection number.

The Control Measure Problem.

Let *U* be a metrizable linear topological space and let *B* be a σ-algebra of sets. A function $\mu : B \to U$ is a vector measure if $\sum_0^\infty a_n = \lim_{n\to\infty} \sum_0^n a_n$ is defined in *U* and is equal to $\mu\left(\bigvee_n a_n\right)$ for every disjoint sequence $\{a_n\}_n$. A σ-additive measure *m* on *B* is a control measure for μ if $\mu(a) = 0$ if and only if $m(a) = 0$ (see [10]).

The Control Measure Problem is equivalent to the question whether every vector measure has a control measure.

In [21], Nigel Kalton and James Roberts found a significant reformulation of the Control Measure Problem. A submeasure on a Boolean algebra is a function *m* that satisfies 2.7 (a), (b), (c) (without continuity (d)).

Definition 2.1. *A* submeasure *m* on *B* is exhaustive if $\lim_n m(a_n) = 0$ for every infinite antichain $A = \{a_n : n \in \omega\}$. It is uniformly exhaustive if for every $\varepsilon > 0$ there exists $n \in \omega$ such that there is no sequence of n disjoint elements $a_1, ..., a_n \in B$ with $m(a_i) \geq \varepsilon$ for all $i = 1, ..., n$.

Every Maharam submeasure is exhaustive (see the proof of Proposition 2.1) and every measure is uniformly exhaustive. The main result of [21] is the following. Two submeasures m and μ are equivalent if $m(a_n) \to 0$ if and only if $\mu(a_n) \to 0$.

Theorem 2.6 (Kalton-Roberts [21]). Every uniformly exhaustive submeasure on a Boolean algebra is equivalent to a finitely additive measure.

Corollary 2.1. The control measure problem is equivalent to the statement: Every exhaustive submeasure on a Boolean algebra is uniformly exhaustive. (2.13)

Proof of Corollary 2.1 from Theorem 2.6. First we assume that every Maharam algebra is a measure algebra, and prove (2.13). Let m be an exhaustive submeasure on a Boolean algebra B. B can be embedded to a complete Boolean algebra C so that m extends to a Maharam submeasure μ in C (see [9]; C is the metric completion of B). Hence C is a Maharam algebra, and by the assumption it has a measure λ. Because μ and λ are equivalent (cf. [10]), μ is uniformly exhaustive. So is its restriction to B, and hence m is uniformly exhaustive.

In the other direction, assume that (2.13) holds. If *B* is a Maharam algebra with Maharam submeasure *m*, then *m* is exhaustive, and therefore uniformly exhaustive by (2.13). By the Kalton-Roberts Theorem, *B* carries a finitely additive measure, and by Kelley's Theorem 2.4, *B* is a measure algebra.

It can be seen that if (2.13) holds for all countable Boolean algebras then it holds for all Boolean algebras.

Back to Maharam submeasures.

The line of reasoning started by Maharam was continued in [3], leading to the following improvement of Maharam's Theorem 2.1:

Theorem 2.7 (Balcar, Glowczyhski, Jech [3]). A complete ccc Boolean algebra *B* is a Maharam algebra if and only if the space (B, τ) is a Hausdorff space.

In December 2003 (see [4]), Balcar and Jech obtained the following result. The space (B, τ) has the G_δ property if $\{0\}$ is a G_δ set, i.e., if there exist open neighborhoods U_n of 0 such that $\bigcap_{n\in\omega} U_n = \{0\}$.

Theorem 2.8 (Balcar, Jech, Pazak [5]). A complete ccc Boolean algebra is a Maharam algebra if and only if

(i) *B* is weakly distributive and
(ii) (B, τ) has the G_δ property.

The G_δ property is weaker than first countability, and so Theorem 2.8 implies Maharam's theorem 2.3. We remark that weak distributivity follows from first countability of τ, but is necessary in Theorem 2.8, as the Cohen algebra also has the G_δ property and is not weakly distributive.

Theorem 2.8, combined with a consistency result of Stevo Todorcevic [38], answers the von Neumann-Maharam question (Problem 2.3):

Theorem 2.9 (Balcar, Jech, Pazak [5]). It is consistent that every weakly distributive complete ccc Boolean algebra is a Maharam algebra.

(Following [4], a manuscript of [40] containing an identical result appeared in February 2004 with no reference to [4]; as Balcar and Jech claim that they presented the result in a seminar that it cannot be independently verified.)

In June 2004 Stevo Todorcevic improved Theorem 2.8 as follows.

Theorem 2.10 (Todorcevic [39]). A complete Boolean algebra *B* is a Maharam algebra if and only if

(i) *B* is weakly distributive, and

(ii) *B* satisfies the σ-finite chain condition.

We shall prove Theorems 2.7-2.10 in sections 2.6-2.8.

2.5. EQUIVALENTS OF WEAK DISTRIBUTIVITY.

In section 2.6 we shall explore the sequential topology on weakly distributive complete ccc Boolean algebras, with the goal of describing under what additional conditions such algebras carry a Maharam submeasure. We shall obtain various equivalent characterizations of Maharam algebras. In the present section we give several equivalent descriptions of weak distributivity.

For $X, Y \subseteq B$ and $a \in B$, we let $X \vee Y = \{x \vee y : x \in X, y \in Y\}$, $a \vee Y = \{a \vee y : y \in Y\}, X \Delta Y = \{x \Delta y : x \in X, y \in Y\}, a \Delta Y = \{a \Delta y : y \in Y\}$.

Theorem 2.11. Let B be a complete ccc Boolean algebra. Each of the following conditions is equivalent to the weak distributivity of B.

(i) Let $a_0^n \leq a_1^n \leq \ldots$ for n = 0, 1, 2, ... such that $\bigvee_k a_k^n = 1$. Then

$$\bigvee_{f:\omega\to\omega} \bigwedge_n a_{f(n)}^n = 1.$$

(ii) Let $a_0^n \leq a_1^n \leq \ldots$ for n = 0, 1, 2, ... such that $\bigvee_k a_k^n = 1$. Then there exist functions $f_k : \omega \to \omega$, k = 0, 1, 2, ... such that

$$\bigvee_k \bigwedge_n a_{f_k(n)}^n = 1.$$

(iii) Let $a_0^n \leq a_1^n \leq \ldots$ for n = 0, 1, 2, ... such that $\bigvee_k a_k^n = 1$. Then there exist functions $f : \omega \to \omega$, such that

$$\lim_n a_{f(n)}^n = 1.$$

Proof. Property (i) modifies definition (2.5) by assuming that $\bigvee_k a_k^n = 1$ for all n, and is easily seen to be equivalent to the definition of weak distributivity.

That (ii) is equivalent to (i) follows from a general property of algebras that satisfy ccc : for every X there is a countable subset $Y \subseteq X$ such that $\bigvee_{x\in X} a_x = \bigvee_{x\in Y} a_x$.

To see how (iii) follows from (ii), let f_k be as in (ii) and let $f(n) = \max\{f_0(n), ..., f_n(n),\}$. Then for each k, $\bigvee_{n\geq k} a_{f(n)}^n \geq \bigvee_{n\in\omega} a_{f_k(n)}^n$, and so

$$\liminf a_{f(n)}^n = 1..$$

Finally, (iii) implies (ii) by taking $\bigvee$ over all finite modifications of f.

Increasing sequences converging to 1 correspond to maximal antichains: if $0 = a_0 \leq a_1 \leq a_2 \ldots$ with $\bigvee_n a_n = 1$, then $\{a_{n+1} - a_n : n \in \omega\}$ is a maximal antichain, and if $\{a_n : n \in \omega\}$ is a maximal antichain, then $a_0, a_0 \vee a_1, a_0 \vee a_1 \vee a_2, \ldots$ is an increasing sequence with supremum 1. Thus we can formulate the weak distributivity in terms of maximal antichains.

Theorem 2.12. Let B be a complete ccc Boolean algebra. Each of the following conditions is equivalent to the weak distributivity of *B:*

(i) Let $A_n = \left\{a_f^n : k \in \omega\right\}, n \in \omega,$ be maximal antichains. Then

$$\bigwedge_{f:\omega\to\omega} \bigvee_{n} \bigwedge_{k \le f(n)} a_k^n = 1.$$

(ii) If A_0, A_1, A_2, … are maximal antichains then there exists a dense set D such that each $d \in D$ meets only finitely many elements of each A_n.

(iii) If A_0, A_1, A_2, … are maximal antichains then each A_n has a finite subset E_n such that

$$\lim_n \bigwedge E_n = 1.$$

Property (i) is a reformulation of (i) in Theorem 2.11, and (ii) is clearly equivalent to (i). Property (iii) is a reformulation of Theorem 2.11 (iii).

Diagonal property.

Theorem 2.11 (iii) can be further reformulated to give the following equivalents.

Theorem 2.13. Let B be a complete ccc Boolean algebra. Either of the following conditions is equivalent to the weak distributivity of B*:*

(i) If $\lim_k a_k^n = 0$ for every n, then there exists an increasing function $f : \omega \to \omega$, such that $\lim_n a_{f(n)}^n = 0$..

(ii) If $\lim_n a_n = a$ and for each n, $\lim_k a_k^n = a_n$, then there exists an increasing function f such that $\lim_n a_{f(n)}^n = a$.

Property (i) is called the diagonal property. It is equivalent to theorem 2.11 (iii). This is immediate for decreasing sequences a_k^n. In general, we use the fact that each sequence converging to 0 can be majorized by a decreasing sequence converging to 0. Property (ii) follows from (i) by using symmetric differences.

An immediate consequence of theorem 2.13 (ii) is that the closure of a set $X \subseteq B$ in the sequential topology is the set of all limits of convergent sequences in X.

The convergence ideal.

Let X be an infinite countable subset of B^+. If $\lim_n a_n = 0$ for some enumeration $\{a_n\}_{n\in\infty}$ of X, then $\lim_n a_n = 0$ for every enumeration of X (see (2.8)(b)). Thus we can write $\lim X = 0$ without ambiguity and talk about sets converging to 0.

Definition 2.2. The convergence ideal $\boldsymbol{I}$ is the collection of all countable sets $X \subseteq B$ that converge to 0.

$\boldsymbol{I}$ is an ideal on $[B^+]^\omega$, i.e., a collection of countable subsets of B^+ closed under unions and subsets. $\boldsymbol{I}$ contains all infinite antichains, by (2.8) (d).

The convergence ideal was first considered (for a Suslin algebra B) by Abraham and Todorcevic in [1], and introduced in general in [2] by Balcar, Franek and Hruska, and in [31] by Quickert.

Definition 2.3. An ideal ***I*** on some $[S]^\omega$ is a *P*-ideal if for every sequence X_n of members of ***I*** there is an $X \in$ ***I*** such that $X_n - X$ is finite for all *n*.

In [1], the authors show that in the special case, the convergence ideal is a *P*-ideal. In [31], Quickert shows that ***I*** is a *P*-ideal for every weakly distributive complete ccc Boolean algebra. It turns out that this property is another equivalent of weak distributivity:

Theorem 2.14 (Quickert). Let *B* be a complete ccc Boolean algebra. *B* is weakly distributive if and only if the convergence ideal ***I*** on *B* is a *P*-ideal.

Proof. It is easy to see that if ***I*** is a *P*-ideal then *B* has the diagonal property. We prove that if *B* has the diagonal property then ***I*** is a *P*-ideal. Let X_n, $n \in \omega$, be sets in ***I***, and let $X_n = \left\{x_k^n\right\}_{k\in\omega}$ for each *n*. For each *n*, let $\left\{y_k^n\right\}_{k\in\omega}$ be the sequence defined by $y_k^n = x_k^0 \vee x_k^1 \vee \ldots \vee x_k^n$. As the sequences $\left\{y_k^n\right\}_k$ converge to 0 (by (2.8)(e), there exists an increasing function *f* such that $\lim_n y_{f(n)}^n = 0$, by the diagonal property.

Let $X = \left\{x_k^n : n \in \omega \text{ and } k \geq f(n)\right\}$. Clearly, $X_n - X$ is finite for each *n*, and we claim that lim *X* = 0. For each *n*, let E_n be the finite set $\left\{x_k^i : i \leq n \text{ and } f(i) \leq k \leq f(n)\right\}$. Since $\bigvee(X - E_n) = \bigvee_{m\geq n} y_{f(m)}^m$, we have $\limsup X = \bigwedge_{m\geq n} \bigvee(X - E_n) = \limsup_n y_{f(n)}^n = 0.$

A Baire Category Theorem

The next equivalent of weak distributivity is reminiscent of the Baire Category Theorem.

Definition 2.4. A set $X \subseteq B$ is downward closed if $a \leq b \in X$ implies $a \in X$. For $a \neq 0$, $B \upharpoonright a$ is the algebra $\{x \in B : x \leq a\}$.

Note that $B \upharpoonright a$ is a closed subspace of (B, τ).

Theorem 2.15. Let B be a complete ccc Boolean algebra. Either of following conditions is equivalent to the weak distributivity of B :

(i) If $U_0, U_1, U_2, \ldots$ are downward closed and $\mathrm{cl}(U_n) = B$ for all n, then $\bigcap_n U_n$ is dense in B.

(ii) If $U_0, U_1, U_2, \ldots$ are downward closed and $\bigcap_n U_n = \{0\}$ then $\bigcap_n \mathrm{cl}(U_n) = \{0\}$.

Proof. First assume that B is weakly distributive and prove (ii). Let U_n, $n \in \omega$, be as in (ii). Toward a contradiction, assume that some $a \neq 0$ is in $\bigcap_n \mathrm{cl}(U_n)$. For each n there exists a sequence $\left\{a_k^n\right\}_{k\in\omega}$ in U_n with a limit a. Using the diagonal property we obtain a sequence $\{b_n\}_{n\in\omega}$ such that $b_n \in U_n$ and $\lim_n b_n = a$. Since $\liminf b_n \neq 0$ there exists some $b \neq 0$ such that for eventually all n, $b_n \geq b$. Now if n is such that $b \notin U_n$, then $b_n \notin U_n$ because U_n is downward closed; a contradiction.

Next we prove that (ii) implies (i). Let U_n be downward closed with $\mathrm{cl}(U_n) = B$ and assume that $\bigcap_n U_n$ is not dense in B. Let $a > 0$ be such that $\bigcap_n U_n \cap B \upharpoonright a = \{0\}$.

For each n, let $V_n = \{a \wedge x : x \in U_n\}$. As the U_n are downward closed, we have $V_n = U_n \cap B \upharpoonright a$, the V_n are downward closed, and $\cap_n V_n = \{0\}$. By (ii), $\cap_n \mathrm{cl}(V_n) = \{0\}$. However, for each n we have $a \in \mathrm{cl}(U_n)$ and so (by definition of closure), $a = a \wedge a \in a \wedge \mathrm{cl}(U_n) = \mathrm{cl}(a \wedge U_n) = \mathrm{cl}(V_n)$.

Finally, let us assume (i) and show that B is weakly distributive. Let A_n, $n \in \omega$, be maximal antichains. For each n, let U_n be the set of all x that meet only finitely many elements of A_n. U_n is downward closed and $\mathrm{cl}(U_n) = B$. Thus $\cap_n U_n$ is dense in B, proving theorem 2.12 (ii).

Bounding forcing.

It is a well known fact in the theory of forcing that weakly distributive complete Boolean algebras yield generic models that have a bounding property.

A function $f : \omega \to \omega$ is bounded by $g : \omega \to \omega$ if $f(n) < g(n)$ for all n; f is eventually bounded by g, $f < g$, if for some n, $f(k) < g(k)$ for all $k \geq n$.

Theorem 2.16. Let B be a complete ccc Boolean algebra. Either of the following properties is equivalent to the weak distributivity of B :

(i) In V, every $f : \omega \to \omega$ is bounded by a function in V.
(ii) If $\dot{f}$ is a B-name for a function from ω to ω then there exists a function g such that $\|\dot{f} < g\| = 1$.

Property (ii) is an analog of theorem 2.12 (iii): A name $\dot{f}$ corresponds to a sequence of maximal antichains $\{\|\dot{f}(n) < k\| : k \in \omega\}$, and $\|\dot{f} < g\| = 1$ if and only if $\lim_n \|\dot{f}(n) < g\| = 1$.

2.6. THE DECOMPOSITION THEOREM.

Our work on sequential topology in [3] and [5] led to the following theorem that we shall use in the next section to analyze Maharam algebras.

Theorem 2.17 (Balcar, Jech, Pazak [5]). Let B be a complete ccc Boolean algebra. There exist disjoint elements m and d such that $m \vee d = 1$, and

(i) the algebra $B \restriction m$ carries a Maharam submeasure;
(ii) in $B \restriction d$ every nonempty open set is topologically dense.

In the theorem, either m or d can be 0. For $a \neq 0$, $B \restriction a$ is the algebra $\{x \in B: x \leq a\}$ and $B \restriction 0 = \{0\}$. We remark that for the Cohen algebra, $d = 1$.

Property (ii) means that the algebra is very non-Hausdorff: any two nonempty open set have nonempty intersection. As a consequence, if (B, τ) is a Hausdorff space then B is a Maharam algebra (Theorem 2.7). We shall now prove the following theorem, which implies theorems 2.7, 2.8 and 2.3.

Theorem 2.18. Let B be a complete ccc Boolean algebra.

(a) If (B, τ) is a Hausdorff space, then
  (i) B is a weakly distributive, and
  (ii) (B,τ) has the G_δ property.

(b) If B is weakly distributive and (B, τ) has the G_δ property, then

(i) $\vee$ is continuous, and

(ii) (B, τ) is a first countable space.

(c) (Maharam) If $\vee$ is continuous and (B, τ) is first countable then B is a Maharam algebra.

Towards the proof of Theorem 2.18, we start with the following observation:

Lemma 2.1. Let B be a complete Boolean algebra. If (B, τ) is a Hausdorff space, then B is weakly distributive.

Proof. Let $a_0^n \leq a_1^n \leq \ldots$ for $n = 0, 1, 2, \ldots$ such that $\bigvee_k a_k^n = 1$. We assume that B is Hausdorff and show that for each $a \neq 0$ there exists function $f\colon \omega \to \omega$ such that $a \wedge \bigvee_n a_{f(n)}^n \neq 0$.

Let $a \neq 0$. There exists an open neighborhood U of a such that $a \notin \mathrm{cl}(U)$. Since $\lim_k (a \wedge a_k^0) = a$, there exists some $k = f(0)$ such that $b_0 = a \wedge a_k^0 \in U$. Inductively, we assume that $b_n = a \wedge a_{f(0)}^0 \wedge \ldots \wedge a_{f(0)}^n \in U$ and since $\lim_k (b_n \wedge a_k^{n+1}) = b_n$, we find a $k = f(n + 1)$ such that $b_{n+1} = b_n \wedge a_k^{n+1} \in U$. We have $a \wedge \bigvee_n a_{f(n)}^n = \lim_n b_n \neq 0$, because $0 \notin \mathrm{cl}(U)$.

We say that B is *nowhere weakly distributive* if $B \mid a$ is not weakly distributive for each $a \neq 0$. The same argument as in Lemma 2.1 proves that if B is nowhere weakly distributive, then $0 \in \mathrm{cl}(U)$ for every nonempty open set U. Then $a \in \mathrm{cl}(U)$ for every $a \in B$ (because $a \in \mathrm{cl}(U)$ if and only if $0 \notin \mathrm{cl}(U \,\Delta\, a)$), and we have the following:

Corollary 2.2. If B is a nowhere weakly distributive complete ccc Boolean algebra then $\mathrm{cl}(U) = B$ for every nonempty open set U.

The next lemma summarizes additional properties of the sequential topology under the weak distributivity. Note that if A is downward closed then $A \vee A = A \,\Delta\, A$.

Lemma 2.2. Let B be a weakly distributive complete ccc Boolean algebra.

(a) For every $A \subseteq B$, the closure of A is the set of all limits of convergent sequences in A.

(b) For every open neighborhood V of O there exists a nonempty open $U \subseteq V$ that is downward closed. Hence

$$\mathcal{N} = \{U : U \text{ is nonempty, open and downward closed set}\}$$

is a neighborhood base of 0.

(c) If $U \in \mathcal{N}$ then $\mathrm{cl}(U) = \bigcap \{U \vee V : V \in \mathcal{N}\} \subseteq U$ and $\mathrm{cl}(U)$ is downward closed.

(d) If $U \in \mathcal{N}$ then $\mathrm{cl}(U) \subseteq U$ and U is open and downward closed.

Proof. (a) follows from 2.13 (ii)

(b) Let $U = B$ - cl(A) where $A = \{a: (\exists b \leq a)\ b \notin V\}$. Using (a), one verifies that U is downward closed and $0 \in U$.

(c) It is easy to see that cl$(U) \subseteq U \Delta V$ for every open neighborhood V of 0 and in fact cl$(U) = \bigcap\{U \Delta V : V \in \mathcal{N}\}$. In particular cl$(U) \subseteq U \Delta V$. Since U and V are downward closed, we have $U \Delta V = U \vee V$.

(d) Easy.

The main technical lemma is this:

Lemma 2.3. [3] Let B be a weakly distributive complete ccc Boolean algebra. For every $U \in \mathcal{N}$ there exists a $V \in \mathcal{N}$ such that $V \subseteq U$.

Proof. Assume that $U \in \mathcal{N}$ is such that the statement fails. We construct sequences V_n, x_n, y_n, z_n as follows: Let $V_0 = U$. For each n, let $x_n, y_n, z_n \in V_n$ be such that $x_n \vee y_n \vee z_n \notin U$, and let $V_{n+1} \subseteq V_n$ be in $\mathcal{N}$ such that $x_n \vee V_{n+1} \subseteq V_n$, $y_n \vee V_{n+1} \subseteq V_n$ and $z_n \vee V_{n+1} \subseteq V_n$, V_{n+1} exists by the one-sided continuity of $\vee$.

Let $X = \bigcap_n \mathrm{cl}(V_n)$ and x = lim supx_n, y = lim sup y_n, z = lim supz_n. X is closed, downward closed and $X \subseteq$ cl$(U) \subseteq U$. For each n and each k we have $x_n \vee x_{n+1} \vee \ldots \vee x_{n+k} \in V_n$, hence $\bigvee_{i \geq n} x_i \in \mathrm{cl}(V_n)$ and therefore $x \in \mathrm{cl}(V_n)$. It follows that $x \in X$, and similarly $y \in X$ and $z \in X$. A similar argument shows that for each n and each k, $x_n \vee \ldots \vee x_{n+k} \vee X \subseteq$ cl(V_n), and so $\bigvee_{i \geq n} x_i \vee X \subseteq \mathrm{cl}(V_n)$.

As cl(V_n) is downward closed and $x \le \bigvee_{i \ge n} x_i$, we have $x \vee X \subseteq$ cl(V_n). It follows that $x \vee X \subseteq X$ and similarly $y \vee X \subseteq X$, $z \vee X \subseteq X$.

Hence $x \vee y \vee z$ is in X and hence in U. But $x \vee v \vee z$ = lim sup($x_n \vee y_n \vee z_n$). For each n, $\bigvee_{i \ge n}(x_i \vee y_i \vee z_i) \notin U$ because U is downward closed, and since U is open, we have $x \vee y \vee z$ = lim sup($x_n \vee y_n \vee z_n$) $\notin U$. *A* contradiction.

Proof of Theorem 2.18 (a).

We first show that for every $b \neq 0$ there is a nonzero c < b and a sequence $\{V_n\}_n$ in $\mathcal{N}$ such that $c \wedge \bigvee(\bigcap_n V_n) = 0$. As the space is Hausdorff, there exists a $V_0 \in \mathcal{N}$ such that $b \notin V_0 \vee V_0$. For each n let $V_{n+1} \in \mathcal{N}$ be such that $V_{n+1} \subseteq V_n$. Let $A = \bigcap_n V_n$ and $a = \bigvee A$.

For each n we have $V_2 \supset V_{n+2}$ and hence $V_2 \supset A$. As $a = \lim_n(a_1 \vee \ldots \vee a_n)$ for some sequence $\{a_n\}_n$ in A, we have $a \in$ cl(V_2). Therefore $a \in V_2 \subseteq V_0$, and since V_0 is downward closed, $b \not\le a$. Thus c $\wedge\ \omega A = 0$, where $c = b - a$.

Now let C be a maximal antichain such that for each $c \in C$ there exists a sequence $\{V_n\}_n$ in $\mathcal{N}$ with $c \wedge \bigvee(\bigcap_n V_n) = 0$.

Then $\{V_n^c : c \in C, n \in \omega\}$ is a. countable set of open neighborhoods of 0, and $\bigcap_c \bigcap_n V_n^c = \{0\}$.

Proof of Theorem 2.18 (b).

Let $\{U_n\}_n$ be a sequence of open neighborhoods of 0 such that $\bigcap_n U_n = \{0\}$. Since B is weakly distributive, we find such U_n that are downward closed, and by Theorem 2.15 (ii), we have $\bigcap_n \mathrm{cl}(U_n) = \{0\}$. We shall show first that the operation $\vee$ is continuous.

Thus assume that $\vee$ is not continuous at 0. There exists a $U \in \mathcal{N}$ such that for every $V \in \mathcal{N}$ there exists $x, y \in V$ with $x \vee y \notin U$.

We construct sequences V_n, x_n, y_n as follows: Let $V_0 = U_0 \bigcap U$. For each n, let $x_n, y_n \in V_n$ such that $x_n \vee y_n \notin U$. By one-sided continuity of $\vee$ there exists a $V_{n+1} \subseteq U_{n+1}$ such that $x_n \vee V_{n+1} \subseteq V_n$ and $y_n \vee V_{n+1} \subseteq V_n$. Let $x = \limsup x_n$ and $y = \limsup y_n$.

For each n and each k we have $x_n \vee x_{n+1} \vee \dots \vee x_{n+k} \in V_n$, because $x_i \vee V_{i+1} \subseteq V_i$ for all i. Hence $\bigvee_{i \geq n} x_i \in \mathrm{cl}(V_n)$, and therefore $x \in \mathrm{cl}(V_n)$. It follows that $x = 0$, and similarly, $y = 0$, hence $x \vee y = 0$.

But $x \vee y = \limsup x_n \vee \limsup y_n = \limsup (x_n \vee y_n)$. Since U is downward closed and $x_n \vee y_n \notin U$, we have $\bigvee_{k \geq n} (x_n \vee y_n) \notin U$, and because U is open, we get $0 = x \vee y \notin U$. A contradiction.

Now, we prove that (B, τ) is first countable. By the continuity of $\vee$ and by the G_δ property there exist $U_n \in \mathcal{N}$ such that $\bigcap_n \mathrm{cl}(U_n) = \{0\}$ and $U_{n+1} \vee U_{n+1} \subseteq U_n$ for every n. We claim that $\{U_n\}_{n \in \omega}$

is a neighborhood base. Assume not. Then there exists a $V \in \mathcal{N}$ such that for every n, $U_n \not\subseteq V$. For each n let x_n be such that $x_n \in U_n - V$.

It follows by induction on k that for each n and each k, $x_{n+1} \vee x_{n+2} \vee \ldots \vee x_{n+k} \in U_n$. Thus $\bigvee_k x_{n+k} \in \mathrm{cl}(U_n)$ and it follows that lim sup $x_n \in \mathrm{cl}(U_m)$ for each m; hence lim $x_n = 0$. This is a contradiction because V is an open neighborhood of 0.

Proof of Theorem 2.18 (c).

(B, Δ, 0) is a topological group and a first countable space. By Kakutani's Theorem, (B, Δ, 0) has an invariant metric, and so B is Maharam, as in the proof of Theorem 2.1.

Remark. In [3] we proved that continuity of $\vee$ is itself sufficient for B to be a Maharam algebra. The condition can be stated as follows: For every $U \in \mathcal{N}$ there exists a $V \in \mathcal{N}$ such that $V \vee V \subseteq U$. Compare this with the condition in Lemma 2.3 which holds for every weakly distributive B.

Proof of theorem 2.1.

In view of Corollary 2.2 it suffices to prove the theorem under the assumption that B is weakly distributive; the proof of the general case combines the proof below with the proof of Corollary 2.2.

Thus assume that B is weakly distributive. Let $D = \bigcap\{\mathrm{cl}(U) : U \in \mathcal{N}\}$, $d = \omega D$ and $m = -d$. D is closed and downward closed, and $D = \{U \vee U : U \in \mathcal{N}\}$. Now if $a \notin D$, in particular if $a \leq m$, then $a \notin U \vee U$ for some $U \in \mathcal{N}$. Hence U and $a \Delta U$ are disjoint, and so $(B \mid m, \tau)$ is a Hausdorff space. Hence $B \mid m$ is a Maharam algebra.

We now show that in $B \mid d$, every nonempty open set is topologically dense. We claim that D is closed under $\vee$. By Lemma 2.3, for every $U \in \mathcal{N}$ there is a $V \in \mathcal{N}$ such that $V \vee V \vee V \subseteq U \vee U$, and therefore there is a $V \in \mathcal{N}$ such that $V \vee V \vee V \vee V \subseteq U \vee U$. Hence $D = \bigcap\{U \vee U : U \in \mathcal{N}\} = \bigcap\{V \vee V \vee V \vee V : V \in \mathcal{N}\}$ and so $D \vee D = D$.

There is a sequence $\{a_n\}_n$ in D such that $d = \lim(a_0 \vee \ldots \vee a_n)$ and since D is closed, we have $d \in D$. It follows that $B \mid d = D$.

We finish the proof by showing that $\mathrm{cl}(G) \supset D$ for every nonempty open set G in $B \mid d$. There exist an $a \in D$ and some $U \in \mathcal{N}$ such that $G \supset (a \Delta U) \cap D$. Since $\mathrm{cl}(U) \supset D$, we have $\mathrm{cl}(a \Delta U) \supset a \Delta D = D$ and so $\mathrm{cl}(G) \supset D$.

2.7. MAHARAM ALGEBRAS.

In this section we present a number of additional necessary and sufficient conditions for a complete ccc Boolean algebra B to carry a Maharam submeasure. It turns out that some of the properties are natural generalizations of the conditions for weak distributivity presented in section 2.5.

By the results of sections 2.3 and 2.6, each of the following is equivalent to being Maharam.

(B, τ) is metrizable (Maharam [27]), (2.14)

(B, τ) is Hausdorff (Balcar, Glowczyhski, Jech [3]), (2.15)

B is weakly distributive and (B, τ) is first countable (2.16)

(Maharam [27]),

B is weakly distributive and (B, τ) has the G_δ property (2.17)

(Balcar, Jech, Pazak [5]).

We remark that the assumption of weak distributivity in (2.16) is not necessary (but is necessary in (2.17)). Todorcevic's paper [39] presents (2.16) and (2.17) from a different point of view.

Uniform weak distributivity and uniformly bounding forcing.

The following two conditions are uniform versions of the corresponding conditions theorem 2.11 (ii) and 2.16 (ii) for weak distributivity.

Theorem 2.19. Let B be a complete ccc Boolean algebra. Either of the following is equivalent to B being Maharam:

(i) There exists a sequence of functions $\{F_n\}_n$ such that for each maximal antichain A, $F_n\{A)$ is a finite subset of A, and if $A_0, A_1, A_2,...$ are maximalantichains then $\lim_n \omega F_n(A_n) = 1$.

(ii) There exists a sequence of functions $\{F_n\}_n$ such that for each B-name $\dot{a}$ for a natural number, $F_n(\dot{a})$ is a natural number, and if $\dot{f}$ is a B-name for a function from ω to ω. then, letting $g(n) = F_n(\dot{f}(n))$, we have $\left\| \dot{f} < g \right\| = 1$.

Proof. As (ii) is reformulation of (i) in terms of Boolean-valued models, let us consider (i). If m is Maharam submeasure, let F_n (A) be a finite $E \subseteq A$ so that $m(\omega E) > 1 - \frac{1}{2^n}$.

Conversely, if B satisfies (i), let U_n be for each n, the set of all $\omega F_n(A)$, for all maximal antichains A. One verifies that 0 is an interior point of Un, and that $\cap U_n = \{0\}$. Hence B has the G_δ property and is weakly distributive since theorem 2.19 (i) implies theorem 2.11 (ii).

Strong Diagonal Property.

Theorem 2.20. Let B be a complete ccc Boolean algebra. B is a Maharam algebra if and only if there exists a family S of sequences converging to 0 such that every sequence with limit 0 has a subsequence in S, and if $\left\{a_k^n\right\}_{k\in\omega}, n = 0,1,2,...$ are members of S, then $\lim_n a_k^n = 0$.

Proof. If m is a Maharam submeasure, let S be the set of all sequences $\{a_n\}_n$ such that $m(a_n) < \frac{1}{2^n}$, for each n.

Conversely, if B has the strong diagonal property, let U_n be, for each n, the set of all x such that x is below the n^{th} term of some sequence $s \in S$. 0 is an interior point of each U_n and that $\cap Un = \{0\}$. Hence B has the G_δ property (and is weakly distributive because it has the diagonal property).

Hausdorff variations. We consider three related properties of the space (B, τ). The first one is a reformulation of Hausdorffness.

Theorem 2.21. Let B be a complete ccc Boolean algebra. Each of the following properties is equivalent to B carrying a Maharam submeasure.

(i) Every $a \neq 0$ has an open neighborhood G such that $0 \notin \mathrm{cl}(G)$.

(ii) Every $a \neq 0$ has an open neighborhood G such that $G \cap X$ is finite, for every X with limit 0 *{i.e.*, $X \in \boldsymbol{I}$ see Definition 2.2).

(iii) Every $a \neq 0$ has an open neighborhood G such that every antichain $A \subseteq G$ is finite.

Proof. (i) states that (B, τ) is a Hausdorff space, and (i) implies (ii) implies (iii). It suffices to show that (iii) implies (i).

Thus assume that the space is not Hausdorff. By the Decomposition Theorem 2.17 there exists a $d \neq 0$ such that in $B \restriction d$, any two nonempty open sets intersect. We claim that every open neighborhood of d includes an infinite antichain.

Let G be any open neighborhood of d. Since $\{x \leq d: x \in G\}$ and $\{x \leq d : d - x \in G\}$ intersect, we find disjoint nonzero $a_1 \in G$ and $a_2 \in G$ below d. Continuing this, we produce an infinite antichain in G.

Fragmentations.

The next three properties involve a decomposition of B^+ into countably many parts (fragmentations) and are related to the three preceding conditions. The first one is a reformulation of the G_δ property, while the third one is the σ-finite chain condition of Horn and Tarski (thus Theorem 2.22 (iii) proves Todorcevic's Theorem 2.10).

Theorem 2.22. Let B be a complete ccc Boolean algebra. If B is weakly distributive then each of the following conditions is equivalent to B being a Maharam algebra:

(i) $B^+ = \bigcup_n S_n$ such that for each n, S_n is closed.

(ii) $B^+ = \bigcup_n S_n$ such that for each n, $S_n \cap X$ is finite for every $X \in \boldsymbol{I}$.

(iii) $B^+ = \bigcup_n S_n$ such that for each n, $S_n \cap A$ is finite for every antichain A.

Proof. Every Maharam algebra has these properties (let $S_n = \{a : m(a) \geq \frac{1}{n}\}$) and clearly, (i) implies (ii) implies (iii). It suffices to show that (iii) implies theorem 2.21 (iii).

Let $\{S_n\}_n$ be as in (iii); we may assume that each S_n is upward closed. Letting $U_N = B - S_n$, we have $\cap_n U_n = \{0\}$, and so $\cap_n \text{Cl}(U_n) = \{0\}$ by Theorem 2.15 (ii), because B is weakly distributive. It follows that every $a \neq 0$ is an interior point of some S_n, and Theorem 2.21 (iii) follows.

Infinite games.

Finally, we show that strategic versions of weak distributivity are equivalent to carrying a Maharam submeasure. We consider three games, one for each of the properties Theorem 2.12 (iii), 2.13 (i) and 2.16 (ii) from section 2.5. The weak distributive game is Fremlin's modification of a game introduced by Jech in 1980 (cf. [18]) and by Charles Gray in his dissertation [14]. The diagonal game and the bounding game are our reformulations. In December 2004 Fremlin proved Theorem 2.23 below for the weak distributivity game, cf. [12].

Each of the following game is an infinite game of two players. Players I and II take turns to successively produce two infinite sequences of moves. We consider the properties of *B* stated in terms of winning strategies.

Weak distributivity game. I plays maximal antichains A_0, A_1, A_2, ..., and II plays finite subsets $E_n \subseteq A_n$. II wins if and only if $\lim_n \omega E_n = 1$.

Diagonal game. I plays sequences $\{a_k^n\}_{k\in\omega}, n = 0,1,2,...$ each converging to 0, and II plays integers $k(0)$, $k(1)$, $k(2)$, … II wins if and only if $\lim_n a_{k(n)}^n = 0$.

Bounding game. I plays *B*-names $\dot{f}(0)$, $\dot{f}(1)$, $\dot{f}(2)$, ... for integers, and II plays integers $g(0)$, $g(1)$, $g(2)$, … II wins if and only if $\|\dot{f} < g\| = 1$.

These games correspond to properties in Theorems 2.12 (iii), 2.13 (i) and 2.16 (ii), and for complete ccc algebras are mutually equivalent, i.e., I (respectively II) has a winning strategy in one game if and only if I (respectively II) has a winning strategy in either game. One can show that if B is a complete ccc algebra then B is weakly distributive if and only if player I does not have a winning strategy, cf. [18].

Theorem 2.23 (Fremlin [12]). Let B be a complete ccc Boolean algebra. B is a Maharam algebra if and only if Player II has a winning strategy in either of the three games.

Proof. Let us consider the diagonal game. If m is a Maharam submeasure. then II has the following winning strategy: at move n, II plays $k(n)$ so that $m\left(a^n_{k(n)}\right) < \frac{1}{2^n}$.

Conversely if II has a winning strategy, then B has the diagonal property and so is weakly distributive. We show that (B, τ) is a Hausdorff space. If not, then by Decomposition Theorem 2.17 there exists some $d \neq 0$ such that in $B \upharpoonright d$, any two nonempty open sets intersect. Without loss of generality we assume that $d = 1$ (because II has a winning strategy in $B \upharpoonright d$).

Let σ_0 be a winning strategy for player II in the diagonal game, and let σ_1 be a winning strategy for II in a related game where I plays sequences converging to 1 and II tries to construct a diagonal sequence with limit 1. Let X_0 and Y_0 be, respectively, the sets of all elements of B given by the first moves of player II using σ_0

(respectively using σ_1). More precisely, $X_0 = \{x_{k(0)} : k(0) = (\sigma_0 (\{x_i\}_{i\in\omega})$ where $\lim_i x_i = 0\}$ and $Y_0 = \{y_{k(0)} : k(0) = (\sigma_1 (\{y_i\}_{i\in\omega})$ where $\lim_i y_i = 1\}$. We claim that 0 is an interior point of X_0 (and 1 is an interior point of Y_0). If not, there is a sequence $\{a_n\}_n$ outside X_0 that converges to 0. But if $k_0 = \sigma_0 (\{a_n\}_n)$, then $a_{k(0)} \in X_0$, a contradiction.

Hence $X_0 \cap Y_0$ is nonempty and let $a_0 \in X_0 \cap Y_0$. There exist sequences $\{x_n^0\}_n$ and $\{y_n^0\}_n$ such that $a_0 = x_{k(0)}^0 = y_{l(0)}^0$, where $k(0) = \sigma_0(\{x_n^0\}_n)$ and $l(0) = \sigma_1(\{y_n^0\}_n)$. Let X_1 and Y_1 be, respectively, the set of all elements of B given by the second moves of player II, using σ_0 and σ_1 respectively, with the first move of I being $\{x_n^0\}_n$ and $\{y_n^0\}_n$, respectively. Again, 0 and 1 are, respectively, interior points of X_1 and Y_2; therefore $X_1 \cap Y_1 \neq \emptyset$ and so there exists sequences $\{x_n^1\}_n$ and $\{y_n^1\}_n$ such that $a_1 = x_{k(1)}^1 = y_{l(1)}^1$, where $k(0) = \sigma_0(\{x_n^0\}_n, \{x_n^1\}_n)$ and $l(1) = \sigma_1(\{y_n^0\}_n, \{y_n^1\}_n)$. We continue in this fashion and construct a sequence a_0, a_1, a_2, … . Since both σ_0 and σ_1 are winning strategies, we have simultaneously $\lim_n a_n = 0$ and $\lim_n a_n = 1$, a contradiction.

2.8. CONSISTENCY OF THE VON NEUMANN-MAHARAM PROBLEM.

We prove Theorem 2.9 and then discuss some examples. The proof uses a general consistency result of S. Todorcevic and one of the equivalences presented in section 2.7.

P-ideal Dichotomy (Todorcevic). Let S be an infinite set. Then for every *P-ideal* $I \subseteq [S]^\omega$, either

(i) $\exists Y \subseteq S$ uncountable such that $[Y]^\omega \subseteq I$, or

(ii) $S = \bigcup_{n\in\omega} S_n$ sucn that for each n, $S_n \cap X$ is finite, for every $X \in I$.

Theorem 2.24 (Todorcevic [38]). The *P*-ideal dichotomy is consistent with *ZFC*.

In [38] it is shown that the *P*-ideal dichotomy follows from the Proper Forcing Axiom (PFA), but is also consistent with GCH. For PFA and its consistency, we refer to [32], [7] or [19].

Theorem 2.9 is now a consequence of the following.

Theorem 2.25 (Balcar, Jech, Pazak [5]). Assuming the *P*-ideal dichotomy, every weakly distributive complete ccc Boolean algebra carries a Maharam submeasure.

Proof. Let *B* be a weakly distributive complete ccc Boolean algebra, and let ***I*** be the convergence ideal for *B*. By Theorem 2.14, ***I*** is a *P*-ideal. Note that condition (ii) for ***I*** in the *P*-ideal Dichotomy is exactly the condition (ii) in Theorem 2.22, which implies that *B* is a Maharam algebra.

Thus it is enough to show that condition (i) in *P*-ideal dichotomy fails for ***I***. This is proved in the following lemma, completing the proof of Theorem 2.25.

Lemma 2.4. Let *B* be a complete ccc Boolean algebra. For every uncountable $Y \subseteq B^+$ there exists a countable $X \subseteq Y$ such that lim sup $X > 0$.

Proof. It suffices to prove this for Y of cardinality $\aleph_1$, so let $Y = \{a_\alpha : \alpha < \omega_1\}$. For each $\alpha < \omega_1$, let $b_\alpha = \omega_{\xi\geq\alpha}\alpha_\xi$. Since the b_α are decreasing and B satisfies ccc, there exists an α_0 such that $b_\alpha = b_{\alpha_0}$ for all $\alpha \geq \alpha_0$. Let $b_\alpha = b_{\alpha_0}$.

For each $\alpha \geq \alpha_0$ there exists an $f(\alpha) > \alpha$ such that $b_\alpha = \omega_{\xi\geq\alpha}a_\xi. = \omega_{\alpha\leq\xi<f(\alpha)}a_\xi$ (again using ccc). Let $\alpha_{n+1} = f(\alpha_n)$ for each n, and $\alpha_\omega = \lim_n \alpha_n$. Letting $X = \{a_\xi : \alpha_0 \leq \xi < \alpha_\omega]$ we have lim sup $X = b$.

We shall conclude the article with a discussion on the independence of the von Neumann-Maharam problem. In Section 2.3 we presented one counterexample, the Suslin algebra. It is an ω-distributive complete ccc Boolean algebra that is not a Maharam algebra, and it is consistent that one exists. We present two other consistent examples of weakly distributive complete ccc algebras that are not Maharam.

Theorem 2.26 (Glowczyhski [13]). Let κ be a regular uncountable cardinal that carries a σ-saturated σ-ideal I containing all singletons and assume that $2^{\aleph_0} > \kappa$, and that Martin's Axiom holds. Then $\mathcal{P}(\kappa)/I$ is a weakly distributive complete ccc Boolean algebra and is not a Maharam algebra.

The assumptions in Theorem 2.26 are consistent: they can be obtained by forcing over a model with a measurable cardinal. Note also that this example shows that the assumption of PFA in Theorem 2.24 cannot be weakened to MA.

Let $B = \mathcal{P}(\kappa)/I$. B is an atomless complete ccc algebra; to show that it has the desired properties, we look first at the atomic algebra $\mathcal{P}(\kappa)$. It is well known that $\mathcal{P}(\kappa)$ has the diagonal property if $\kappa < \mathfrak{b}$, and $(\mathcal{P}(\kappa), \tau)$ is sequentially compact if $\kappa < \mathfrak{s}$ (cf. [3] for details as well definitions of the cardinal invariants $\mathfrak{b}$ and $\mathfrak{s}$). As a consequence of MA we have $\mathfrak{b} = \mathfrak{s} = 2^{\aleph_0} > \kappa$, and so $\mathcal{P}(\kappa)$ has the diagonal property and is sequentially compact.

It is easy to see that these two properties are preserved under quotients by a σ-ideal. Thus B has the diagonal property and (B, τ) is sequentially compact. Thus B is weakly distributive, and we claim that B is not a Maharam algebra.

Assume that B is a Maharam algebra. Then (B, τ) is metrizable, and because it is sequentialy compact, it is a compact metric space. By [5], B is isomorphic to $\mathcal{P}(\omega)$, a contradiction.

In his Problem List [11] D. Fremlin asked the following question:

'Suppose that every completely countably generated subalgebra of a given complete Boolean algebra B is measurable. Must B be measurable?'

A consequence of Theorem 2.25 is that the P-ideal dichotomy implies the affirmative answer to Fremlin's question:

Assume that every weakly distributive complete ccc Boolean algebra is Maharam, and let *B* be a complete Boolean algebra such that every complete subalgebra is a measure algebra. B is ccc and weakly distributive because every countably generated subalgebra is. Hence *B* is a Maharam algebra. Then *B* is a measure algebra, by the following observation of Fremlin:

Lemma 2.5. If *B* is a Maharam algebra such that every countably generated complete subalgebra is a measure algebra then *B* is a measure algebra.

Proof. Let *m* be a Maharam submeasure on *B*. For every countably generated subalgebra *C*, the restriction of *m* to *C* is an exhaustive submeasure, and since *C* is a measure algebra, *m* is uniformly exhaustive, by [21]. It follows that *m* is uniformly exhaustive, and by [21] again, *B* is a measure algebra.

The following recent example proves the consistency of the negative answer to Fremlin's question:

Theorem 2.27 (Farah, Velickovic [8]). Assume there is a cardinal λ such that $\lambda^{\aleph_0} = \lambda, 2^{\lambda} = \lambda^+$ holds. Then there is a complete Boolean algebra *B* of size λ^+ such that *B* is not a Maharam algebra but every subalgebra of size $\leq \lambda$ is a measure algebra.

Under the given assumptions, one constructs *B* as the union of a chain $\{B_\alpha : \alpha < \lambda^+\}$ of measure algebras, each of size at most λ, in such a way that *B* is not a measure algebra. Since *B* is the union of such a chain, it satisfies ccc and is weakly distributive. By Lemma 8.5, *B* is not a Maharam algebra.

The assumptions of Theorem 2.27 hold unless there exists an inner model with a measurable cardinal κ of Mitchell order κ^{++}. It follows that the consistency of the von Neumann-Maharam Problem (Theorem 2.9) implies the consistency of a measurable cardinal κ, of Mitchell order κ^{++}.

A final comment: the consistency of the von Neumann-Maharam problem for small algebras does not require large cardinals. This is because for complete Boolean algebras of cardinality at most $2^{\aleph_0}$ the *P*-ideal dichotomy is consistent with ZFC alone, cf. [1].

REFERENCES

[1] Abraham, U. and Todorcevic, S. (1997): Partition properties ω_1 compatible with CH, Fundamenta Mathematicae, vol. 152, pp. 165-181.

[2] Balcar, B.; Franek, F. and Hruska, J. (1999): Exhaustive zero-convergence structures on Boolean algebras. Acta Universitatis Carolinae. Mathematica et Physica, vol. 40, no. 2, pp. 27-41.

[3] Balcar, B.; Glowczynski, W. and Jech, T. (1998): The sequential topology on complete Boolean algebras, Fundamenta Mathematicae, vol. 155, no. 1, pp. 59-78.

[4] Balcar, B.; Jech, T. and Pazak, T. (2003): Complete ccc Boolean algebras, the order sequential topology and a problem of von Neumann, Preprint, http://arxiv.org/abs/math.LO/0312473, December 2003.

[5] ———— (2005): Complete ccc Boolean algebras, the order sequential topology and a problem of von Neumann, The Bulletin of the London Mathematical Society, vol. 37, pp. 885-898.

[6] Banach, S. and Kuratowski, K. (1929): Sur une Generalisation du Probleme de la Mesure, Fundamenta Mathematicae, vol. 14, pp. 127-131.

[7] Baumgartner, J. E. (1983): Iterated forcing. Surveys in set theory, London Mathematical Society Lecture Note Series, vol. 87, Cambridge University Press, Cambridge, pp. 1-59.

[8] Farah I. and Velickovic, B. : Von Neumann's problem and large cardinals. The Bulletin of the London Mathematical Society.

[9] Fremlin, D. H. (1989): Measure algebras. Handbook of Boolean algebras, vol. 3, North-Holland Publishing Company, Amsterdam, pp. 877-980.

[10] ———— (2000-2004): Measure theory, vol. 1-4, Torres Fremlin, Colchester.

[11] ———— (2004): Problems. Unpublished notes, January 2004.

[12] ———— (2004): Maharam algebras, Unpublished notes, December 2004.

[13] Glowczynski, W. (1991): Measures on Boolean algebras. Proceedings of the American Mathematical Society, vol. III, no. 3, pp. 845-849.

[14] Gray, C. (1982): Iterated forcing from the strategic point of view, Berkeley.

[15] Hewitt, E. and Ross, K. A. (1979): Abstract harmonic analysis, vol. I, Springer-Verlag, Berlin.

[16] Horn, A. and Tarski, A. (1948): Measures in Boolean algebras, Transactions of the American Mathematical Society, vol. 64, pp. 467-497.

[17] Jech, T. (1967): Non-provability of Souslin's hypothesis, Commentationes Mathematicae Universitatis Carolinae, vol. 8, pp. 291-305.

[18] ———— (1984): More game-theoretic properties of Boolean algebras. Annals of Pure and Applied Logic, vol. 26, pp. 11-29.

[19] ———— (2002): Set theory, the third millennium edition. Springer Monographs in Mathemat ics, Springer-Verlag, Berlin.

[20] Kakutani, S. (1936): Uber die Metrisation der topologischen Gruppen, Proceedings of the Imperial Academy, Tokyo, vol. 12, pp. 82-84.

[21] Kalton, N. J. and Roberts, J. W. (1983): Uniformly exhaustive submeasures andnearly additive set functions, Transactions of the American Mathematical Society, vol. 278, pp. 803-816.

[22] Kantorovic, L. V.; Vulikh, B. Z. and Pinsker, A. G. (1950): Functional analysis in partially. ordered spaces, in Russian.

[23] Kelley, J. L. (1959): Measures on Boolean algebras. Pacific Journal of Mathematics, vol. 9, pp. 1165-1177.

[24] Koppelberg, S. (1989): Handbook of Boolean algebras, vol. 1, North-Holland Publishing Company, Amsterdam.

[25] Kurepa, D. (1935): Ensembles ordonnes et ramifies, Publicationes Mathematicae, University of Belgrade, vol. 4, pp. 1-138, Zbl. 014.39401.

[26] Maharam, D. (1942): On homogeneous measure algebras, Proceedings of the National Academy of Sciences of the United States of America, vol. 28, pp. 108-111.

[27] ———— (1947): An algebraic characterization of measure algebras. Annals of Mathematics. Second Series, vol. 48, pp. 154-167.

[28] Mauldin, D. (editor) (1981): The Scottish Book, Birkhauser Boston, Massachusetts.

[29] Miller, E. W. (1943): A note on Souslin's problem, American Journal of Mathematics, vol. 65, pp. 673-678.

[30] Monk, J. D. (editor) (1989): Handbook of Boolean algebras, North-Holland Publishing Company, Amsterdam.

[31] Quickert, S. (2002): CH and the Sacks property, Fundamenta Mathematicae, vol. 171, no. 1, pp. 93-100.

[32] Shelah, S. (1982): Proper forcing, Lecture Notes in Mathematics, vol. 940, Springer-Verlag, Berlin.

[33] Solovay, R. M. and Tennenbaum, S. (1971) : Iterated Cohen extensions and Souslin's problem, Annals of Mathematics. Second Series, vol. 94, pp. 201-245.

[34] Suslin, M. (1920): Probleme 3, Fundamenta Mathematicae, vol. 1, p. 223.

[35] Szpilrajn, E. (Marczewski), (1934): Remarques sur les fonctions completement additives d'ensemble et sur les ensembles jouissani de lapropriete de Baire, Fundamenta Mathematicae, vol. 22, pp. 303-311.

[36] Talagrand, M. (2006): Maharam's problem, Fundamental Mathematicae, vol. 185, pp. 385-401.

[37] Tennenbaum, S. (1968): Souslin's problem. Proceedings of the National Academy of Sciences of the United States of America, vol. 59, pp. 60-63.

[38] Todorcevic, S. (2000): A dichotomy for P-ideals of countable sets, Fundamenta Mathematicae, vol. 166, no. 3, pp. 251-267.

[39] ———— (2004): A problem of von Neumann and Maharam about algebras supporting continu ous submeasures, Fundamenta Mathematicae, vol. 183, pp. 169-183.

[40] VelickoviC, B. (2005): CCC forcing and splitting reals, Israel Journal oj Mathematics, vol. 147.

[41] Vladimirov, D. A. (2002): Boolean algebras in analysis. Mathematics and its Applications, vol. 540, Kluwer Academic Publishers, Dordrecht, Translated from the Russian manuscript, Foreword and appendix by S. S. Kutateladze.

[42] von Neumann, J. (1936-1937): Continuous geometry, The Institute for Advanced Study. Notes by L. Roy Wilcox on lectures given during 1935-36 and 1936-37 at the Institute for Advanced Study.

[43] ———— (1960): Continuous geometry, Princeton University Press, Princeton, New Jersey.

*****

CHAPTER III

TOPOLOGY DETERMINED BY CONVERGENT SEQUENCES IN A COMPLETE BOOLEAN ALGEBRA

3.0. INTRODUCTION

We investigate the sequential topology τ_s on a complete Boolean algebra *B* determined by algebraically convergent sequences in *B*. We show the role of weak distributivity of *B* in separation axioms for the sequential topology. The main result is that a necessary and sufficient condition for *B* to carry a strictly positive Maharam submeasure is that *B* is ccc and that the space (B, τ_s) is Hausdroff. We also characterize sequential cardinals.

We deal with sequential topologies on complete Boolean algebras from the point of view of separation axioms.

Our motivation comes from the still open Control Measure Problem of D. Maharam (1947, [13]). Maharam asked whether every σ-complete Boolean algebra that carries a strictly positive continuous submeasure admits a σ-additive measure.

Let us review basic notions and facts concerning Maharam's problem. More details and further information can be found in Fremlin's work [5].

Let B be a Boolean algebra. A submeasure on B is a function $\mu : B \to \mathbb{R}^+$ with the properties

(i) $\mu(0) = 0$,

(ii) $\mu(a) \leq \mu(b)$ whenever $a \leq b$ (monotonicity),

(iii) $\mu(a \vee b) \leq \mu(a) + \mu(b)$ (subadditivity).

A submeasure μ on B is

(iv) exhaustive if $\lim \mu(a_n) = 0$ for every sequence $\{a_n : n \in \omega\}$ of disjoint elements,

(v) strictly positive if $\mu(a) = 0$ only if $a = 0$,

(vi) *a* (finitely additive) measure if $\mu(a \vee b) = \mu(a) + \mu(b)$ for any disjoint a and b.

If B is a σ-complete algebra, a submeasure μ on B is called a Maharam submeasure if it is continuous, i.e. $\lim \mu(a_n) = 0$ for every decreasing sequence $\{a_n : n \in \omega\}$ such that $\varpi\{a_n : n \in \omega\} = 0$. It is easy to see that a measure on a σ-complete algebra is continuous if and only if it is σ-additive.

We consider the following four classes of Boolean algebras.

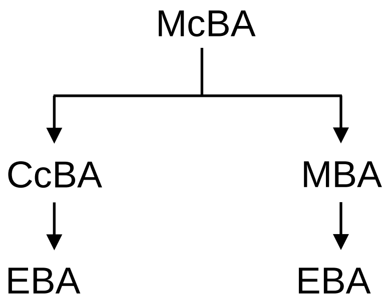
MBA : the class of all Boolean algebras that carry a strictly positive finitely additive measure.

McBA : the class of all measure algebras, i.e. complete Boolean algebras that carry a strictly positive σ-additive measure.

EBA : the class of all Boolean algebras that carry a strictly positive exhaustive submeasure.

CcBA : the class of all complete algebras that carry a strictly positive continuous submeasure.

The diagram below shows the obvious relations between



these classes:

The following theorem, whose proof is scattered throughout Fremlin's work [5], gives additional information. Note that the relations between the classes with measure are the same as between the classes with submeasure.

Theorem 3.1. (i) The class MBA consists exactly of all subalgebras of algebras in McBA.

(ii) The class EBA consists exactly of all subalgebras of algebras in CcBA.

(iii) The class McBA consists of all algebras in MBA that are complete and weakly distributive.

(iv) The class CcBA consists of all algebras in EBA that are complete and weakly distributive.

The problem whether CcBA = McBA is the problem of Maharam mentioned above. It follows from Theorem 3.1 that it is equivalent to the problem whether EBA = MBA.

The class MBA is closed under regular completions : Let *B* be a Boolean algebra and let μ, be a finitely additive strictly positive measure. It follows from [11] that μ, can be extended to a strictly positive measure on the completion $\bar{B}$.

Similarly, the class EBA is closed under regular completions : Let *B* be a Boolean algebra and let μ, be a strictly positive exhaustive submeasure. By [5], *B* can be embedded into a complete Boolean algebra *A* such that μ can be extended to a strictly positive exhaustive submeasure on *A*. By Sikorski's Extension Theorem ([12], p. 70), the completion $\bar{B}$ embeds in *A*, and so $\bar{B}$ also carries a strictly positive exhaustive submeasure.

Consider an algebra $B \in$ CcBA and let μ be a strictly positive Maharam submeasure on *B*. The submeasure determines a topology on *B* : (B, Θ_μ) is a metric space with the distance defined by $\Theta_\mu(a, b) = \mu(a \Delta b)$ for any $a, b \in B$. If *v* is another such submeasure then Θ_μ and Θ_v are equivalent; they determine the same topology on *B*. In [13], Maharam studied a sequential topology on complete Boolean algebras from the point of view of metrizability.

We study sequential topologies on complete Boolean algebras in a more general setting. Our goal is to show that the sequential topology τ_s on a ccc complete Boolean algebra *B* is

Hausdoroff if and only if B carries a strictly positive Maharam submeasure. Following [1] and [15] we say that a cardinal κ is a sequential cardinal if there exists a continuous real-valued function on the space $(\mathcal{P}(\kappa), \tau_s)$ which is not continuous with respect to the product topology. We prove that κ is a sequential cardinal if and only if κ is uncountable and there is a nontrivial Maharam submeasure on the algebra $\mathcal{P}(\kappa)$.

3.1. SEQUENTIAL TOPOLOGY

We review some notions from topology.

Definition 3.1. Let (X, τ) be a topological space. The space X is

(i) sequential if a subset $A \subseteq X$ is closed whenever it contains all limits of τ-convergent sequences of elements of A;

(ii) Fréchet if for every $A \subseteq X$,

$$\mathrm{cl}_\tau(A) = \{x \in X : (\exists\ \langle x_n : n \in w\rangle \subseteq A)\ x_n \underset{\tau}{\to} x\}.$$

It is clear that every Fréchet space is sequential.

Now, consider a complete Boolean algebra B; σ-completeness is sufficient for the following definition. For a sequence $\langle b_n : n \in \omega\rangle$ of elements of B we define

$$\overline{\lim} b_n = \bigvee_{k\in\omega} \bigwedge_{n\geq k} b_n \text{ and } \underline{\lim} b_n = \bigwedge_{k\in\omega} \bigvee_{n\geq k} b_n.$$

We say that a sequence $\langle b_n\rangle$ algebraically converges to an element $b \in B$ in symbols, $b_n \to b$, if $\overline{\lim} b_n = \underline{\lim} b_n = b$.

A sequence $\langle b_n \rangle$ algebraically converges if and only if there exist an increasing sequence $\langle a_n \rangle$ and a decreasing sequence $\langle c_n \rangle$ such that $a_n \leq b_n \leq c_n$ for all $n \in \omega$, and $\bigvee_{n\in\omega} a_n = \bigwedge_{n\in\omega} c_n$.

Definition 3.2. We summarize basic properties of $\rightarrow$:

(i) every sequence has at most one limit;

(ii) for a constant sequence $(x : n \in \omega)$, we have $(x : n \in \omega) \rightarrow x$;

(iii) $x_n \rightarrow 0$ iff $\overline{\lim} x_n = 0$;

(iv) if the x_n's are pairwise disjoint then $x_n \rightarrow 0$;

(v) $\overline{\lim}(x_n \vee y_n) = \overline{\lim} x_n \vee \overline{\lim} y_n$;

(vi) if $x_n \rightarrow x$ and $y_n \rightarrow y$ then $x_n \vee y_n \rightarrow x \vee y$ and $-x_n \rightarrow -x$;

(vii) if $\langle x_n \rangle$ is increasing then $x_n \rightarrow \omega_{n\in\omega} x_n$.

Definition 3.3. Sequential topology on B. Consider all topologies τ on B with the following property:

$$\text{if } x_n \rightarrow x \text{ then } x_n \underset{\tau}{\rightarrow} x.$$

There is a largest topology with respect to inclusion among all such topologies. We denote it by τ_s and call it the sequential topology on B.

The topology τ_s can be described as follows, by definining the closure operation: For any subset A of the algebra B let $u(A) = \{x : x$ is the limit of a sequence $\{x_n\}$ of elements of $A\}$.

The closure of a set A in the topology τ_s is obtained by iteration of u:

$$\mathrm{cl}_{\tau_s}(A) = \bigcup_{\alpha<\omega_1} u^{(\alpha)}(A),$$

where $u^{(\alpha+1)}(A) = u(u^{(\alpha)}(A))$, and $u^{(\alpha)}(A) = \bigcup_{\beta<\alpha} u^{(\beta)}$ for a limit α.

It is clear that the topology τ_s is T_1, i.e. every singleton is a closed set. Moreover, (B, τ_s) is a Fréchet space if and only if $\mathrm{cl}(A) = u(A)$ for every $A \subseteq B$.

We remark that a sequence $\{x_n\}$ converges to x topologically if and only if every subsequence of $\{x_n\}$ has a subsequence that converges to x algebraically.

Example (Measure algebras). Let B be a complete Boolean algebra carrying a strictly positive σ-additive measure μ. For any $a, b \in B$, let

$$\Theta(a, b) = \mu(a \,\Delta\, b);$$

Θ is a metric on B and the topology given by Θ coincides with the sequential topology. Hence (B, τ_s) is metrizable.

Maharam's Control Measure Problem is equivalent to the question of whether there exist complete Boolean algebras other than the algebras in the class McBA for which the sequential topology is metrizable.

Properties of the topology τ_s

Proposition 3.1. (i) The operation of taking complement is continuous (and hence a homeomorphism).

(ii) For a fixed a, the function $a \vee x$ is a continuous function of x.

(iii) For a fixed a, the function $a \,\Delta\, x$ is continuous.

The operation $\vee$ is generally not a continuous function of two variables. As a consequence of (iii), the space (B, τ_s) is homogeneous: given $a, b \in B$, there is a homeomorphism f such that $f(a) = b$, namely $f(x) = (x \,\Delta\, b) \,\Delta\, a$. The topology τ_s is determined by the family $\mathcal{N}_0$ of all neighborhoods of 0 as for every $a \in B$ and every set W, W is a neighborhood of *a if and only if* $a \,\Delta\, W \in \mathcal{N}_0$.

As a consequence of homogeneity of (B, τ_s), B does not have isolated points unless B is finite.

Lemma 3.1. Let B be a σ-complete algebra. Let $\{u_n\}_{n=0}^{\infty}$ be an antichain in B, and let U be a neighborhood of 0. Then there exists a k such that $B \,|\, \omega_{n\exists k}\, u_n \subset U$.

Proof. If not, then for every k there exists an element x_k below $\omega_{n\exists k}\, u_n$ such that $x_k \notin U$. But then the sequence $\{x_k\}_k$ converges to 0 and so, because $U \in \mathcal{N}_0$, there exists some k_0 such that $x_k \in U$ for all $k > k_0$; a contradiction.

A subset D of a Boolean algebra B is dense if for every $b \in B$, $b \neq 0$, there is some $d \in D$, $d \neq 0$, such that $d \leq b$. We call D downward closed if $a < d \in D$ implies $a \in D$.

A downward closed dense set is called open dense. Since we consider a topology on B we shall call dense and open dense sets algebraically dense and algebraically open dense to avoid confusion with the corresponding topological terms.

Corollary 3.1. (i) Every neighborhood of 0 contains all but finitely many atoms.

(ii) If B is atomless then every neighborhood of 0 contains an algebraically open dense subset of B.

(iii) If B is atomless and ccc, then for every $U \in \mathcal{N}_0$ there exists a k such that

$$u_0 \vee u_1 \ldots \vee u_k \vee u = 1$$

where $u = \omega_{n > k} u_k \in U$.

Proof. (i) is clear.

(ii) Let V be a neighborhood of 0. If V does not contain an algebraically open dense set then B - V is algebraically dense below some $u \neq 0$ and hence contains a pairwise disjoint set $\{x_n\}_n$. But then $\lim x_n = 0$ and so there is some n such that $x_n \in V$; a contradiction.

(iii) As U is algebraically dense in B, there exists a maximal antichain of B included in U, and by ccc the antichain is countable: $\{u_n\}_n \subset U$. There exists a k so that $u = \omega_{n>k} u_n \in U$, and then $u_0 \vee u_1 \vee ... \vee u_k \vee u = 1$.

Proposition 3.2. If B is atomless and ccc, then (B, τ_S) is connected.

Proof. Assume that there are two disjoint nonempty clopen sets X and Y with $X \cup Y = B$ and $0 \in X$, and let $a \in Y$. Let C be a maximal chain in B such that $\inf C = 0$ and $\sup C = a$. Let $x = \sup(C \cap X)$; by ccc, x is the limit of a sequence in $C \cap X$ and therefore $x \in X$. Let $y = \inf(Y \cap \{c \in C : c \geq x\})$. Using the ccc again we have $y \in Y$, and clearly $x < y$. By maximality, both x and y are in C. Since B is atomless, there exists some z with $x < z < y$. This contradicts the maximality of C.

Lemma 3.2 (i) An ideal I on a σ-complete Boolean algebra B is a closed set in the sequential topology if and only if it is a σ-complete ideal.

(ii) If I is a σ-ideal on B then the sequential topology on the quotient algebra B/I is the quotient topology of τ_s given by the canonical projection.

(iii) If τ_s is Fréchet then so is the quotient topology.

3.2. FRÉCHET SPACES

We shall now consider those complete Boolean algebras for which the sequential topology is Fréchet. We will show that this is equivalent to an algebraic property. First we make the following observation:

Proposition 3.3. If (B, τ_s) is a Fréchet space then for every $V \in \mathcal{N}_0$ there is some $U \subseteq V$ in $\mathcal{N}_0$ such that U is downward closed.

Proof. If $V \in \mathcal{N}_0$, consider the set

$$X = \{a \in B : \text{there exists some } b \leq a \text{ such that } b \notin V\},$$

and let $u(X)$ be the set of all limits of sequences in X. As τ_s is Fréchet, $u(X)$ is the closure of X. We shall prove that the set $U = B - u(X)$ is downward closed and contains 0.

For the first claim it suffices to show that $a \in u(X)$ and $a < b$ implies $b \in u(X)$. Thus let $a = \lim a_n$ with $a_n \in X$. It follows that $b = \lim(a_n \vee b)$, and since $a_n \vee b \in X$, we have $b \in u(X)$.

To see that $0 \notin u(X)$, assume that $\{a_n\} \subseteq X$ and $\lim a_n = 0$. Then there are $x_n \leq a_n$ in $B - V$, but this is impossible because $\lim x_n = 0$. Hence 0 is not in $u(X)$.

Thus if (B, τ_s) is Fréchet, its topology is determined by the set $\mathcal{N}_0^{\mathrm{d}}$ of all $U \in \mathcal{N}_0$ that are downward closed. $\mathcal{N}_0^{\mathrm{d}}$ is a neighborhood base of 0.

Definition 3.4. Let κ be an infinite cardinal. A Boolean algebra B is (ω, κ)-weakly distributive if for every sequence $\{P_n\}$ of maximal antichains, each of size at most κ, there exists a dense set Q with the property that each $q \in Q$ meets only finitely many elements of each P_n. B is weakly distributive if it is (ω, ω)-weakly distributive.

If B is a κ^+-complete Boolean algebra then B is (ω, κ)-weakly distributive if and only if it satisfies the following distributive law:

$$\bigwedge_n \bigvee_\alpha a_{n\alpha} = \bigvee_{f:\omega\to[\kappa]^{<\omega}} \bigwedge_n \bigvee_{\alpha\in f(n)} a_{n\alpha}.$$

We recall two frequently used cardinal characteristics.

Definition 3.5. The splitting number is the least cardinal s of a family S of infinite subsets of ω such that for every infinite $X \subseteq \omega$ there is some $s \in S$ such that both $X \cap S$ and $X - S$ are infinite. (S 'splits' X.)

The bounding number is the least cardinal **b** of a family $\mathcal{F}$ of functions from ω to ω such that $\mathcal{F}$ is unbounded; i.e. for every $g \in \omega^\omega$ there is some $f \in \mathcal{F}$ such that $g(n) \leq f(n)$ for infinitely many n.

The following characterization of Fréchet spaces (B, τ_s) uses the cardinal invariant **b** and is similar to several other results using **b**, such as in [2]. A consequence of Theorem 3.2 is that $(P(\kappa),\tau_s)$ is a Frechet space if and only if $\kappa < \mathbf{b}$.

Theorem 3.2. Let B be a complete Boolean algebra. The sequential space (B, τ_s) is Fréchet if and only if B is weakly distributive and satisfies the **b**-chain condition.

We first reformulate the condition stated in Theorem 3.2. Let B be a complete Boolean algebra. We call a matrix $\{a_{mn}\}$ increasing if each row $\{a_{mn} : n \in \omega\}$ is an increasing sequence with limit 1. Note that B is weakly distributive if and only if for every increasing matrix

$$\bigvee_{f\in\omega^\omega} \underline{\lim}\, a_{m,f(m)} = 1.$$

Lemma 3.3. A complete Boolean algebra B is weakly distributive and satisfies the **b**-chain condition if and only if for every increasing matrix $\{a_{mn}\}$ there exists a function $f \in \omega^\omega$ such that $\lim a_{mf(m)} = 1$.

Proof. First let B be weakly distributive and satisfy **b**-c.c., and let $\{a_{mn}\}$ be an increasing matrix. By the **b**-chain condition there exists a set $F \subset \omega^\omega$ of size less than **b** such that $\omega_{f \in F} \underline{\lim}\, a_{m,f(m)} = 1$. Let $g : \omega \to \omega$ be an upper bound of F under eventual domination. Since the matrix is increasing, we have $\underline{\lim}\, a_{m,f(m)} \leq \underline{\lim}\, a_{m,\, g(m)}$ for every $f \leq F$. Therefore $\lim a_{m,g(m)} = 1$.

Conversely, assume that the condition holds. Then B is weakly distributive, and we verify the **b**-chain condition. Thus let W be a partition of 1; we prove that $|W| < \mathbf{b}$. Let $\{f_u : u \in W\}$ be any family of functions from ω to ω indexed by elements of W. For each m and each n we let

$$a_{mn} = \omega\{u \in W : f_u(m) < n\}.$$

The matrix $\{a_{mn}\}$ is increasing and therefore there exists a function $g : \omega \to \omega$ such that $\lim a_{m,g(m)} = 1$. Since W is an antichain, it follows that for any $u \in W$ there is some m_u such that $u < a_{m,g(m)}$ for every $m \geq m_u$. Hence $f_u(m) < \mathrm{g}(m)$ for every $m \geq m_u$ and it follows that g is an upper bound of the family $\{f_u : u \in W\}$. Therefore every family of functions of size $|W|$ is bounded and so $|W| < \mathbf{b}$.

Proof of Theorem 3.2. We wish to show that the condition in Lemma 3.3 is necessary and sufficient for the space (B, τ_s) to be Fréchet. To see that the condition holds if (B, τ_e) is Fréchet, we recall [13] that for

(B, τ_s), being Fréchet is equivalent to the following statement: whenever $\{x_{mn}\}$, $y_m\}$ and z are such that $\lim_n x_{mn} = y_m$ for each m and $\lim_m y_m = z$, then there is an $f : \omega \to \omega$ such that $\lim_n x_{m, f(m)} = z$.

To show that the condition implies that (B, τ_s) is Fréchet, let $\{x_{mn}\}$, $\{y_m\}$ and z be as above. For each m and each n let $u_{mn} = X_{mn} \Delta (-Y_m)$, and let $a_{mn} = \varpi_{k\geq n}\ u_{mk}$. For each m, $\lim_n u_{mn} = 1$; the matrix $\{a_{mn}\}$ is increasing, with each row converging to 1 and so there exists some $f : \omega \to \omega$ such that $\lim a_{m,f(m)} = 1$. It follows that $\lim_m \varpi_{k\geq f(m)}\ u_{mk} = 1$, and so $\lim(x_{m},f_{(m)}\ \Delta\ (\text{-z})) = \lim(x_{m,\ f(m)}\ \Delta\ (-y_m)) = \lim\ u_{m,f(m))} = 1$. Hence $\lim x_{m,f(m)} = z$.

We conclude with the following observation that we shall use in Section 3.4.

Lemma 3.4. (a) For every set $A \subseteq B$, $\text{cl}(A) = \cap\{A\ \Delta\ V : V \in \mathcal{N}_0\}$

(b) If (B, τ_s) is Fréchet and A is downward closed then $\text{cl}(A) \cap\{A \vee V : V \in \mathcal{N}_0^d\}$, and $\text{cl}(A)$ is downward closed.

Proof. (a) For any $x \in B$, $x \in \text{cl}(A)$ iff for all $V \in \mathcal{N}_0$, $(V\ \Delta\ x) \cap A \neq 0$, i.e. there exist $v \in V$ and $a \in A$ such that $v\ \Delta\ x = a$. The latter is equivalent to $x = a\ \Delta\ v$, or $x \in A\ \Delta\ V$.

(b) If both A and V are downward closed then $A \vee V = A\ \Delta\ V$.

Corollary 3.2. For every $U \in \mathcal{N}_0$, $\text{cl}(U) \subseteq U$. If (B, τ_s) is Fréchet, then $\text{cl}(U) \subseteq U$ for every $U \in \mathcal{N}_0^d$.

3.3. SEPARATION AXIOMS

We will now discuss separation axioms for the topology τ_s. We immediately see that the sequential topology on B is T_1. The space is Hausdroff if and only if every point $b \neq 0$ can be separated from 0, which is equivalent to the statement that for every $b \neq 0$ there exists some $V \in \mathcal{N}_0$ such that $b \notin V$.

Theorem 3.3. If (B, τ_S) is a Hausdroff space then B is (ω, ω_1)-weakly distributive.

We first prove a weaker statement, namely that being Hausdroff implies weak distributivity:

Lemma 3.5. If B is not weakly distributive then there exists an $a \neq 0$ such that $c \in \text{cl}(U)$ for every $c \leq a$ and every $U \in \mathcal{N}_0$. Hence (B, τ_s) is not Hausdroff.

Proof. Assume that B is not (ω, ω)-weakly distributive. There is some $a \neq 0$ and there exists an infinite matrix $\{a_{mn}\}$ such that each row is a partition of a, and for any nonzero $x \leq a$ there is some m such that $x \wedge a_{mn} \neq 0$ for infinitely many n.

Let $c \leq a$ and let U be an arbitrary neighborhood of 0. We will show that $c \in \text{cl}(U)$. For every m and every n let $y_{mn} = c \wedge \omega_{i \geq n}\, a_{mi}$. Since the sequence $\{y_{0n}\}$ converges to 0 there exists some no such that $y_{0n_0} \in U$; let $x_0 = y_{0n_0}$. Next we consider the sequence $\{y_{1n} \vee x_0\}$. This sequence converges to x_0 and so there exists some n_1 such that $x_1 \in U$

where $x_1 = y_{1n_1} \vee x_0$. We proceed by induction and obtain a sequence $\{n_m\}$ and an increasing sequence $\{x_m\}$ of elements of U. This sequence converges to c because otherwise, if we let $b \neq 0$ be the complement of $\omega_n x_n$ in c, then $b \leq \bigvee_m \bigvee_{i<n_m} a_{mi}$ and so b meets only finitely many elements in each row of the matrix. Hence $c \in \mathrm{cl}(U)$.

Proof of Theorem 3.3. Let (B, τ_s) be a Hausdroff space. To prove that B is (ω, ω_1)-weakly distributive, let

$$A = \{a_{n\alpha} : n \in \omega, \alpha \in \omega_1\}$$

be a matrix such that each row is a partition of 1. Denote by X the set of all those $x \in B$ that meet at most countably many elements of each row of A. As B is (ω, ω)-weakly distributive, for every nonzero $x \in X$ there is a nonzero $y \leq x$ that meets only finitely many elements of each row of A. Thus we complete the proof by showing that $\omega X = 1$.

Assume otherwise; without loss of generality we may assume that every $x \neq 0$ meets uncountably many elements of at least one row of A. Then the matrix A represents a Boolean-valued name for a cofinal function from ω into ω_1. Thus B collapses ω_1 and therefore there exists a matrix

$$\{b_{n\alpha} : n \in \omega, \alpha \in \omega_1\}$$

such that each row and each column is a partition of 1 (the name for a one-to-one mapping of ω onto ω_1). We get a contradiction to Hausdroffness by showing that 1 is in the closure of every $V \in \mathcal{N}_0$.

Let $V \in \mathcal{N}_0$ be arbitrary. By Lemma 3.3 there is for every $\alpha \in \omega_1$ some $n_\alpha \in \omega$ such that $v_\alpha = \bigvee_{i \geq n_\alpha} b_{i\alpha} \in V$. Thus there exists some n and an infinite set $\{\alpha_k\}_k$ such that $n_{a_k} = n$ for all k. Now, by Definition 3.2(v),

$$\overline{\lim_k} \bigvee_{i<n} b_{i\alpha_k} = \bigvee_{i<n} \overline{\lim_k} b_{i\alpha_k} = 0.$$

Therefore $\lim_k v_{\alpha_k}$ = 1 and so 1 is in the closure of V.

For (ω, ω_1)-weak distributivity we refer to Namba's work [14] which shows that it may or may not be equivalent to (ω, ω)-weak distributivity. If $\mathbf{b} = \omega_1$ then (ω, ω)-weak distributivity and (ω, ω_1)-weak distributivity are equivalent, and there is a model of ZFC in which they are not equivalent. Below (in Example 3.1(i)) we give another example of a complete Boolean algebra that is (ω, ω)-weakly distributive but not (ω, ω_1)-weakly distributive.

Theorem 3.3 cannot be extended by replacing ω_1 with ∞ : Example 3.3(ii), due to Prikry [16], provides a complete Boolean algebra that is Hausdroff (therefore weakly distributive) but not (ω, κ)-weakly distributive, for a measurable κ.

In view of Theorems 3.2 and 3.3 the question arises about the relative strength of being a Hausdroff space and being a Fréchet space. Example 3.1 below shows that Hausdroff does not imply Fréchet : the space $(P(\mathbf{b}),\tau_s)$ is Hausdroff but not Fréchet.

For the other direction, see Examples 3.2 and 3.3. If T is a Suslin tree then $(B(T), \tau_s)$ is Fréchet but not Hausdroff.

Example 3.1. For every infinite cardinal κ the space $(P(\kappa), \tau_s)$ is Hausdroff. This is because each principal ultrafilter on κ is a closed and open subset of $P(\kappa)$.

We identify $P(\kappa)$ with 2^κ (via characteristic functions). For each $\alpha \in \kappa$ the set $\{X \subseteq \kappa : \alpha \in X\}$ and its complement $\{X \subseteq \kappa : \alpha \notin X\}$ are closed under limits of sequences and so are both closed and open. This implies that the topology τ_s extends the product topology, and the space $(P(\kappa),\tau_s)$ is a totally disconnected Hausdroff space. If $\kappa = \aleph_0$ then τ_s is equal to the product topology. To see this, let $U \subseteq P(\omega)$ be an open set in the sequential topology and let $A \in U$. For each n let S_n denote the basic open set (in the product topology) $\{X \subseteq \omega : X \cap n = A \cap n\}$. It suffices to show that U contains some S_n as a subset. If not, there exists for each n some $X_n \in S_n - U$. But $A = \lim_n X_n$, and since the complement of U is closed, $A \notin U$; a contradiction.

When κ is an uncountable cardinal, the space $(P(\kappa), \tau_s)$ is not compact and so τ_s strictly stronger than the product topology.

By [16] space $(P(\kappa), \tau_s)$ is sequentially compact if and only if $\kappa < s$, the splitting umber.

By [7] $(P(\kappa), \tau_s)$ is regular if and only if $\kappa = \omega$. See Corollary 3.3.

Example 3.2. (Aronszajn trees). We show that the Boolean algebra associated with a Suslin tree is an example of a Fréchet space that is not Hausdroff. We point out that in ZFC, the only known examples of algebras that are Fréchet spaces are measure algebras.

Let T be an Aronszajn tree and assume that each node has at least two immediate successors. Let $B(T)$ denote the complete Boolean algebra that has upside down T as a dense set. We will show that $(B(T), \tau_s)$ is not a Hausdroff space. This shows that the converse of Theorem 3.1 is not provable : if T is a Suslin tree then $B(T)$ is a ccc ω-distributive Boolean algebra.

We prove that 0 and 1 cannot be separated by open sets : we show that $1 \in V$ for every open neighborhood V and 0. Let $V \in \mathcal{N}_0$. For every $\alpha \in \omega_1$, the αth level T_α of the tree is a countable partition of 1 and so there exists a finite set $u_\alpha \subseteq T_\alpha$ such that $x_\alpha = \omega(T_\alpha - u_\alpha) \in V$. Let $y_\alpha = \omega u_\alpha$. We claim that there is a β such that $y_\beta \in V$; this will complete the proof as $1 = x_\beta \Delta y_\beta \in V$.

Let $f : [\omega_1]^2 \to \{0, 1\}$ be the function defined as follows : $f(\alpha, \beta) = 0$ if $y_\alpha \wedge y_\beta = 0$ and $f(\alpha, \beta) = 1$ otherwise. By the Dushnik-Miller Theorem there exists a set $I \subseteq \omega_1$, either homogeneous in color 0 and of size $\aleph_0$, or homogeneous in color 1 and of size $\aleph_1$. The latter case is impossible because the $u_\alpha s$ are disjoint finite sets in an Aronszajn tree (see [9], Lemma 24.2). Hence there is an infinite set $\{\alpha_n : n \in w\}$ such that the y_{α_n} are pairwise disjoint. Thus the sequence $\{y_{\alpha_n}\}$ converges to 0 and so there exists some n such that $y_{\alpha_n} \in V$.

Example 3.3. (using large cardinals).

(i) assume that there exists a nontrivial $\aleph_2$-saturated σ-ideal I on $P(\omega_1)$, and assume that $\mathbf{b} = \aleph_2$. Both these assumption are consequences of Martin's Maximum (MM) with I = the non-stationary ideal.

Let $B = P(\omega_1)/I$. Then B is a complete Boolean algebra and satisfies that $\aleph_2$-chain condition. Since $\mathbf{b} = \aleph_2$, the space $(P(\omega_1),\tau_s)$ is Fréchet, and so by Lemma 1.8, (B, τ_s) is Fréchet. Therefore B is weakly distributive.

Since forcing with B collapses is $\aleph_1$, B is not (ω, ω_1)-weakly distributive, and hence (B, τ_s) is not Hausdoff.

The space $(P(\omega_1),\tau_s)$ is separable : this follows from MM, specifically from $\mathbf{p} = \aleph_2$ (cf. [4], [17] and [18]). Hence (B, τ_s) is separable, and so the complete Boolean algebra B is countably generated.

This example is in the spirit of [7] where a similar example is presented using MA and a measurable cardinal.

(ii) Let κ be a measurable cardinal, and let B be the complete Boolean algebra associated with Prikry forcing. B is not (ω, κ)-weakly distributive as it changes the cofinality of κ to w. But the space (B, τ_s) is Hausdoff : for any $a \in B^+$ there is a κ-complete ultrafilter on B containing a (cf. [15]). Every such ultrafilter is a clopen set in (B, τ_s).

Thus Hausdroffness does not imply (ω, ∞)-weak distributivity of (B, τ_s). We do not know if the large cardinal assumption is necessary.

A topological space is regular if points can be separated from closed sets; equivalently, for every point x and its neighborhood U there exists an open set V such that $x \in V$ and $\mathrm{cl}(V) \subseteq U$. The space (B, τ_s) is regular if and only if for every $U \in \mathcal{N}_0$ there is some $V \in \mathcal{N}_0$ such that $\mathrm{cl}(V) \subseteq U$.

A result proved independently in [7] states that the atomic algebra $P(\omega_1)$ is not regular. The following lemma uses the method employed in these chapters.

Lemma 3.6. In the space $(P(\omega_1), \tau_s)$ for every $V \in \mathcal{N}_0$ there exists a closed unbounded set $C \subset \omega_1$ such that $\omega_1 - \beta \in \mathrm{cl}(V)$ for every $\beta \in C$.

Proof. Let $V \in \mathcal{N}_0$. Let $\{A_{\alpha n} : \alpha \in \omega_1, n \in \omega\}$ be an Ulam matrix, i.e. a double array of subsets of ω_1 with the following properties :

$$A_{\alpha n} \cap A_{\alpha m} = \emptyset \qquad (n \neq m),$$
$$A_{\alpha n} \cap A_{\beta m} = \emptyset \qquad (\alpha \neq \beta),$$
$$\bigcup_{n\in\omega} A_{\alpha n} = \omega_1 - \alpha$$

By Lemma 3.1 there exists for each α some k_α such that $X_\alpha = \bigcup_{n\geq k_\alpha} A_{\alpha n}$ is in V. There exist some k and an uncountable set W such that $k_\alpha = k$ for every $\alpha \in W$. Let C be the set of all limits of increasing sequences of ordinals in W. We claim that $\omega_1 - \beta \in \mathrm{cl}(V)$ for every $\beta \in C$.

Let $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots$ be in W such that $\beta = \lim_n \alpha_n$. Note that $\overline{\lim}\, n \bigcup_{i<k} A_{\alpha_n i} = \bigcup_{i<k} \overline{\lim}_n A_{\alpha_n i} = \emptyset$, and hence $X = \lim n X_{\alpha_n} = \omega_1 - \beta$. Therefore $X \in \mathrm{cl}(V)$.

Corollary 3.3. The space $(P(\omega_1), \tau_s)$ is not regular.

Proof. Let U be the set of all $x \subset \omega_1$ whose complement is uncountable. By Lemma 3.6, U does not contain $\mathrm{cl}(V)$ for any $V \in \mathcal{N}_0$.

Corollary 3.4. If a complete Boolean algebra B does not satisfy the countable chain condition then (B, τ_s) is not regular.

Proof. B contains $P(\omega_1)$ as a complete subalgebra, therefore as a closed subspace. Hence it is not regular.

Corollary 3.5. Let $B = P(\omega_1)$, or more generally, let B be a complete Boolean algebra that does not satisfy the countable chain condition. If $\{U_n\}_n$ is a countable subset of $\mathcal{N}_0$ then $\cap_n \mathrm{cl}(U_n)$ is uncountable.

Proof. This follows easily from Lemma 3.6 when $B = P(\omega_1)$. In the general case, (B, τ_s) contains $P(\omega_1)$ as a subspace and each $U_n \cap P(\omega_1)$ is an open neighborhood of $\emptyset$.

Corollary 3.6. Let B be a complete Boolean algebra, and assume that in the space (B, τ_s) there exists a countable family $\{U_n\}_n$ of neighborhoods of 0 such that $\cap_n \mathrm{cl}(U_n) = \{0\}$. Then B satisfies the countable chain condition and (B, τ_s) is Fréchet.

Proof. B satisfies ccc by Corollary 3.5. Also, (B, τ_s) is clearly Hausdroff and so B is weakly distributive by Lemma 3.5. Hence, by Theorem 3.2, B is a Fréchet space.

We conclude this section with some remarks:

A Fréchet space is Hausdroff if and only if

$\cap\{V : V \in \mathcal{N}_0\} = \{0\}$.

Even more is true: If the space (B, τ_s) is Fréchet and Hausdroff, then for every k,

$$\bigcap_k \{V : V \in \mathcal{N}_0\} = \{0\}.$$

This is a consequence of the following:

Lemma 3.7. Let B be a σ-complete Boolean algebra such that (B, τ_s) is Fréchet. Then for every $U \in \mathcal{N}_0^d$ there exists a $V \in \mathcal{N}_0^d$ such that $V \subseteq U$.

In the next section we use this consequence of Lemma 3.7.

Corollary 3.7. If B is as in Lemma 3.7 and $U \in \mathcal{N}_0^d$ then there exists a $V \subseteq U$ in $\mathcal{N}_0^d$ such that $\mathrm{cl}(V) \vee \mathrm{cl}(V) \subseteq U$.

(To see that this follows from Lemma 3.7, use $\mathrm{cl}(V) \subseteq V$.)

Proof (of Lemma 3.7). Assume that for every $V \in \mathcal{N}_0^d$ there exist x, y and z in V such that $x \vee y \vee z \notin U$.

Let $V_0 = U$; by induction we define neighborhoods V_n and points x_n, y_n, z_n as follows: For each n let $x_n, y_n, z_n \in V_n$, be such that $x_n \vee y_n \vee z_n \notin U$. Then let $V_{n+1} \subseteq V_n$, be in $\mathcal{N}_0^d$ and such that the sets $x_n \vee V_{n+1}$, $y_n \vee V_{n+1}$ and $z_n \vee V_{n+1}$ are all included in V_n; such a neighborhood exists by the one-sided continuity of V.

Let $X = \cap_n \mathrm{cl}(V_n)$ and $\bar{x} = \overline{\lim}_n x_n, \bar{y} = \overline{\lim}_n y_n$ and $\bar{z} = \overline{\lim}_n z_n$. The set X is topologically closed and downward closed, and $X \subseteq \mathrm{cl}(U) \subseteq U$.

We claim that $\bar{x}, \bar{y}, \bar{z} \in X$. Thus let us prove that $\bar{x} \in \mathrm{cl}(V_n)$ for each n. We have $x_n \in V_n$, and by induction on $k > 0$ we see that $x_n \vee x_{n+1} \vee \ldots \vee x_{n+k} \in V_n$. Thus $V_{i \geq n}\, x_i \in \mathrm{cl}(V_n)$ and $\bar{x} \in \mathrm{cl}(V_n)$.

Next we claim that $\overline{x} \vee X \subseteq X$ (and similarly for $\overline{y}, \overline{z}$). Let n be arbitrary; we show that $\overline{x} \vee X \subseteq \mathrm{cl}(V_n)$. For any k we have $x_n \vee \ldots \vee x_{n+k} \vee V_{n+k+1} \subseteq V_n$, and by the one-sided continuity of $\vee$ it follows that $x_n \vee \ldots \vee x_{n+k} \vee \mathrm{c1}(V_{n+k+1}) \subseteq \mathrm{cl}(V_n)$. Hence $x_n \vee \ldots \vee x_{n+k} \vee X \subseteq \mathrm{cl}(V_n)$, and so $V_{i \geq n} x_i \vee X \subseteq \mathrm{cl}(V_n)$. As $\overline{x} \leq \bigvee_{i \geq n} x_i$ and $\mathrm{cl}(V_n)$ is downward closed, we have $\overline{x} \vee X \subseteq \mathrm{cl}(V_n)$.

Now it follows that $\overline{x} \vee \overline{y} \vee \overline{z}$ is in X and hence in U. But $\overline{x} \vee \overline{y} \vee \overline{z} = \overline{\lim}_n(\overline{x}_n \vee \overline{y}_n \vee \overline{z}_n)$. As the complement of U is upward closed, we have $\omega_{i \geq n}(x_i \vee y_i \vee z_i) \notin U$ for each n, and because U is topologically open, we have $\overline{x} \vee \overline{y} \vee \overline{z} \notin U$, a contradiction.

3.4. METRIZABILITY.

We will show that for complete ccc Boolean algebras, Hausdroffness of the sequential topology is a strong property: it implies metrizability, and equivalently, the existence of a strictly positive Maharam submeasure. We remark that the assumption of completeness is essential.

Theorem 3.4. If B is a complete Boolean algebra, then the following are equivalent:

(i) B is ccc and (B, τ_s) is a Hausdroff space,

(ii) there exists a countable family $\{U_n\}_n$ of open neighborhoods of 0 such that $\bigcap_n \mathrm{cl}(U_n) = \{0\}$,

(iii) the operation $\vee$ is continuous at (0, 0), i.e. for every $V \in \mathcal{N}_0$ there exists a $U \in \mathcal{N}_0$ such that $U \subseteq V$,

(iv) (B, τ_s) is a regular space,

(v) (B, τ_s) is a metrizable space,

(vi) B carries a strictly positive Maharam submeasure.

The equivalence of (v) and (vi) is proved in [13], and (v) implies (i). We shall prove in this section that properties (i)-(iv) are equivalent and imply (vi). First we claim that each of the four properties implies that *B* satisfies ccc, and that the space (B, τ_s) is Fréchet.

If *B* is ccc and Hausdroff, then by Theorems 3.2 and 3.3 it is Fréchet.

Property (ii) implies Fréchet by Corollary 3.6, and property (iv) implies (i) (and hence Fréchet) by Corollary 3.4.

To complete the claim, Lemma 3.9-3.11 and Corollary 3.7 below prove that (iii) implies Fréchet. Let *B* be a complete Boolean algebra and assume that $\vee$ is continuous at (0, 0).

Lemma 3.8. *B* satisfies the countable chain condition.

If *B* does not satisfy ccc then (B, τ_s) contains $P(\omega_1)$ as a closed subspace. Thus the lemma is a consequence of the following lemma closely related to Corollary 3.3:

Lemma 3.9. In $(P(\omega_1), \tau_s)$ the operation $\bigcup$ is not continuous at (0, 0).

Proof. Let *U* be the set of all $x \subset \omega_1$ whose complement is uncountable. *U* is an open neighborhood. We will show that for every $V \in \mathcal{N}_0$ there exist *Y* and *Z* in *V* such that $Y \bigcup Z \notin U$. Thus let $V \in \mathcal{N}_0$.

By Lemma 3.6 there exists an *X* such that $X \notin U$ while $X \in \mathrm{cl}(V)$. By Corollary 3.2 there exist *Y* and *Z* in *V* such that $X = Y \Delta Z$. But $Y \Delta Z \subseteq Y \bigcup Z$ and therefore $Y \bigcup Z \notin U$.

Lemma 3.10. B is weakly distributive.

Proof. Assume that B is not weakly distributive. By Lemma 3.5 there exists some $a \neq 0$ such that $a \in \mathrm{cl}(V)$ for every $V \in \mathcal{N}_0$.

Let $U = \{x \in B : x \not\geq a\}$; then U is a neighborhood of 0. We claim that $V \lceil\ U$ for every $V \in \mathcal{N}_0$, contradicting the continuity of V. Thus let $V \in \mathcal{N}_0$ be arbitrary.

We have $a \in \mathrm{cl}(V)$. By Corollary 2.7, $a \in V \Delta V$ and so there exist x and y in V such that $a = x \Delta y$. If we let $b = x \vee y$ then $b \geq a$ and therefore $b \notin U$. But $b \in V$, completing the proof.

Corollary 3.8. (B, τ_s) is Fréchet.

Proof. Use Theorem 3.2.

For the rest of section 3.4, we assume that B is a complete Boolean algebra that satisfies the countable chain condition, and that the space (B, τ_s) is Fréchet. In particular, $\mathcal{N}_0^{\mathrm{d}}$ is a neighborhood base, so we shall only consider those neighborhoods of 0 that are downward closed.

To prove that (i)–(iv) are equivalent, we first observe that (iii) implies (iv):

Proposition 3.4. If $\vee$ is continuous at (0, 0) then (B, τ_s) is regular.

Proof. Let $V \in \mathcal{N}_0$. By homogeneity, it suffices to find an open U such that $\mathrm{cl}(U) \subseteq V$. Since $\vee$ is continuous at (0, 0) and since (B, τ_s) is Fréchet, by Corollary 2.7 there exists $U \in \mathcal{N}_0^{\mathrm{d}}$ such that $\mathrm{cl}(U) \subseteq U \subseteq V$.

As (iv) implies (i), it remains to show that (i) implies (ii) and that (ii) implies (iii). Lemma 3.11 proves the latter:

Lemma 3.11. Assume that (B, τ_s) satisfies (ii). Then the operation $\vee$ is continuous at (0, 0).

Proof. As (B, τ_s) is Fréchet, the set $\mathcal{N}_0^{\mathrm{d}}$ of all downward closed open neighborhoods of 0 is a neighborhood base. Thus let us assume that there exists $U \in \mathcal{N}_0^{\mathrm{d}}$ such that for every $V \in \mathcal{N}_0$ there exist x and y in V with $x \vee y \notin U$.

Let $\{V_n\}_n$ in $\mathcal{N}_0^{\mathrm{d}}$ be such that $\bigcap_n \mathrm{cl}(V_n) = \{0\}$. We construct a descending sequence of neighborhoods U_n in $\mathcal{N}_0^{\mathrm{d}}$ as follows: Let $U_0 = V_0 \cap U$. Given U_n let $x_n, y_n \in U_n$ be such that $x_n \vee y_n \notin U$. By (separate) continuity of $\vee$ there exists a set $U_{n+1} \in \mathcal{N}_0^{\mathrm{d}}$ such that $x_n \vee U_{n+1} \subset U_n$ and $y_n \vee U_{n+1} \subset U_n$; moreover, we may assume that U_{n+1} is included in V_{n+1}.

Let $\bar{x} = \overline{\lim} x_n$ and $\bar{y} = \overline{\lim} y_n$. First we claim that $\bar{x} = \bar{y} = 0$ and therefore $\bar{x} \vee \bar{y} = 0 \in U$.

We have $\overline{x} = \bigvee_n z_n$ where $z_n = z_n \bigvee_k x_{n+k}$. It suffices to prove that for each n, $\varpi_m z_m$ is in the closure of U_n, and for that it is enough to show that $z_m \in \mathrm{cl}(U_n)$ for each $m \geq n$.

Let n be arbitrary and let $m \geq n$. As for each k we have $x_{m+k} \vee U_{m+k+1} \subset U_{m+k}$, it follows (by induction on k) that $x_m \vee x_{m+1} \vee \ldots \vee x_{m+k} \in U_m \subset U_n$. Hence $z_m \in \mathrm{cl}(U_n)$.

Now we get a contradiction by showing that $\overline{x} \vee \overline{y} \notin U$. We have $\overline{x} \vee \overline{y} = \overline{\lim}(x_n \vee y_n) = \bigvee_n z_n$ where $z_n = \bigvee_{k \geq n}(x_k \vee y_k)$. As U is a downward closed open set and $x_k \vee y_k \notin U$ for each k, we have $z_n \notin U$ for each n and therefore $\bigvee_n z_n \notin U$.

We now prove that (i) implies (ii):

Lemma 3.12. Let B be a complete ccc Boolean algebra such that (B, τ_s) is a Hausdroff space. Then there exists a sequence $\{U_n\}_n$ in $\mathcal{N}_0$ such that $\bigcap_n \mathrm{cl}(U_n) = \{0\}$.

Proof. For any given $b \in B^+$ we shall find a sequence $\{V_n\}_n$ in $\mathcal{N}_0^{\mathrm{d}}$ such that $c_b = b - \omega(\bigcap_n \mathrm{cl}(V_n)) \neq 0$. Then the set of all such c_b is algebraically dense and therefore there exists a partition $\{c_k\}_k$ of 1 and sequences $\left\{V_n^k\right\}_n$ with $\bigvee\left(\bigcap_n \mathrm{cl}\left(V_n^k\right)\right) \wedge c_k = 0$. Now if we let $U_n = V_n^0 \bigcap V_n^1 \bigcap \ldots \bigcap V_n^n$ for each n, we get a sequence with the desired properties.

Thus let b $\neq$ 0. We construct the sequence $\{V_n\}_n$. For every set $S \subseteq B$ let $S^{(n)}$ denote the a sequence in S.

As the space is Hausdroff, there exists a $V_0 \in N_0^{\mathrm{d}}$ such that $b \notin V_0$. By Lemma 3.7 and Corollary 3.7 there exists for each n some $V_{n+1} \in N_0^{\mathrm{d}}$ such that $\mathrm{cl}(V_{n+1})\omega\mathrm{cl}(V_{n+1}) \subseteq V_n$, and $V_{n+1}^{(3)} \subseteq V_n^{(2)}$. Let $X = \cap_n \mathrm{cl}(V_n)$ and $a = \omega X$.

In order to prove that $b - a \neq 0$, it suffices to show that $a \in V_0$, because that set is downward closed and b is outside it. By ccc, $a = \lim_n a_n$ where $a_n \in X^{(n)}$ for each n. We claim that $X^{(n)} \subseteq V_2$ for each n. Then $a \in \mathrm{cl}(V_2) \subseteq V_2^{(4)} \subseteq V_1^{(3)} \subseteq V_0^{(2)}$.

The claim is proved as follows (we may assume that n is even) :

$$x^{(n)} \subseteq \left(\mathrm{cl}(V_{n+1})\right)^{(n)} \subseteq V_n^{(n)} \subseteq \ldots \subseteq V_2^{(2)}.$$

This completes the proof of the equivalence of properties (i)-(iv). We make the following remark:

Corollary 3.9. Let B be a complete Boolean algebra such that (B, τ_s) is a regular space. Then the Boolean operations $\wedge$, - and Δ are continuous, and $(B, \Delta, 0, \tau_s)$ is a topological group. Moreover, (B, τ_s) is a completely regular space.

Proof. As (B, τ_s) is Fréchet, 0 has a neighborhood base $\mathcal{N}_0^{\mathrm{d}}$. Because V is continuous at (0, 0), Δ is also continuous at (0, 0). From that it easily follows that Δ is continuous (at every $(u, v) \in B \times B$) and that $(B, \Delta, 0, \tau_s)$ is a topological group. Consequently, $\vee$ and $\wedge$ are also continuous everywhere. Finally, every regular topological group is completely regular (cf. 8).

We now prove (vi), assuming that (B, τ_s) is regular.

Lemma 3.13. (a) There exists a sequence $\{u_n\}_n$ of elements of $\mathcal{N}_0^{\mathrm{d}}$ such that $\mathrm{cl}(U_{n+1}) \subset U_{n+1} \vee U_{n+1} \subset U_n$ for every n and such that $\cap_n U_n = \{0\}$.
(b) Moreover, $\{U_n\}_n$ is a neighborhood base of 0.

Proof. (a) By continuity of $\vee$ there is a sequence $\{U_n\}_n$ in $\mathcal{N}_0^d$ such that $U_{n+1} \subset U_n$ for every n. By (ii) we may assume that $\cap_n U_n = \{0\}$.

(b) We prove that the U_n form a neighborhood base. Assume not. Then there exists a $V \in \mathcal{N}_0$ such that $U_n \not\subseteq V$ for every n. For each n let x_n be such that $x_n \in U_n - V$.

It follows by induction on k that $x_{n+1} \vee x_{n+2} \vee \ldots \vee x_{n+k} \in U_n$ for each n and each k. Thus $\omega_k\ x_{n+k} \in \mathrm{cl}(U_n)$ and it follows that $\overline{\lim} x_n \in U_m$ for each m; hence $\overline{\lim} x_n = 0$ and so $\lim x_n = 0$. This is a contradiction because V is a neighborhood of 0.

We are now ready to prove (vi). Let $\{U_n\}_n$ be a neighborhood base of 0 as in Lemma 4.10, with $U_0 = B$. Let $\mathbf{D}$ be the set of all rational numbers of the form $r = \sum_{i=1}^{k} 2^{-n_i}$ where $\{n_1, \ldots, n_k\}$ is a finite increasing sequence of positive integers. For each $r \in \mathbf{D}$ as above, let $V_r = U_{n_1} \vee \ldots \vee U_{n_k}$, and let $V_1 = U_0 = B$. For each $a \in B$, we define

$$\mu(a) = \inf\{r \in \mathbf{D} \cup \{1\} : a \in V_r\}.$$

Lemma 3.14. The function μ is a strictly positive Maharam submeasure.

Proof. We repeatedly use the following fact that follows by induction on k : For every increasing sequence $\{n_1, ..., n_k\}$ of nonnegative integers, $U_{n_1+1} \vee ... \vee U_{n_k+1} \subseteq U_{n_1}$.

First, if $a \leq b$ then $\mu(a) \leq \mu(b)$; this is because for all $r \leq s \in \mathbf{D}$, if $r \leq s$ then $V_r \subseteq V_s$.

Second, $\mu(a \vee b) \leq \mu(a) + \mu(b)$ for all a and b; this is because $V_r \vee V_s \subseteq V_{r+s}$ for all r and s such that $r + s < 1$.

Third, the submeasure μ is strictly positive: if $a \neq 0$ then there exists a positive integer n such that $a \notin U_n = V_{1/2^n}$, and so $\mu(a) \geq 1/2^n$.

Next we show that μ is continuous: if $\{a_n\}_n$ is a descending sequence converging in B to 0 then for every k eventually all a_n are in U_k, hence $\mu(a_n) < 1/2^k$ for eventually all n, and so $\lim_n \mu(a_n) = 0$.

Finally, the topology induced by the submeasure μ coincides with τ_s: this is because $U_n \subseteq \{a \in B : \mu(a) \leq 1/2^n\} \subseteq \bigcap_{k>n}(U_n \vee U_k) = \mathrm{cl}(U_n) \subseteq U_{n-1}$ for each $n > 0$.

3.5. SEQUENTIAL CARDINALS

We now turn our attention to the atomic Boolean algebra $P(\kappa)$ where κ is an infinite cardinal. We compare two topologies on $P(\kappa)$: the product topology τ_c (when $P(\kappa)$ is identified with the product space $\{0,1\}^{\kappa}$) and the sequential topology τ_s.

If f is a real-valued function on B we say that f is sequentially continuous if it is continuous in the sequential topology τ_s on B. Equivalently, $f(a_n)$ converges to $f(a)$ whenever a_n converges algebraically to a.

As τ_s is stronger than τ_c, every real-valued function on $P(\kappa)$ that is continuous in the product topology is sequentially continuous. Following [1] we say that κ is a sequential cardinal if there exists a discontinuous real-valued function that is sequentially continuous.

A submeasure μ on $P(\kappa)$ is nontrivial if $\mu(\kappa) > 0$ and $\mu(\{\alpha\}) = 0$ for every $\alpha \in \kappa$. If μ is a Maharam submeasure on $P(\kappa)$ then it is a sequentially continuous function. If μ is nontrivial then it is discontinuous in the product topology, because it takes the value 0 on the dense set $[\kappa]^{\aleph_0}$. Thus if $P(\kappa)$ carries a nontrivial Maharam submeasure then κ is a sequential cardinal. In particular, the least real-valued measurable cardinal is sequential. Keisler and Tarski asked in [10] whether the least sequential cardinal is real-valued measurable.

It follows from Theorem 3.6 below that if the Control Measure Problem has a positive answer then so does the Keisler-Tarski question.

We use the following theorem of G. Plebanek ([14], Theorem 3.5). A σ-complete Boolean algebra B carries a Mazur functional if there exists a sequentially continuous real-valued function f on B such that $f(0) = 0$ and $f(b) > 0$ for all $b \neq 0$.

Theorem 3.5 (Plebanek [14]). If κ is a sequential cardinal then there exists a σ-complete proper ideal H on $P(\kappa)$ containing all singletons and such that the algebra $P(\kappa)/H$ carries a Mazur functional.

Theorem 3.6. An infinite cardinal is sequential if and only if the algebra P(κ) carries a nontrivial Maharam submeasure.

Proof. Let κ, be a sequential cardinal. By Theorem 3.5 the σ-complete algebra $B = P(\kappa)/H$ carries a Mazur functional f. First we claim that B satisfies the countable chain condition, and hence is a complete algebra. If not, there is an uncountable antichain, and it follows that there is some $\varepsilon > 0$ and there are infinitely many pairwise disjoint elements a_n, $n = 0, 1, 2, \ldots$, such that $|f(a_n)| \geq \varepsilon$ for all n. This contradicts the sequential continuity of f as $\lim_n a_n = 0$.

For each n, let U_n be the set of all $a \in B$ such that $|f(a)| < 1/n$. The U_n are neighborhoods of 0 and satisfy property (ii) of Theorem 3.4.

By Theorem 3.4, B carries a strictly positive Maharam submeasure. This submeasure induces a strictly positive Maharam submeasure on $P(\kappa)$ that vanishes H and therefore on singletons. Thus $P(\kappa)$ carries a nontrivial Maharam submeasure.

REFERENCES

[1] Antonovskii, M. and Chudnovsky, D. (1976): Some questions of general topology and Tikhonov semifields II, Russian Math. Surveys 31, 69-128.

[2] Blass, A. and Jech, T. (1986): On the Egoroff property of pointwise convergent sequences of functions, Proc. Amer. Math. Soc. 98, 524-526.

[3] Engelking, R. (1985): General Topology, 2nd ed., PWN, Warszawa.

[4] Fremlin, D. H. (1984): Consequences of Martin's Axiom, Cambridge Univ. Press.

[5] _______ (1989): Measure algebras, in: Handbook of Boolean Algebras, Vol. 3, J. D. Monk and R. Bonnet (eds.), North-Holland, Amsterdam, 877-980.

[6] _______ (1993): Real-valued measurable cardinals, in: Set Theory of the Reals, H. Judah (ed.), Amer. Math. Soc., 151-304.

[7] Glówczyński, W. (1991): Measures on Boolean algebras, Proc. Amer. Math. Soc. 111, 845-849.

[8] Hewitt, E. and Ross, K. (1963): Abstract Harmonic Analysis, Springer.

[9] Jech, T. (1978): Set Theory, Academic Press.

[10] Keisler, H. J. and Tarski, A. (1964): From accessible to inaccessible cardinals, Fund. Math. 53, 225-308.

[11] Kelley, J. L. (1959): Measures on Boolean algebras, Pacific J. Math. 9, 11651177.

[12] Koppelberg, S. (1989): General Theory of Boolean Algebras, Vol. 1 of Handbook of Boolean Algebras, J. D. Monk and R. Bonnet (eds.), North-Holland, Amsterdam.

[13] Maharam, D. (1947): An algebraic characterization of measure algebras, Ann. of Math. 48, 154-167.

[14] Namba, K. (1972): Independence proof of (ω, ω_1)-WDL from (ω, ω)-WDL, Comment. Math. Univ. St. Paul. 21 (2), 47-53.

[15] Plebanek, G. (1993): Remarks on measurable Boolean algebras and sequential cardinals, Fund. Math. 143, 11-22.

[16] Prikry, K. (1971): On σ-complete prime ideals in Boolean algebras, Colloq. Math. 22, 209-214.

[17] Rothberger, F. (1944): On families of real functions with a denumerable base, Ann. of Math. 45, 397-406.

[18] Todorčeviĉ, S. (1994): Some partitions of three-dimensional combinatorial cubes, J. Combin. Theory Ser. A 68, 410-437.

*****

CHAPTER IV

COMPACT TOPOLOGICAL ORTHOMODULAR LATTICES

4.0. INTRODUCTION

A lattice is order-topological iff its order convergence is topological and makes the lattice operations continuous. We show that the following properties are equivalent for any complete orthomodular lattice *L*: (i) *L* is order-topological, (ii) *L* is continuous (in the sense of Scott), (iii) *L* is algebraic, (iv) *L* is compactly atomistic, (v) *L* is a totally order-disconnected topological lattice in the order topology.

A special class of complete order-topological orthomodular lattices, namely the compact topological orthomodular lattices, are characterized by various algebraic conditions, for example, by the existence of a join-dense subset of so-called hypercompact elements.

In this chapter we continue the investigation of order convergence and related topologies on orthomodular lattices (OMLs). Our results explain why atomic OMLs behave much better than arbitrary ones, not only from the algebraic, but also from the topological point of view: the complete atomic and meet-continuous OMLs are just the order-topological ones, i.e., those complete OMLs which have topological order convergence and form a topological lattice with respect to the order topology; in this situation, the orthocomplementation automatically becomes continuous, being a dual automorphism. Moreover, we shall find that these OMLs are precisely the algebraic (= compactly generated complete) ones, and that it even suffices to postulate continuity in the sense of Scott [15] in order to ensure atomicity. This will be achieved by applying earlier results of the Erné [6] to the blocks, i.e., to the maximal Boolean subalgebras of the given OML (see, e.g., [17, Ch. 1, §4]). Among other characterizations, we shall find that a complete OML is order-topological iff it is a totally separated (and, moreover, a totally order-disconnected) topological lattice in its order topology.

Important examples of order-topological complete OMLs are supplied by compact topological OMLs (see Choe and Greechie [3], Pulmannova and Riecanova [22-23]). However, our theory essentially extends the compact case, because there are interesting order-topological complete OMLs which fail to be compact (cf. Example 4.4.3). In contrast to this fact, an order-topological complete Boolean algebra is always compact (see [6, Corollary 19]). Several purely algebraic characterizations of compact (order-) topological OMLs will be given in Section 4.3.

The noncomplete case is not yet settled entirely, but we are able to establish some necessary and sufficient conditions for an OML to have a MacNeille completion which is an order-topological (hence atomic) OML.

4.1. TOPOLOGICAL AND ALGEBRAIC PROPERTIES OF LATTICES

If a subset X of a lattice or poset L has a join (least upper bound), this will be denoted by $\omega\ X$; dually, $\varpi\ X$ denotes the meet (greatest lower bound) of X. In accordance with [6, 13, 14, 19], we define order convergence of filters on L as follows. For any filter (base) ⟩ on L, put

$$\rangle^{\uparrow} = \bigcup\{Y^{\uparrow} : Y \in \rangle\} \text{ and } \rangle\downarrow = \bigcup\{Y\downarrow : Y \in \rangle\}$$

where $Y^{\uparrow}$ denotes the set of all upper bounds and $Y\downarrow$ denotes the set of all lower bounds for Y. Then we say ⟩ order converges or *o*-converges to x if $x = \omega\rangle\downarrow = \varpi\ Y^{\uparrow}$. The order topology τ_o is the finest topology τ on L such that o-convergence implies τ-convergence. The following explicit description of the order topology can be found in [8, Theorem 0.5] or in [14, Theorem 4.6].

Lemma 4.1. A subset U of a lattice L is open in τ_o iff for each directed subset Y of L and for each filtered (i.e., down-directed) subset Z of L with $\omega.Y = \varpi Z \in U$, there exist elements $y \in Y$ and $z \in Z$ such that the whole interval $[y, z] =\{x \in L: y \leq x \leq z\}$ is contained in U.

From this characterization it is evident that τ_o is always finer than the interval topology τ_i, the coarsest topology such that every principal filter $[y)$, every principal ideal $(z]$ and, consequently, every interval $[y, z]$ is closed. The following facts on the interval topology are well known (see, for example, [14, 2.4-2.6]):

Lemma 4.2. (1) An ultrafilter on a lattice o-converges to a point if this point is the unique τ_i-limit of the ultrafilter. Hence, if the interval topology is T_2 then it coincides with the order topology.

(2) The interval topology of a lattice *L* is compact iff *L* is complete.

Now let us turn to some important algebraic properties of lattices. We emphasize that no a priori completeness assumption is made in the subsequent general considerations. A lattice *L* is called meet-continuous (cf. [2, 8]) or upper continuous (cf. [5]) if every ideal *Y* of *L* possessing a join enjoys the distribution law

$$x \wedge \omega\, Y = \omega\, \{x \wedge y : y \in Y\} \ (x \in L).$$

Join-continuous lattices are defined dually, and if both continuity properties are satisfied, we shall speak of *o*-continuous lattices, because they may be characterized by continuity of the lattice operations with respect to *o*-convergence (see [6, Proposition 5] or [8, Theorem 2.14]); furthermore, *o*-continuity is also equivalent to continuity of the unary operations $\vee_y : x \propto x \vee y$ and $\wedge_y : x \propto x \wedge y$ with respect to the order topology. However, τ_o-continuity of the binary operations $\vee$ and $\wedge$ is stronger than *o*-continuity (see below). We also note that a lattice is meet-continuous iff for each ideal *J* possessing a join and for each element $x \leq \omega\, J$, there is an ideal $I \subseteq J$ with $x = \omega\, I$.

A stronger type of "continuity", due to Scott, is defined as follows (cf. [6, 15]). A lattice *L* is called *S*-continuous (in [10]:

s_3-continuous) if each element x of L is the join of its way-below ideal, that is, the intersection of all ideals possessing a join above x. By the remark at the end of the previous paragraph, every *S*-continuous lattice is meet-continuous. Elements which belong to their own way-below ideal are called compact. Equivalently, an element y of an arbitrary lattice L is compact iff for each subset X of L possessing a join with $y \leq \omega X$, there is some finite subset F of X with $y \leq \omega F$. The complete *S*-continuous lattices are just the continuous lattices in the sense of Scott, but we avoid the latter terminology in order to prevent confusion with von Neumann's definition of continuous lattices, i.e., complete *o*-continuous lattices.

A subset Y of a lattice L is join-dense if for any two elements $x, z \in L$ with $x \not\leq z$, there is some $y \in Y$ with $y \leq x$ but $y \Leftarrow z$. Thus Y is join-dense in L if each element of L is a join of elements from Y. Meet-density is defined dually. A lattice L is said to be compactly generated if the set of all compact elements is join-dense in L. Similarly, L is atomistic if the set of all atoms is join-dense in L, and L is compactly atomistic if the set of all compact atoms is join-dense in L (and consequently, every atom of L is compact). Thus, without any completeness assumptions, we have the implications

compactly atomistic $\Rightarrow$ *compactly generated* $\Rightarrow$ *S-continuous* $\Rightarrow$ *meet-continuous*

but none of these implications can be inverted in general. However, for atomistic lattices all four properties are equivalent, because atoms of meet-continuous lattices are compact.

A topological lattice is a lattice endowed with a topology making the binary-operations of join and meet continuous, while in a semitopological lattice, only continuity of the unary operations $\vee_y$ and $\wedge_y$, is required. Thus a lattice *L* is *o*-continuous iff (L, τ_o) is a semitopological (but not necessarily a topological) lattice. For example, the complete Boolean lattice of all regular open subsets of the real line fails to be a topological lattice in its order topology, though being *o*-continuous (see [13, Corollary 4.11]).

By an order-topological lattice we mean a topological lattice whose topological filter convergence agrees with order convergence (here we get the Hausdorff separation property for free because order limits are unique). Thus a lattice is order-topological iff it is *o*-continuous and has topological order convergence. From [6, Theorem 2], we cite:

Theorem 4.1. A lattice *L* is order-topological iff *L* and its dual are *S*-continuous.

For later purposes, we add some statements concerning the relativization of order convergence and order topology; these facts belong to the folklore of topological lattice theory. The proof is straightforward.

Lemma 4.3. A sublattice *S* of a complete lattice *L* is closed in the order topology and contains the universal bounds 0, 1 iff *S* is subcomplete, i.e., closed in *L* under the formation of arbitrary joins and meets. In this case, the following relativization properties hold:

(1) Order convergence on S is inherited from order convergence on L.

(2) The order topology on S is induced from the order topology on L.

(3) If order convergence is topological on L then also on S.

(4) If L is order-topological then so is S.

Remarks. At several places in Birkhoff's Lattice Theory [2], the term "topological lattice" means "*o*-continuous lattice". Since in that monograph, no distinction is made between order convergence and convergence in the order topology, some statements like "Any complete Boolean lattice is a topological lattice under order convergence" are a bit misleading. It is a rather surprising matter of fact that in all nonatomic Boolean algebras, order convergence differs from convergence in the order topology (see [6, Theorem 4]).

Second, it should be noted that even a complete orthomodular lattice with topological order convergence need not be *o*-continuous (see Example 4.1). However, any lattice with topological order convergence is "locally *o*-continuous" in the following sense: if I and J are ideals with $x = \omega I = \omega J$ then also $x = \omega(I \wedge J) = \omega(I \cap J)$, and dually (see [14, Corollary 4.2]).

Third, a few additional comments concerning net-theoretical convergence are in order. If a net order converges (in the sense of Birkhoff [2, Ch. X]) to some point in a lattice L then so does the associated filter, but not conversely, unless L is complete. For example, in the atomic (hence order-topological) Boolean algebra of all finite or cofinite subsets of a fixed uncountable set, no sequence of atoms singletons) order converges to 0, although the corresponding filter does and, consequently, the sequence converges to 0 in the order topology (cf. Example 4.2). This example also disproves the claim in [2, p. 244] that order convergence of nets in an arbitrary lattice or poset would be obtained by relativization from the MacNeille completion.

In the setting of complete lattices, however, the passage between nets and filters is unproblematic. For example, in a statement like "order convergence is topological (hence identical with convergence in the order topology)", it does not matter whether we refer to filters or to nets, provided we are dealing with complete lattices. Moreover, a complete lattice is order-topological in the language of filters iff it is order-topological in the language of nets. But, unfortunately, this holds no longer for noncomplete lattices, as the above example shows.

Finally, we should mention one positive observation concerning filters and nets on arbitrary (possibly noncomplete) lattices: the order topology τ_o is not only the finest topology such that order convergence of filters implies τ-convergence, but also the finest topology τ such that order convergence of nets implies τ-convergence (see [8, Theorem 0.4]).

4.2. ORDER CONVERGENCE AND TOPOLOGIES ON ORTHOMODULAR LATTICES

After these rather general preliminaries, let us focus on more specific types of lattices. Recall that an ortholattice is a lattice together with a self-inverse dual automorphism $^\perp$ such that $x^\perp$ is always a complement of x (for this, it suffices to postulate $x^\perp \Leftarrow x$ unless x is the greatest element). Clearly, for ortholattices, join.- meet- and o-continuity are equivalent properties. By Theorem 4.1, an ortholattice is S-continuous iff it is order-topological.

If Y is a join-dense subset of an ortholattice L then

$$Y^\perp = \{y^\perp : y \in Y\}$$

is a meet-dense subset of L. We define a topology τ_Y on L by declaring the principal filters

$$[y) = \{x \in L : x \geq y\}$$

and the principal ideals

$$(y^\perp] = \{x \in L : x \leq y^\perp\}$$

with $y \in Y$ as subbasic open sets for τ_Y. Notice that τ_Y is the weakest topology on L making the sets $[y)$ for $y \in Y$ open and the dual automorphism $^\perp$ continuous.

An ordered topological space is totally order-disconnected iff for any two points x, y with $x \Leftarrow y$, there exists a clopen upper set U containing x but not y (where U is an upper set iff $u \in U$ implies $[u) \subseteq U$). Clearly, any such space is totally disconnected and T_2.

Our first result on ortholattices actually holds for arbitrary lattices possessing a dual automorphism (see [12, Proposition 3.3 and Theorem 3.4]).

Proposition 4.1. The following conditions on a subset Y of an ortholattice L are equivalent:

(a) Y is join-dense and consists of compact elements.

(b) The space (L, τ_Y) is totally order-disconnected, and τ_Y is coarser than τ_o.

(c) τ_Y-convergence agrees with order convergence.

(d) τ_Y is the order topology.

Henceforth L denotes an orthomodular lattice (OML), that is, an ortholattice satisfying the orthomodular law

$$x \geq y \Rightarrow x = y \vee (x \wedge y^{\perp}),$$

and A denotes the set of all atoms of L. (In [24-26] and related papers, the corresponding topology τ_A is denoted by τ_Ψ). We are now prepared for the main result:

Theorem 4.2. The following statements on a complete orthomodular lattice L are equivalent:

(a) L is meet-continuous and atomic.

(b) L is compactly atomistic.

(c) *L* is compactly generated.

(d) *L* is S-continuous.

(e) *L* is order-topological.

(f) *L* is meet-continuous, and each block of *L* is atomic, hence a power set lattice.

(g) τ_A-convergence agrees with order convergence.

(h) τ_A is the order topology.

Proof. (a) implies (b) since every atomic OML is already atomistic (cf. [17, p. 140]).

The implications (b) ⇒ (c) ⇒ (d) are clear. For (d) ⇔ (e), see Theorem 4.1.

The equivalence of (b), (g) and (h) has been established in Proposition 4.1.

Condition (f) implies (a) since a is an atom in L if a is an atom in some block of L (see [17, p. 39]). Hence it remains to prove the implication (e) ⇒ (f).

As remarked earlier, order-topological lattices are meet-continuous. We use the fact that any block of a complete OML is subcomplete (see [17, pp. 28 and 39]). Hence, by Lemma 4.3, each block B is an order-topological Boolean lattice, and by [6, Theorem 4], this forces B to be atomic, hence isomorphic to the power set of its atoms.

Notice that the completeness assumption has been used at one step only, namely in order to guarantee that the order convergence (and consequently, the order topology) of each block is obtained by relativization. We do not know whether this is true for arbitrary OMLs, though even in noncomplete OMLs, blocks are closed under all existing joins and meets.

However, it is possible to establish several natural and purely algebraic criteria for a lattice to have a MacNeille completion which is an order-topological OML. Basic for this purpose is the study of completion-invariant properties, i.e., properties that are satisfied by a lattice or poset iff they hold for its MacNeille completion (cf. [7, 11]). Unfortunately, the orthomodular law is not completion-invariant (see [1]). But, combining it with certain other algebraic postulates, we arrive at the desired completion-invariant notions. Thus we strengthen the continuity properties and call a lattice L strongly meet-continuous if for each ideal J and each $x \in J_{\downarrow}^{\uparrow}$ there is an ideal $I \subseteq J$ with $x = \omega I$. The strong way-below ideal of an $x \in L$ is the intersection of all ideals J with $x \in J_{\downarrow}^{\uparrow}$; a lattice is strongly S-continuous (s_1-continuous in [10]) if each element is the join of its strong way-below ideal. Similarly, an element is strongly compact if it belongs to its strong way-below ideal, and a lattice is strongly compactly generated if the strongly compact elements form a join-dense subset. It is not hard to verify that each of the previously defined strong properties is completion-invariant, implies the corresponding weak property, and coincides with the latter on complete lattices (for details, see [7, 9, 11]). Moreover, a lattice is strongly S-continuous iff it is both S-continuous and strongly meet-continuous. These facts lead to the announced extension of Theorem 4.2 to the noncomplete case.

Corollary 4.1. The following statements on an arbitrary orthomodular lattice *L* are equivalent:

(a) *L* is strongly meet-continuous and atomic.
(b) *L* is strongly compactly atomistic.
(c) *L* is strongly compactly generated.
(d) *L* is strongly *S*-continuous.
(e) *L* is strongly meet-continuous and order-topological.
(f) The MacNeille completion of *L* is an order-topological (and orthomodular) lattice.

For the implication (b) ⇔ (f), see [25, Theorem 3.4] and [27, Theorem 3.3].

By a famous theorem due to Kaplansky [18] (see also [17, §17]), every complete modular ortholattice is an *o*-continuous OML. Thus, in this case, Theorem 4.2 amounts to:

Corollary 4.2. For a complete modular ortholattice *L*, the following conditions are equivalent:

(a) The greatest element of *L* is a join of atoms.
(b) *L* is atomi(sti)c.
(c) *L* is compactly generated.
(d) *L* is *S*-continuous.
(e) *L* is order-topological.
(f) *L* is geometric.

For the equivalence of (a), (b) and (c), see [5, 4.3]. The term "geometric" is here used in the sense of Grätzer [16, IV-3], namely for compactly atomistic, semimodular complete lattices, while Birkhoff refers to these as “atomic matroid lattices” and reserves the term “geometric lattices” for semimodular atomistic lattices of finite height (see [2, Ch. IV and VIII] and [5, Ch. 14]). Notice also that in [2], “atomic” means “atomistic”.

We know from Proposition 4.1 and Theorem 4.2 that any compactly atomistic complete OML is a totally order-disconnected topological lattice in its order topology. Now we are going to prove a strong converse of that implication. Recall that in an arbitrary topological space, the quasicomponent of a point *x* is the intersection of all clopen sets containing *x*, and that the space is called totally separated if every quasicomponent is a singleton; in other words, if any two distinct points may be separated by a clopen set. Clearly, total order-disconnectedness implies total separatedness, which in turn implies total disconnectedness; and it is well known that for compact topological lattices, all three properties are equivalent to zero-dimensionality (+ T_1). There is a close connection between such strong topological separation properties and the algebraic properties of atomicity and compact generation (see [12] for a thorough investigation of similar phenomena in general lattices).

Proposition 4.2. If an element *x* of a complete orthomodular lattice *L* has a τ_o-clopen neighborhood disjoint from 0 then (*x*] contains an atom; the converse is true if *L* is o-continuous. Hence, in an *o*-continuous complete orthomodular lattice, the quasi-component of 0 with respect to the order topology is the union of all atomless principal ideals.

Proof. Suppose $x \in U$ for some τ_o-clopen set $U \subseteq L\backslash\{0)$. By Lemma 1.1, U is closed under filtered meets, and by Zorn's lemma, it has a minimal element u. Let B be a block containing u, yet assume that $(u] \cap B$ would not contain any atom of B. Then we could select a sequence (x_n) of pairwise orthogonal nonzero elements in $(u]\cap B$ as follows. Choose $x_0 \in B$ with $0 < x_0 < u$, and suppose there have been found $x_0, \ldots x_n \in B$ with $0 < x_n < x_0^{\perp} \wedge \ldots \wedge x_{n-1}^{\perp} \wedge u$; then $0 < y_n = x_0^{\perp} \wedge \ldots \wedge x_n^{\perp} \wedge u$ by the orthomodular law. As y_n belongs to the subalgebra B but is not an atom, we find an element $x_{n+1} \in (u] \cap B$ with $0 < x_{n+1} < y_n$. Obviously, the elements x_n obtained in this way are pairwise orthogonal. But any sequence of pairwise orthogonal elements in a complete Boolean algebra order converges to 0. Hence, by completeness and o-continuity of B, the sequence $(x_n^{\perp} \wedge u)$ order converges to $u \in U \cap B$. But since $U \cap B$ is open in the order topology of B (the latter being induced by the order topology of L; cf. Lemma 1.4), some $x_0^{\perp} \wedge u$ must be a member of $U \cap B$, contradicting the hypothesis that u is minimal in $U \cap B$ and $x_n^{\perp} \wedge u < u$. Hence $(u] \cap B$ must contain an atom of B which is then also an atom of L.

Conversely, if L is o-continuous and a is an atom in $(x]$ then a is compact, and consequently $[a)$ is a τ_o-clopen neighborhood of x disjoint from 0.

The second part of the proposition is merely a reformulation of the first.

Corollary 4.3. For any complete orthomodular lattice *L*, the conditions in Theorem 4.2 are equivalent to each of the following statements:

(i) (L, τ_o) is a totally order-disconnected (semi)topological lattice.

(j) (L, τ_o) is a totally separated (semi)topological lattice.

(k) (L, τ_o) is a (semi)topological lattice whose least element constitutes a quasi-component.

Proof. By Proposition 4.1, the conditions (e) and (h) of Theorem 4.2 together imply (i): the implications (i) (j) (k) are trivial, and (k) implies (a) of Theorem 4.2, by Proposition 4.2. Recall that *o*-continuity of *L* is equivalent to saying that (L, τ_o) is a semitopological lattice.

4.3. COMPACT TOPOLOGICAL ORTHOMODULAR LATTICES

Various results on compact topological OMLs (see, e.g., [3, 22]) are easy consequences of Proposition 4.1 and Theorem 4.2, once one has checked that such lattices must be complete and order-topological. A convenient way to the latter conclusion is to prove atomicity by passing to the blocks and using known facts on Boolean algebras (see [3, Corollary 2]); then [25, Theorem 2.3] applies. A slightly different (and less elementary) approach would be to argue that any compact topological OML must be totally disconnected (see [3, Lemma 3]) and then to invoke the Fundamental Theorem on Compact Totally Disconnected Semilattices (see [15, VI-3.13]) which guarantees that the OML in question is algebraic, hence order-topological, by Theorem 4.2. However, the known proofs of total disconnectedness involve, as a first step, the verification of atomicity. Choe and Greechie have shown that even every locally compact and locally convex topological OML is totally disconnected (see [4]). But, in contrast to the compact case. the topology of such a topological OML is not uniquely determined in general.

Frequently, topological orthomodular lattices are assumed to satisfy the Hausdorff separation axiom (see, e.g. [22, 27]). We shall see below that this is a rather mild restriction: indeed, it will suffice to postulate closedness of one single element. A similar phenomenon is well known from group theory: a topological group having one closed element is already Hausdorff. Since Boolean algebras may be regarded as Boolean rings, it is not surprising that the same situation holds for topological Boolean algebras (see [13, Corollary 4.9]). The orthomodular case, however, is not so obvious.

In what follows, we mean by a topological orthomodular lattice an OML equipped with a topology (not necessarily T_2) making the orthocomplementation and the lattice operations continuous. It suffices to postulate continuity of one lattice operation, because either of them can be expressed in terms of the other and orthocomplementation.

Proposition 4.3. If (L, τ) is a topological orthomodular lattice with at least one closed singleton then its topology τ is T_2 and finer than T_A. Moreover, for $x \not\leq y$ there exists an open upper (= increasing) set U containing x and a disjoint open lower (= decreasing) set V containing y.

Proof. If a singleton $\{b\}$ is closed then so are the sets $\{b^\perp\}$, $(b]$, $(b^\perp]$, and $(b]\cap(b^\perp] = \{0\}$, by continuity of the operations of join and complementation. If a is an atom then $[a)$ and $(a^\perp] = [a)^\perp$ are τ-open, whence $\tau_A \subseteq \tau$ (cf. [3, Lemma 4]).

Now assume $x \Leftarrow y$. Then we have $z = x \wedge y < x$, and by the orthomodular law, $x \wedge z^\perp \neq 0$. Hence $L\backslash\{0\}$ is an open neighborhood of $x \wedge z^\perp$, and we find open sets T, W such that $x \in T$, $z^\perp \in W$, and $T \wedge W \subseteq L\backslash\{0\}$. Moreover, we may assume that T and W are upper sets, since $\uparrow T$ and $\uparrow W$ are again open, and $0 \notin Z \uparrow T \wedge \uparrow W$. Then V is an open lower set containing y (indeed, $x \wedge y = z \in W^\perp$, and $u \leq v \in V$ implies $x \wedge u \leq x\ v \in W^\perp$, hence $x \wedge u \in W^\perp$ and $u \in V$). Finally, using continuity of the binary meet once more, we may choose an open upper set U with $x \in U$ and $U \wedge U \subseteq T$. Then U does not meet V, because otherwise, $v \in U \cap V$ would entail $x \wedge v \in T \cap W^\perp$, which leads to the contradiction $0 = x \wedge v \wedge (x \wedge v)^\perp \in T \wedge W \subseteq L\backslash\{0\}$.

A related result was obtained by Choe and Greechie in [3, Lemma 1]: If a topological orthomodular lattice has at least one isolated point then it is discrete.

We call an element y of a lattice hypercompact (cf. [12]) if the complement of $[y)$ is a finite union of principal ideals. Of course, this entails (strong) compactness of y. A lattice is hypercompactly generated if its hypercompact elements form a join-dense subset.

Lemma 4.4. In an atomic OML, an atom a is hypercompact iff the set $N_a = \{b \in A : a \Leftarrow b^\perp\}$ is finite.

Proof. If this is the case then the equation $L \backslash [a) = \bigcup\{(b^\perp]: b \in N_a\}$ yields hypercompactness of a. Conversely, if $L \backslash [a) = \bigcup\{(c]: c \in F\}$ for some finite set $F \subseteq L$ then we may assume that each $c \in F$ is a coatom, by coatomicity of L; but then each atom b with $a \Leftarrow b^\perp$ must coincide with some $c^\perp$.

In [24], an atomic OML has been called almost orthogonal if each atom has the above property equivalent to hypercompactness.

The next theorem provides various characterizations of compact T_2-topological OMLs, including some of the results from [3, 22-27].

Theorem 4.3. The following statements on an orthomodular lattice L are equivalent:

(a) L is complete and hypercompactly generated.

(b) L is complete, atomic and almost orthogonal.

(c) L is complete, meet-continuous, and the interval topology is T_2.

(d) L is complete, and $\tau_i = \tau_c = \tau_A$.

(e) τ_A is a compact topology such that all intervals are closed.

(f) L is a compact order-topological lattice.

(g) (L, τ) is a compact topological OML for some T_1 (T_2, T_3, T_4) topology τ.

If such a topology exists then the space (L, τ) is totally order-disconnected, and τ agrees with each of the topologies τ_i, τ_o and τ_A.

Proof. The equivalence of (a) and (b) is clear by Theorem 4.2 and Lemma 4.4.

The implications (e) $\Rightarrow$ (a) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) $\Rightarrow$ (f) hold in arbitrary lattices, as shown in [12, Theorem 3.13].

(c) $\Rightarrow$ (d): By Lemma 4.2 (1), one has $\tau_i = \tau_o$. Using relativization to the blocks and known facts on Boolean lattices, one shows that a complete OML with T_2 interval topology is atomic (see [6, Theorem 4] or [28, p. 75]). Hence τ_o agrees with τ_A, by Theorem 4.2.

(f) $\Rightarrow$ (g): This is clear because orthocomplementation is always τ_o-continuous.

(g) $\Rightarrow$ (d): By Proposition 4.3, the topology τ is T_2. Now an application of [25, Theorem 2.3] or [23, Lemma 2.1] yields the equation $\tau = \tau_i = \tau_o = \tau_A$.

In all, we have established the following implication circle: (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a).

The completeness assumptions in Theorem 3.3 can be bypassed via the MacNeille completion (cf. [23, Theorem 1.1]):

Corollary 4.4. The following statements on an orthomodular lattice L are equivalent:

(a) L is hypercompactly generated.
(b) L is atomic and almost orthogonal.
(c) L is meet-continuous, atomic, and the interval topology is T_2.
(d) The MacNeille completion of L is a compact T_1 (T_2, totally disconnected) OML.

4.4. EXAMPLES

Let us conclude our study with a few instructive examples, making the position of the previous results more precise.

Example 4.1. Let L be the horizontal sum of two atomic (complete) Boolean algebras B_1 and B_2 with more than two elements; thus L is the disjoint union of B_1 and B_2 with identified greatest and least elements, respectively. Then L is an atomistic (complete) OML which is not *o*-continuous, hence not order-topological unless L is finite. Furthermore, order convergence of filters is topological on L if and only if B_1 or B_2 is finite. In fact, if, say, B_1 is finite then each $x \in B_1 \setminus \{0, 1\}$ is an isolated point in the order topology on L (see Lemma 4.1), while the points of B_2 have open neighborhood bases of intervals $[y, z]$ where y is a finite join of atoms and z is a finite meet of coatoms in B_2. As B_1 is finite, these (co)atoms are not only (co)compact in B_2 but also in L. By [14, Corollary 4.13] this ensures that order convergence of filters is topological on L. On the other hand, if both B_1 and B_2 are infinite then the filterbases consisting of all principal dual ideals generated by compact elements of B_1 and B_2, respectively, order converge to the greatest element of L, while the intersection of the corresponding filters does not. Hence order convergence cannot be topological (cf. [6, Examples 4 and 5]).

Example 4.2. Let $(L_i\colon i \in I)$ be any family of finite OMLs with discrete topologies. The direct product $L = \Pi_{i\in 1} L_i$ with componentwise lattice operations and orthocomplementation is then again a complete OML. Equipped with the product topology. L is a compact order-topological OML, and so is every subcomplete OML of L (see Lemma 4.3).

Let S denote the sub-OML of L consisting of all finite or cofinite elements, i.e., finite joins of atoms or finite meets of coatoms. If I is infinite then S is not complete but still atomic and order-topological; in particular, order convergence of filters is topological on S; but if I is uncountable, order convergence of nets is not topological on S (see [27, Example 4.1]).

Example 4.3. Let H be a real Hilbert space, and let L be the atomic complete OML of all closed linear subspaces of H. If H is finite-dimensional then L is an order-topological OML with discrete, hence noncompact order topology. On the other hand, if H is of infinite dimension then L is not even meet-continuous. In fact, the following statements are equivalent:

(a) H is finite-dimensional.

(b) For all $X, Y \in L$, $X + Y$ is closed (hence the join of X and Y in L).

(c) The atoms of L are compact.

(d) L is compactly generated.

(e) L is S-continuous.

(f) L is order-topological.

(g) L is meet-continuous.

(h) L is modular.

The implication chain (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (g) $\Rightarrow$ (c) is clear. For the equivalence of (a), (b) and (h), see [17, p. 68]. For (c) $\Rightarrow$ (b), observe that any $y \in (X \vee Y)\backslash(X + Y)$ generates a subspace that is a noncompact atom of L, since $X \vee Y$ is the join of the directed set of all finite-dimensional (hence closed) subspaces of $X + Y$.

REFERENCES

[1] Adams, D.H. (1969) : The completion by cuts of an orthocomplemented modular lattice, Bull. Austral. Math. Soc. 1, 279-280.

[2] Birkhoff, G. (1973) : Lattice Theory, American Mathematical Society Colloquium Publications 25, American Mathematical Society, Providence, RI, 3rd ed.

[3] Choe, T.H. and Greechie, R.J. (1990) : Profinite orthomodular lattices, Preprint, McMaster University, Hamilton, Ont. and Kansas State University, Manhattan, KS.

[4] Choe, T.H. and Greechie, R.J. (1990) : Representations of locally compact orthomodular lattices, Preprint, McMaster University, Hamilton, Ont., and Kansas State University, Manhattan, KS.

[5] Crawley, P. and Dilworth, R.P. (1973) : Algebraic Theory of Lattices, Prentice-Hall, Englewood Cliffs, NJ.

[6] Erné, M. (1980) : Order-topological lattices, Glasgow Math. J. 21, 57-68.

[7] Erné, M. (1980) : Verallgemeinerungen der Verbandstheorie: I Halbgeordnete Mengen und das Prinzip der Vervollständigungs-Invarianz; II m-Ideale in halbgeordneten Mengen und Hiillenraumen, Preprint 109 und Habilitationsschrift, Universität Hannover, Hannover.

[8] Erné, M. (1981) : Topologies on products of partially ordered sets III, Algebra Universalis 13, 1-23.

[9] Erné, M. (1979) : A completion-invariant extension of the concept of continuous lattices, in: B. Banaschewski and R. E. Hoffmann, eds., Continuous Lattices, Bremen, Lecture Notes in Mathematics 871 (Springer, Berlin, 1981) 45-60.

[10] Erné, M. (1979) : Scott convergence and Scott topology in partially ordered sets II, in: B. Banaschewski and R. E. Hoffmann, eds., Continuous Lattices, Bremen, Lecture Notes in Mathematics 871 (Springer, Berlin, 1981) 61-96.

[11] Erné, M. (1991) : The Dedekind-MacNeille completion as a reflector, Order 8, 159-173.

[12] Erné, M. (1992) : Lattice topologies with interval bases, Universität Hannover, Hannover.

[13] Erné M. and Gatzke, H. (1982) : Convergence and continuity in partially ordered sets and semilattices, in: R. E. Hoffmann and K.H. Hofmann, eds., Continuous Lattices and Their Applications, Bremen, Lecture Notes in Pure and Applied Mathematics 101 (Marcel Dekker, New York, 1985) 9-40.

[14] Erné M. and Weck, S. (1980) : Order convergence in lattices, Rocky Mountain J. Math. 10, 805-818.

[15] Gierz, G.; Hofmann, K.H.; Keimel, K.; Lawson, J.D.; Mislove, M. and Scott, D.S. (1980) : A Compendium of Continuous Lattices, Springer, Berlin.

[16] Grätzer, G. (1978) : General Lattice Theory, Birkhäuser, Basel.

[17] Kalmbach, G. (1983) : Orthomodular Lattices, Academic Press, London.

[18] Kaplansky, J. (1955) : Any orthocomplemented complete modular lattice is a continuous geometry, Ann. of Math. 61, 524-541.

[19] Kent, D.C. (1966) : On the order topology in a lattice, Illinois J. Math. 10, 90-96.

[20] Lawson, J.D. (1973) : Intrinsic topologies in topological lattices and semilattices, Pacific J. Math. 44, 593-602.

[21] Pulmannová, S. and Riečanovčá, Z. (1991) : Logics with separating sets of measures, Math. Slovaca 41, 167-177.

[22] Pulmannová, S. and Riečanovčá, Z. (1991) : Compact topological orthomodular lattices, in: Contributions to General Algebra 7, Hölder-Pichler-TempsIcy, Wien, 277-282.

[23] Pulmannová, S. and Riečanovčá, Z. (1992) : Modular almost orthogonal quantum logics, Internat. J. Theoret. Phys. 31, 881-888.

[24] Riečanovčá, Z. (1989) : Topologies in atomic quantum logics, Acta Univ. Carolin. Math. Phys. 30, 143-148.

[25] Riečanovčá, Z. (1991) : Applications of topological methods to the completion of atomic orthomodular lattices, Demonstratio Math. 24, 331-341.

[26] Riečanovčá, Z. (1992) : Measures and topologies on atomic quantum logics, in: Topology, Measures and Fractals, Mathematical Research 66, Akademie Verlag, Berlin, 154-161.

[27] Riečanovčá, Z. (1993) : Topological and order-topological orthomodular lattices, Bull. Austral. Math. Soc. 47, 509-518.

[28] Sarymsakov, T.A.; Ajupov, S.A.; Chadžijev, Z. and Čilin, V.J. (1983) : Uporiadočennyje Algebry, FAN, Taškent.

*****