SOME FIXED POINT RESULTS IN DISLOCATED AND DISLOCATED QUASI METRIC SPACES

THESIS

SUBMITTED TO
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BY

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UNDER THE SUPERVISION OF
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MAY 2013

Dedicated

To The Memory of

My Parents

Harikala Devi Panthi

and

Devakinandan Panthi

Student's Declaration

I hereby declare that, the research work **Some Fixed Point Results in Dislocated and Dislocated Quasi Metric Spaces** submitted here for the requirement of Doctor of Philosophy **(Ph.D.)** degree in mathematics to the Department of Natural Sciences, School of Science, Kathmandu University, in May 2013, is original work done by me and has not been published or submitted elsewhere for the award of a degree or diploma from other institution. Any work done by others and cited within this thesis has been given due acknowledgement and listed in the reference section.

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This is to certify that the thesis entitled **Some Fixed Point Results in Dislocated and Dislocated Quasi Metric Spaces** which is being submitted by Mr. Dinesh Panthi in fulfilment for the award of Doctor of Philosophy (**Ph.D.**) degree in mathematics of Kathmandu University, Nepal is a record of his own work carried out by him under my guidance and supervision.

The matter embodied in this thesis has not been submitted for the award of any degree.

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Certificate of Approval

The thesis entitled **Some Fixed Point Results in Dislocated and Dislocated Quasi Metric Spaces** that is being submitted to Kathmandu University, Dhulikhel, Kavre for the award of degree of Doctor of Philosophy (Ph.D.) in mathematics is original, and is a record of bonafide research work carried out by Mr. Dinesh Panthi in the Department of Natural Sciences, School of Science, Kathmandu University, during the last three years.

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Synopsis of Thesis

The theory of fixed point is an extensive field of research in mathematics having various applications. Fixed point theory has played a central role in the development of non-linear functional analysis and provided a power tool in demonstrating the existence and uniqueness of solutions to various mathematical models representing phenomena arising in different fields such as in Engineering, Economics , Game Theory and Nash Equilibrium, Steady State Temperature Distribution, Epidemics, Flow of Fluids, Chemical Reactions, Neutron Transport Theory, Haar Measures, Abstract Elliptic Problems, Invariant Subspace Problems, Approximation Problems, Logic Programming, Neural Networks.

In 1986, S.G. Matthwes [96] initiated the concept of dislocated metric space in the context of metric domains in which the self-distance for any point need not be equal to zero. In 2000, P. Hitzler and A. K. Seda presented modified forms of metric space including dislocated metric space and pointed important questions on topological aspects. In 2006, F. M. Zeyada, G. H. Hassan and M. A. Ahmed [162] introduced the notion of Dislocated Quasi-Metric Space. Since then, a number of fixed point theorems have been established by several authors in these spaces.

In this thesis, we have established common fixed point theorems for single pair and two pairs of mappings in dislocated metric space and two fixed point theorems in dislocated quasi- metric space which generalize and unify some well-known similar results in the literature.

Chapter wise cameo description of the present study is as follows:

CHAPTER ONE deals with the general introduction of fixed point theory. It includes some fundamental concepts and notations relevant to the development of fixed point theory. A brief survey of the development of the fixed point theory in metric space has been presented and some of the well-known theorems have been stated with a review work on some applications of fixed point theorems.

CHAPTER TWO is intended to study the fixed point theorems of asymptotic contractions. It deals with basic definitions and the chronological development of some fixed point theorems of asymptotic contractions.

CHAPTER THREE is intended to obtain some common fixed point theorems with suitable examples in dislocated metric space. It includes basic definitions and some theorems which are relevant for the establishment of our theorems.

CHAPTER FOUR is intended to establish fixed point theorems in dislocated quasi-metric space. It also includes basic definitions and some results which have relevance with our theorems.

List of publications included in the thesis

Papers in Journals

1. A Common Fixed Point Theorem in Dislocated Metric Space,
Applied Mathematical Sciences **(Bulgaria, EU)**, **6** (91)(2012), 4497-4503.
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2. A Common Fixed Point Theorem For Four Mappings in Dislocated Quasi-Metric Space, *International Journal of Mathematical Sciences and Engineering Applications* **(India)**, **6**(1) (2012), 417-424.
MR No. 2977390.

3. A Common Fixed Point of Weakly Compatible Mappings in Dislocated Metric Space, *Kathmandu University Journal of Science, Engineering and Technology,* **8** (2012), 25-30.

4. A Common Fixed Point Theorem for Two Mappings in Dislocated Metric Space, *Yeti Journal of Mathematics*, **1** (1)(2012), 30-34.

5. On Developments of Meir-Keeler Type Fixed Point Theorems, *Nepal Journal of Science and Technology*, **10**(2009), 141 - 147.

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Abstract

The notion of dislocated metric space was first time introduced by S. G. Matthews in 1986 under the name of metric domains. Dislocated metric space is one of the important extensions of metric space. In 2006, F. M. Zeyada, G. H. Hassan and M. A. Ahmed introduced dislocated quasi metric space which is another important extension of metric space.This thesis investigates some new fixed point theorems in dislocated and dislocated quasi- metric spaces which extend and unify some well-known similar results in the literature. A survey work on some fixed point theorems of asymptotic contractions in metric space has also been presented.

Contents

Chapter 1

Introduction

In this chapter, we give a brief introduction of fixed point theory, some fundamental concepts and notations relevant to the development of fixed point theory. A brief survey of the development of the fixed point theory under Meir - Keeler type contraction in metric space has been presented and some of the well-known theorems have been stated.

1.1 Introduction

Historically, the concept of fixed point was initiated by H. Poincare in 1886. The concept of metric space was introduced by M. Frechet in 1906 which furnished the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of *distance* appears. The first fixed point theorem is due to L.E.J. Brouwer in 1912. The fixed point theory has played an important role in the problems of nonlinear functional analysis which is the blend of analysis, topology and algebra.

A fixed point theorem is one which ensures the existence of a fixed point of a mapping T under suitable assumptions both on X and T. Many non-linear equations can be solved using fixed point theorems. In fact, fixed point theorems has applications in non linear integral, differential equations, game theory, optimization theory and boundary value problems etc.

Fixed Point Theory is divided into three major areas namely (i)Topological Fixed Point Theory (ii) Metric Fixed Point Theory and (iii) Discrete Fixed Point Theory. Historically, the boundary lines between the three areas was defined by the discovery of three major theorems: (i) Brouwer's Fixed Point Theorem (ii) Banach's Fixed Point Theorem and (ii) Tarski's Fixed Point Theorem.

Apart from establishing the existence of a fixed point, it often becomes necessary to prove the uniqueness of the fixed point. Besides, from computational point of view, an algorithm for calculating the value of the fixed point to a given degree of accuracy is desirable. Often this algorithm involves the iterates of the given function. In essence, the question about the existence, uniqueness and approximation of fixed point provide three significant aspect of the general fixed point principle.

Banach's contraction principle is perhaps one of the few most significant theorems that answers all these three questions of existence, uniqueness and constructive algorithm convincingly. A deeper, though especial result is Brouwer's fixed point theorem which states that any continuous function mapping a closed ball

$\overline{B(a,r)}$ of $\mathbb{R}^n$ in to itself has a fixed point. In general, Brouwer's fixed point theorem ensures neither the uniqueness of the fixed point nor the convergence of the iterates. While, the early proofs of Brouwer's theorem rely on algebraic-topological ideas based upon analytical arguments. A brief survey of the development of Brouwer's fixed point theorem has been presented in the paper [61].

Most of the mathematics is focussed on the solutions of the various equations involving numbers. For example, for given numbers a and b with $a \neq 0$, the linear equation $ax + b = 0$ has a unique solution. On the other hand, the quadratic equation $ax^2 + bx + c = 0$ may not have real solutions for real numbers a, b and c with $a \neq 0$. However, it will always have a pair of solutions in the system of complex numbers.

More generally, one can consider an equation of the form $S(x) = 0$ where S is a real valued function of a real variable. For $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = S(x) + x$, obviously, a solution of $S(x) = 0$ is a solution of $T(x) = x$ and conversely. An element x_0 for which $T(x_0) = x_0$ is called a **fixed point** of T. Thus, the problem of solving the equation $S(x) = 0$ is equivalent to finding the fixed point of the associated function T. In order to find zeros of a function S, one should seek the fixed point of related function such as T. The procedure of setting zeros is known as iteration method or the fixed point method of solving equation. The study of existence of fixed point falls within the topology and algebraic topology. A topological space is said to posses the fixed point property if every continuous mapping of the space into itself has a fixed point. Of course, if a topological space has the fixed point property, any other topological space homeomorphic to the first will also possess the fixed point property.

We will recall the following definitions.

Definition 1.1.1. *Let X be a non empty set and d be a real function from $X \times X$ into $\mathbb{R}^+$ such that for all $x, y, z \in X$, we have*

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

then, d is called a **metric** or distance function and the pair (X, d) is called a **metric space**. It is a topological space which provides the general setting in which we study the convergence of a sequence and continuity of a function.

Definition 1.1.2. *A sequence $\{x_n\}$ in a metric space (X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there corresponds $n_0 \in \mathbb{N}$ such that for all $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.*

Definition 1.1.3. *A sequence $\{x_n\}$ in metric space is said to be convergent to a point $z \in X$ if for given $\epsilon > 0$, there exists a positive number $n_0 \in \mathbb{N}$ such that $d(x_n, z) < \epsilon$ for all $n \geq n_0$.*

In this case, z is called limit of $\{x_n\}$ and we write $x_n \to z$.

Definition 1.1.4. *A metric space (X, d) is called complete if every Cauchy sequence in it is convergent to a point in X.*

Definition 1.1.5. *A metric space X is said to be compact if every sequence in it has a convergent subsequence.*

Definition 1.1.6. *Let X and Y be metric spaces with metrices d_1 and d_2 respectively, then a function $T : X \to Y$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$ there exist some $\delta > 0$ such that $d_2(Tx, Tx_0) < \varepsilon$ for all $x \in X$ satisfying $d(x, x_0) < \delta$. Also, this happens if and only if*
$Tx_n \to Tx$ whenever $x_n \to x$ in X.

Definition 1.1.7. *Let d be a metric on X. For $x \in X$ and $r > 0$, the set $U_d(x, r) = \{\ y \in X : d(x, y) < r\}$ is called the open ball about x of radius r.*

Definition 1.1.8. *Let $E \subset X$. An element $x \in X$ is called a limit point (or an accumulation point) of E if for every $r > 0$ there is some y in $U_d(r, x) \cap E$ with $y \neq x$.*

Definition 1.1.9. *A subset F of metric space X is called a closed set if it contains each of its limit points.*

Definition 1.1.10. *Two self mappings S and T on a metric space X are said to be commuting if,*

$$STx = TSx \qquad \forall x \in X.$$

Two self mappings S and T on a metric space are said to be commuting at a point z in X if $STz = TSz$.

Also, S and T are said to be non commuting if there is no such point z in X where S and T commute.

Definition 1.1.11. *Two self mappings S and T on a metric space X are said to be weakly commuting if,*

$$d(STx, TSx) \leq d(Sx, Tx) \quad \forall x \in X$$

The mapppings S and T are said to be weakly commuting at a point z in X if, $d(STz, TSz) \leq d(Sz, Tz)$.

The notion of weakly commuting mappings was introduced by S. Sessa [147] in 1982. He introduced the common fixed points of non commuting generalized contraction mappings.

Definition 1.1.12. *Two self mappings S and T on a metric space (X, d) are called compatible if,* $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, *whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \quad for \quad some \quad t \in X.$$

Two self mappings S and T on a metric space X are called non compatible if they are not compatible. Clearly, S and T will be non compatible if there corresponds at least a sequence $\{x_n\}$ such that

$$\lim_{n\to\infty} d(STx_n, TSx_n)$$

is either non zero or non existent. G. Jungck [75] generalized the notion of weak commutativity mappings and also weakly commuting mappings due to Sessa [147] by introducing the concept of compatible maps in 1986, which is also called the asymptotically continuous by Tiwari and Singh [160] in an independent work. It may be observed that the compatible mappings commute at their coincidence points. Also, from above definitions, it is clear that commuting mappings and weakly commuting mappings are compatible, but the converses are not necessarily true.

Definition 1.1.13. *Let S and T be mappings from a metric space (X, d) into itself. Then, S and T are said to be weakly compatible if they commute at their coincident point; that is, $Sx = Tx$ for some $x \in X$ implies $STx = TSx$.*

Definition 1.1.14. *A mapping S of a metric space X into itself is called non-expansive if $d(Sx, Sy) \leq d(x, y)$ for all $x, y \in X$ and $x \neq y$*

Definition 1.1.15. *Let S and T be self mappings on a set X, then a point z in X is called a common fixed point of S and T if $Sz = z = Tz$. Also the point z is called a coincidence point of S and T provided $Sz = Tz$.*

Example 1.1.16. *Let (X, d) be a metric space with*
X = [0, 1] and $d(x, y) = |x - y|$.
Let $S, T : X \to X$ be defined by $Sx = \frac{x}{3}$ and $Tx = \frac{x}{5}$.
Then, 0 is the common fixed point as well as coincidence point of the mappings S and T.

Definition 1.1.17. *Let (X, d) be a metric space.*
Let $T : X \to X$ and let $p \in \mathbb{N}$, then T is said to generalized p-contractive if

$$d(T^p x, T^p y) < diam\{x, y, T^p x, T^p y\} \quad \forall x, y \in X, x \neq y.$$

Definition 1.1.18. *A contraction T is called power contraction if T^k is a contraction mapping for some integer $k > 1$.*

1.2 Historical Developments of Some Fixed Point Theorems in Metric Space

Historically, the most important result in the fixed point theorem is due to L. E. J. Brouwer which asserts that every self continuous mapping of a closed unit ball in $\mathbb{R}^n$, the n - dimensional Euclidean Space, possess a fixed point. A particular case of Brouwer's theorem can be stated as follows:

Theorem 1.2.1. *The closed unit interval* $[0, 1]$ *on the real line posses a fixed point property, i.e each continuous mapping of* $[0, 1]$ *into itself has a fixed point.*

The application of topological theorems to analysis, involves infinite dimensional space of functions or sequences. The usual procedure is to extend a theorem from finite dimensional space to an infinite dimensional space. The infinite dimensional analogue of Brouwer's result was given by J. Schauder [145] in 1930.

Theorem 1.2.2. *Any compact convex nonempty subset of a normed linear space has the fixed point property for continuous mapping.*

Brouwer's and Schauder's fixed point theorems are fundamental theorems in the area of fixed point theory and its applications. Schauder's theorem is of great importance in the numerical treatment of equations in analysis. In 1935, A. N. Tychnoff extended Brouwer's result to a compact convex subset of a locally convex linear topological space.

Theorem 1.2.3. *Any compact convex nonempty subset of a locally convex Hausdorff real topological vector space has the fixed point property for continuous mapping.*

Perhaps the most frequently cited and most widely applied fixed point theorem is due to S. Banach which appeared in his Ph.D. thesis(1920, published in 1922)

Theorem 1.2.4. *Let* (X, d) *be complete metric space and* $T : X \to X$ *be a mapping such that* $d(Tx, Ty) \leq kd(x, y)$ *for some* $0 \leq k < 1$ *and all* $x, y \in X$*. Then,* T *has a unique fixed point in* X*. Moreover, for any* $x_0 \in X$*, the sequence of iterates* x_0, Tx_0, $T(Tx_0), \ldots$ *converges to the fixed point of* T*.*

when $d(Tx, Ty) \leq kd(x, y)$ for some $0 \leq k < 1$ and for all $x, y \in X$, then T is called a *contraction.* A contraction shrinks distances by a uniform factor k less then 1 for all pairs of points. The above theorem is called the contraction mapping theorem or Banach's fixed point theorem. An elementary account of the contraction mapping theorem and some applications, including its role in solving non linear ordinary differential equations, is in [21].

Banach contraction principle is simple in nature and its proof does not involve much of topological machinery. The proof is constructive, that is, the existence of the fixed point is established by constructing the point as the limit of the sequence of the iterates tending to the fixed point. The construction of the sequence $\{x_n\}$ and the study of its convergence are known as the method of successive approximation.

The following is the example of Banach Contraction Principle.

Example 1.2.5. *The cosine function defined as* $T(x) = cosx$

is a contraction function and has a fixed point.

Let (X, d) be a metric space with d as usual metric. Let $X = [0, 1]$ and define a function $T : X \to X$ by $T(x) = cosx$. The graph of $cosx$ and $y = x$ intersect once over $[0, 1]$, which shows that cosine function has a fixed point in $[0, 1]$.
Since, $cos1 \approx 0.54$, $cos[0, 1] \subset [0, 1]$.
For any differentiable function T, by mean value theorem,
$T(x) - T(y) = T'(t)(x - y) forsome\ t \in (x, y)$.
Now, $cosx - cosy = -Sin(t).(x - y)$ for some t.
$\Rightarrow |cosx - cosy| = |-sint||x - y|$.
Since, sine function increases on $[0, 1]$,so $|sint| \leq sin1 \approx 0.84147$.
So, $|cosx - cosy| \leq 0.8415|x - y|$.
Therefore, cosine is a contraction mapping on $[0, 1]$. To get fixed point by iteration, we press cosine bottom repeatedly on a calculator taking any seed value in $[0, 1]$, and we get $p \approx 0.739$, as a fixed point.

Banach contraction mapping theorem has long been used as one of the most important tools in the study of nonlinear problems. It provides an impressive illustration of the unifying power of functional analysis in an analytic method and of the usefulness of fixed point theorems in analysis. Therefore, numerous generalizations of this theorem have been obtained during the past four decades by weakening its hypothesis while retaining the convergence property of successive iterates to the unique fixed point of the mapping. The importance of these generalizations are notions of non expansive and contractive mappings.

Another important direction of generalization of this principle concerns the common fixed point of pair of mappings or sequence of mappings satisfying contractive type conditions.

One of the most interesting generalizations of the Banach Contraction Principle consists of replacing the Lipschitz constant k by some real valued function whose values are less than unity.

One of the first extension of Banach's contraction principle to become widely known is the following theorem due to E. Rakoth [132] in 1962.

Theorem 1.2.6. *Let (X, d) be a complete metric space and suppose $T : X \to X$ satisfies*

$$d(Tx, Ty) \leq \alpha(d(x, y)).d(x, y) \qquad \forall x, y \in X$$

where $\alpha : [0, \infty) \to [0, \infty)$ is monotonically decreasing. Then, T has a unique fixed point z, and for all $x_0 \in X$ we have,

$$T^n x_0 \to z \quad as \quad n \to \infty.$$

Rakoth's theorem is related to the following theorem by M. Edelstein [35] in 1962

Theorem 1.2.7. *Let (X, d) be a non empty compact metric space and suppose $T : X \to X$ satisfies*

$$d(Tx, Ty) < d(x, y) \quad \forall x, y \in X$$

Then T has a unique fixed point z, and for all $x_0 \in X$ we have $T^n x_0 \to z$ as $n \to \infty$.

D. F. Baily [9] in 1966, extended the result of Edelstein to compact metric space in the following theorem.

Theorem 1.2.8. *Let (X, d) be a compact metric space and* $T : X \to X$ *be continuous. If there exists n = n(x,y) with*

$$d(T^n x, T^n y) < d(x, y)$$

for $x \neq y$*, then* T *has a unique fixed point.*

A subsequent generalization of Rakotch's result was obtained by D. W. Boyd and J. S. Wong [13] in 1969.

Theorem 1.2.9. *Let (X, d) be a non empty complete metric space and suppose* $T : X \to X$ *satisfies*

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X$$

where $\phi : [0, \infty) \to [0, \infty)$ *is uppersemicontinuous from the right, and satisfies* $0 \leq \phi(t) < t$ *for* $t > 0$*. Then* T *has a unique fixed point* z *and for all* $x_0 \in X$ *we have* $T^n x_0 \to z$ *as* $n \to \infty$*.*

A quantitative variant of the Boyd - Wong [13] theorem was proved by F. E. Browder [19].

Theorem 1.2.10. *Let (X, d) be a non empty, bounded, complete metric space and suppose* $T : X \to X$ *satisfies*

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \forall x, y \in X,$$

where $\phi : [0, \infty) \to [0, \infty)$ *is monotone nondecreasing and continuous from the right, such that* $\phi(t) < t$ *for* $t > 0$*. Then there exists a unique* $z \in X$ *such that for all* $x_0 \in X$ *we have* $T^n(x_0) \to z$ *as* $n \to \infty$*. Moreover, if* d_0 *is diameter of X, then,*

$$d(T^n x_0, z) \leq \phi^n d_0, \quad \text{and} \quad \phi^n d_0 \to 0 \quad \text{as} \quad n \to \infty.$$

In 1969, A. Meir and E. Keeler [97] generalized the Boyd-Wong Theorem.

Theorem 1.2.11. *Let (X, d) be a non empty, complete metric space and suppose T: $X \to X$ satisfies the condition,*
given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for all $x, y \in X$ with $x \neq y$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon. \tag{1.1}$$

Then, T has a unique fixed point z, and for all $x_0 \in X$, we have $T^n x_0 \to z$ as $n \to \infty$.

A mapping T: X $\to$ X on a metric space (X, d) which satisfies the condition (1.1) is called a Meir - Keeler contraction. In order to compare the Boyd - Wong condition with Meir- Keeler condition, the latter has been characterized by T. C. Lim [90] in the following theorem.

Theorem 1.2.12. *Let (X, d) be a non empty metric space, and let T: $X \to X$ be a mapping. Then, T is a Meir-Keeler contraction if and only if there exists a (nondecreasing and right continuous) function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ and $\phi(s) > 0$ for $s > 0$, such that*

$$d(Tx, Ty) < \phi(d(x, y)), \quad x \neq y, \quad \forall x, y \in X,$$

and such that for every $s > 0$ there exists a $\delta > 0$ such that $\phi(t) \leq s$ for all $t \in [s, s + \delta]$.

The theorems due to Boyd and Wong [13] and Meir -Keeler [97] are the results of considerable significance and each of these theorems has been extended and generalized by various authors. Some significant generalizations of Boyd and Wong theorem are due to Park and Rhoades[126], Singh and Kasahara[151], Hussain and Seghal [52], Singh and Meade [153], Jachymski [55], Pant[116] , Pant and Pant [112], Pant et.al.[108]. Similarly some of the well known generalations of Meir-Keeler theorem are those due to Park and Bae [125], Park and Rhoades [126], Rao and Rao [135], Jungck [75], Pant [[109],[114],[116],[117],[119]], Jungck et.al. [74], Pant et.al.[[101],[105],[108]]

The fixed point theory for non expansive mappings has been one of the main research areas of non linear functional analysis since 1950s. Some well known results in the theory of nonexpansive mappings are probably the theorems established independently by Browder [18], Gohde [46] and Kirk [82].

In 1968, R. Kannan [78] established the following fixed point theorem.

Theorem 1.2.13. *Let (X, d) be a non empty complete metric space. Let $T : X \to X$ be a mapping such that there exists an $\alpha \in [0, \frac{1}{2})$ for which*

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty) \qquad \forall x, y \in X$$

then there exists a unique fixed point to which all Picard iteration sequences converge.

In 1973, G. E. Hardy and T. D. Roggers [47] obtained the following fixed point theorem under the generalized contractive condition.

Theorem 1.2.14. *Let T be a self mapping on a complete metric space X such that,*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] + b[d(x, Ty) + d(y, Tx)] + c\, d(x, y)$$

for all x, y in X where $0 \leq 2a+2b+c < 1$. *Then T has a unique fixed point.*

In 1975, B. K. Dass and S. Gupta[32] generalized Banach Contraction Mapping Theorem through rational expressions.

Theorem 1.2.15. *Let T be a mapping of a metric space X into itself such that,*

1. $d(Tx,\ Ty) \leq \alpha \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \beta\ d(x,y)$
   for all $x,\ y \in X,\ \alpha > 0,\ \beta > 0, \alpha + \beta < 1$
2. *for some* $x_0 \in X$, *the sequence of iterates* $\{T^n(x_0)\}$ *has a subsequence* $\{T^{n_k}(x_o)\}$ *with* $\xi = \lim_{n\to\infty} T^{n_k}(x_o)$ *then* ξ *is a unique fixed point of T.*

In 1977, D. S. Jaggi [57] established the following fixed point theorem using rational type contractive condition in complete metric space which generalizes the Banach Contraction Mapping Theorem..

Theorem 1.2.16. *Let T be a continuous self map defined on a complete metric space (X, d). Further let T satisfies the following contractive conditions*

$$d(Tx,Ty) \leq \alpha \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \beta d(x,y) \tag{1.2}$$

for all $x, y \in X, x \neq y$ *for some* $\alpha, \beta \in [0,1)$ *with* $\alpha + \beta < 1$, *then T has a unique fixed point.*

In 1977, B.E. Rhoades [140] proved the following fixed point theorem based on generalized p-contraction.

Theorem 1.2.17. *Let (X, d) be a non empty compact metric space, and $p \in \mathbb{N}$. Let $T : X \to X$ be continuous and generalized p-contractive. Then, T has a unique fixed point z, and for every $x_0 \in X$ we have,* $\lim_{n\to\infty} T^n x_0 = z$.

In 1980, S. P. Singh [152] established the following improved version of the Kannan's theorem.

Theorem 1.2.18. *Let T be a continuous mapping of a metric space X into itself such that*

$$d(Tx, Ty) < \frac{1}{2}[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$, $x \neq y$, If for some $x \in X$, the sequence of iterates$\{T^n x\}$ has a subsequence converging to z, then $\{T^n x\}$ converges to z and z is the unique fixed point of T.

In 1988, B. E. Rhoades [138] compared some contraction conditions and also considered generalizations of their conditions to the cases where the condition holds for various iterates of the functions. The comparison was shown by P. Collaco and J. Carvalho in [30] in 1997.

Now, we present some common fixed point theorems in metric space.

In 1976, G. Jungck [76] obtained a well known generalization of Banach contraction principle to obtain common fixed points of commuting mappings. Jungck introduced the following contractive condition so called *Jungck Contraction*

$$d(Sx, Sy) \leq kd(Tx, Ty),$$

$0 \leq k < 1$ for all $x, y \in X$ for a pair of self maps S and T in a complete metric space X and established the following theorem.

Theorem 1.2.19. *Let $S, T : X \to X$ be a pair of commuting continuous self maps satisfying the condition,*

$$d(Sx, Sy) \leq k\, d(Tx, Ty), \quad 0 \leq k < 1.$$

then S and T have a unique common fixed point whenever $S(X) \subset T(X)$.

In 1983, B. Fisher [40] established a common fixed point theorem for four self mappings A, B, S and T in a metric space (X, d) satisfying

$$A(X) \subset T(X), B(X) \subset S(X)$$

and the condition,

$$d(Ax, By) \leq k\, max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty)\}, 0 \leq k < 1.$$

then A,B,S and T have a common fixed point. In 1984, B.E. Rhoades extended a theorem of Park and Rhoades [126] involving a pair of mapping satisfying a Meir-Keeler type contractive condition for three mappings.

Theorem 1.2.20. *Let f be a continuous self-map of a complete metric space (X, d), $g, h \in C_f$ the class of continuous and satisfying the following condition,*
For each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\varepsilon \leq max(d(gx, hy), d(fx, gx), d(fy, hy), \frac{[d(fx, hy) + d(fy, gx)]}{2}) < \varepsilon + \delta$$

implies $d(fx, fy) < \varepsilon$. Then, either there exists a point of coincidence of f and g or f and h, or f, g and h have a unique common fixed point.

In 1984, A. Ganguli and in 1985, I.H.M. Rao and K.P.R.Rao [135] extended Meir - Keeler type definitions for three mappings.

In 1986, G. Jungck [75] obtained the following fixed point theorem for four continuous mappings on a compact metric space.

Theorem 1.2.21. *Let A, B, S and T be continuous self mappings of a compact metric space (X, d) with*
$A(X) \subset T(X) \quad and \quad B(X) \subset S(X)$.
If A, B, S and T be compatible pairs and

$$d(Ax, By) < max(m(x, y))$$

where,

$$m(x, y) = \{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{1}{2}[d(By, Sx) + d(Ax, Ty)]\}$$

Then, A, B, S and T have a unique common fixed point.

In 1986, R.P. Pant [119] simultaneously and independently established following common fixed point theorem satisfying Meir - Keeler type contractive condition with δ to be non decreasing.

Theorem 1.2.22. *Let A, B, S and T be commuting self mapping of a complete metric space (X, d) satisfying*
$A(X) \subset T(X) \quad and \quad B(X) \subset S(X)$ *and the condition*
given $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$, $\delta(\epsilon)$ being non decreasing

such that

$$\begin{aligned} \epsilon &\leq max(d(Sx,Ty), d(Ax,Sx), d(By,Ty)) < \epsilon + \delta \\ &\Rightarrow d(Ax,By) < \epsilon \end{aligned}$$

If one of the mappings A, B, S and T is continuous, then A, B, S and T have a unique common fixed point.

Also, J. Jachymski [54], K. Jha and V. pant [65], V. Popa [131], K.Jha [68], K. Jha, R. P. Pant and S. L. Singh [69], K. Jha, R. P. Pant and G.Porru [70], K. Jha [72] have established some common fixed point theorems for four mappings satisfying Meir-Keeler type contractive condition. Also, the common fixed points for four mappings satisfying contractive condition were extended for sequences of mappings by Jungck et.al [74], J. Jachymski [54], R. P. Pant [116].

In 2007, K. Jha [72] established the following fixed point theorem for sequence of mappings involving two pairs of weakly compatible mappings under a Lipschitz type contractive condition.
Let $\{A_i\}$, $i = 1, 2, 3, ...,$ S *and* T be self mappings of a metric space (X, d). In the sequel, let us denote

$$\begin{aligned} M_{1i}(x,y) &= max\{(Sx,Ty), d(A_1x,Sx), d(A_iy,Ty), \\ &\quad \frac{1}{2}[d(Sx,A_iy) + d(A_1x,Ty)]\} \end{aligned}$$

Theorem 1.2.23. *Let* $\{A_i\}$, $i = 1, 2, 3, ...,$ *S and T be self mappings of a metric space (X, d) such that,*

1. $A_1X \subset TX, A_iX \subset SX$ *for* $i > 1$

2. Given $\epsilon > 0$*, there exists a* $\delta > 0$ *such that for all x, y in X,* $\epsilon < M_{12}(x,y) < \epsilon + \delta \Rightarrow d(A_1x, A_2y) \leq \epsilon$*,*

3.

$$\begin{aligned} d(A_1x, A_iy) &< \alpha[d(Sx,Ty) + d(A_1x, Sx) + d(A_iy, Ty) \\ &+ d(Sx, A_iy) + d(A_ix, Ty)] \end{aligned}$$

for $0 \leq \alpha \leq \frac{1}{3}$.

If one of A_iX*, SX, or TX is a complete subspace of X and if the pairs* (A_1, S) *and* (A_k, T)*, for some* $k > 1$*, are weakly compatible, then all the* A_i*, S and T have a unique common fixed point.*

As fixed point theorems are statements containing sufficient conditions that ensure existence of a fixed point, so one of the central concerns in fixed point theory is to find a minimal set of sufficient conditions which ensures the existence of a common fixed point. Also common fixed point for generalized contractions necessarily require a commutativity condition, a condition on the range of mappings, a contractive condition, and continuity of one or more mappings or in general, a Lipschitz type contractive condition. In view of these essential requirements, in the present investigation, we address the following central question concerning common fixed point theorems.

Given a pair of self mappings on a metric space (X, d) satisfying a contractive condition, what minimal assumption on commutativity, continuity and contractive condition guarantee the existence of a common fixed point ? Following this course, recently R. P. Pant [115] has established some common fixed point theorems under minimal type conditions. Also, the review work on the fixed point results under Meir-Keeler contractive condition in metric space has been published in [64].

1.3 Some Applications of Fixed Point Theorems

John Von in 1932, pointed out how fixed point theory could be utilized to prove the existence of equilibrium in economic models in a seminar on the topic *On a system of economic equations and a generalization of Brouwer's fixed point theorem.* This concept has lead to win Nobel Prizes in economics for Kenneth Arrow in 1972 and Gerard Debreu in 1983. Applications of fixed point to Game theory also led to win a Nobel Prize in economics for John Nash in 1994 [4].

The theory of fixed point is a very extensive field which has wide applications. Fixed point theory has played a central role in the problems of non-linear functional analysis and provided a power tool in demonstrating the existence and uniqueness of solutions to various mathematical models representing phenomena arising in different fields such as in Engineering, Economics , Game Theory and Nash Equilibrium, Steady State Temperature Distribution, Epidemics, Flow of Fluids, Chemical Reactions, Neutron Transport Theory, Haar Measures, Abstract Elliptic Problems, Invariant Subspace Problems, Approximation Problems, Logic Programming, Neural Networks.

Banach's fixed point theorem has applications in many branches of mathematics such as analysis, differential functions, ODE and integral equations, Image compression, Google's pagerank algorithem, Newton- Rapshon iteration, system of linear algebraic equations, system of ordinary differential equations, Boundary value problems, Uryshon-Voltera equations, Musielak-Orlicz space settings,quasi-linear integrodifferential equations and Chomology. The paper of K. Jha [58] deals with some applications of Banach fixed point theorem.

Chapter 2

Fixed Point Results of Asymptotic Contractions

This chapter includes introduction, basic definitions and chronological developments of the fixed point theorems of asymptotic contractions in metric space with further research scope.

2.1 Introduction

After the establishment of Contraction Principle by Stephen Banach in 1922, R. Caccioppli in 1930, suggested the concept of asymptotic contraction based on Banach Contraction Principle. In 1962, E. Rakotch, probably, was the first for the extension of weaker form of Banach Contraction Principle for the contraction constant. In 1962, D. W. Boyd and J. S. W. Wong obtained a more general condition. In 2003, W. A. Kirk introduced an asymptotic version of Boyd - Wong Contraction. Since then, many extensions of weaker forms of contraction conditions for fixed points have been established by many authors. Asymptotic fixed point theory deals with conditions describing a behavior of iterates of a mapping. This chapter includes a survey work on fixed points of asymptotic contraction in metric space. In this chapter, the notations ϕ^n and ϕ_n are used to denote the n times iteration of the function ϕ and the sequence of function respectively.

The paper on this survey work has been accepted for publication in [122].

Now we start with the following definitions:

Definition 2.1.1. *[159] A metric space (X, d) satisfies the condition of* **TCS- convergence** *if and only if* $x \in X$ *and* $d(T^n x, T^{n+1}x) \to 0$ *as* $n \to \infty$ *implies that* $\{ T^n x\}_{n \in \mathbb{N}}$ *has a convergent subsequence.*

Definition 2.1.2. *[159] Let* X *be a set and* $T : X \to X$*.For* $x \in X$*, the set* $O_T(x) = \{ x, Tx, T^2x, ...\}$ *is called the orbit of* x*.*

Definition 2.1.3. *[159] A function* $f : X \to \mathbb{R}$ *is T-orbitally lower semicontinuous at the point p if and only if for all sequences* $\{ x_n\}_{n \in \mathbb{N}}$ *such that* $x_n \to p$ *follows that* $f(p) \leq liminf_{n \to \infty} f(x_n)$*.*

Definition 2.1.4. *[159] A mapping* $T : X \to X$ *is said to be orbitally continuous if* $\xi, x \in X$ *are such that* ξ *is a cluster point of* O_T *then* $T(\xi)$ *is a cluster point of* $T(O_T)$*.*

Definition 2.1.5. *[84] Let (X, d) be a metric space. A mapping* $T : X \to X$ *is said to be asymptotic contraction if,* $d(T^n x, T^n y) \leq \phi_n(d(x, y))$ *for all* $x, y \in X$ *where*

$$\phi_n : [0, \infty) \to [0, \infty) \quad and \quad \phi_n \to \phi \in \Phi \tag{2.1}$$

uniformly on the range of d, where Φ *is the class of functions.*

Now, we give an example satisfying the above condition.

Example 2.1.6. *Let* $M = \{n^{-1} \cup \{0\} : n \in \mathbb{N}\}$ *and* $d(M, d)$ *be a metric space with usual metric* d*. Define* $T : M \to M$ *by* $T(0) = 0$ *and* $T(n)^{-1} = (n+1)^{-1}$*. For* $t \in \mathbb{R}$*, define*
$\phi_n(t) = n^{-1} \quad \forall n \in \mathbb{N}$ *and* $\phi(t) = 0 \quad \forall t$*.*
Clearly $\phi(t) < t \quad \forall t > 0$ *and* $\phi_n \to \phi$ *uniformly on* M*.*
Put $x_0 = 1$ *then* $O_T(1) = \{1^{-1}, 2^{-1}, 3^{-1}, ...\} = \{n^{-1}\}$ *which is bounded. Since, all the assumptions are satisfied and* $T^k(x) \to 0$ *as* $k \to \infty$*. Hence,* 0 *is the fixed point of* T*.*

In 2004, Philipp Gerhardy [44] gave the following generalized definition of asymptotic contraction.

Definition 2.1.7. *[44] A function* $T : X \to X$ *on a metric space (X, d)is called an asymptotic contraction if for each* $b > 0$*, there exists a moduli* $\eta^b : (0, b] \to (0, 1)$ *and* $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ *and the following hold :*

1. *there exists a sequence of functions* $\phi_n : (0, \infty) \to (0, \infty)$ *such that for all* $x, y \in X$*, for all* $\varepsilon > 0$ *and for all* $n \in \mathbb{N}$
   $b \geq d(x, y) \geq \varepsilon \to d(T^n x, T^n y) \leq \phi_n(\varepsilon).d(x, y)$
2. *for each* $0 < l \leq b$ *the function* $\beta^b_l := \beta^b(l, .)$ *is a modulus of uniformconvergence for* ϕ_n *on [l, b], i.e.,*
   $\forall \varepsilon > 0 \quad \forall s \in [l, b] \quad \forall m, n \geq \beta^b_l(\varepsilon)(|\phi_m(s) - \phi_n(s)| \leq \varepsilon)$*,*
   and
3. *defining* $\phi := lim_{n \to \infty} \phi_n$*, then for each* $\varepsilon > 0$ *we have that* $\eta^b(\varepsilon) > 0$ *and* $\phi(s) + \eta^b(\varepsilon) \leq s$ *for each* $s \in [\varepsilon, b]$*.*

where there is no ambiguity, superscript b from the moduli η^b, β^b are removed.

Definition 2.1.8. *[90] A function ϕ from $[0, \infty)$ into itself is called an L-function if $\phi(0) = 0$, $\phi(s) > 0$ for $s \in (0, \infty)$, and for every $s \in (0, \infty)$ there exists $\delta > 0$ such that $\phi(t) \leq s$ for all $t \in [s, s + \delta]$*

Definition 2.1.9. *[157] Let (X, d) be a metric space. Then, a mapping T on X is said to be an asymptotic contraction of Meir-Keeler type (ACMK, for short) if there exists a sequence $\{\phi_n\}$ of functions from $[0, \infty)$ into itself satisfying the following:*

1. *$limsup_n \, \phi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$,*
2. *For each $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\phi_\nu \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$, and*
3. *$d(T^n x, T^n y) < \phi_n(d(x, y))$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $x \neq y$.*

Definition 2.1.10. *[158] Let (X, d) be a metric space. Then, a mapping T on X is said to be an asymptotic contraction of final type (ACF, for short) if the following hold:*

1. *$lim_{\delta \to +0} sup\{lim_{n\to\infty} sup \, d(T^n x, T^n y : d(x, y)) < \delta\} = 0$*
2. *for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $\varepsilon < d(x, y) < \varepsilon + \delta$, there exists $\nu \in \mathbb{N}$ such that $d(T^\nu x, T^\nu y) \leq \varepsilon$,*
3. *for x, $y \in X$, with $x \neq y$,there exits $\nu \in \mathbb{N}$ such that $d(T^\nu x, T^\nu y) < d(x, y)$, and*

4. *for* $x \in X$ *and* $\varepsilon > 0$, *there exists* $\delta > 0$ *and* $\nu \in \mathbb{N}$ *such that*
   $\varepsilon < d(T^i x, T^j x) < \varepsilon + \delta$ *implies* $d(T^\nu \circ T^i x, T^\nu \circ T^j x) \leq \varepsilon$ *for all* $i, j \in \mathbb{N}$.

Definition 2.1.11. *[24] Let* Ψ *be the class of functions* $\psi : [0, \infty) \to [0, \infty)$ *with the properties*
(i) ψ *is the Lebesque - integrable on each interval [0, a) with* $a > 0$
(ii)$\int_0^\varepsilon \psi(t) > 0$ *for each* $\varepsilon > 0$.

Let (X, d) be a metric space. Then, a mapping T on X is said to be an asymptotic contraction of integral Meir - Keeler type (ACIMK, for sort) if there exists a sequence $\{\phi_n\}$ *of functions from* $[0, \infty)$ *into itself satisfying the following*

1. $limsup_{n \to \infty} \phi_n(\varepsilon) \leq \varepsilon$ *for all* $\varepsilon > 0$,
2. *for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *and* $s \in \mathbb{N}$ *such that* $\phi_s(t) \leq \varepsilon$ *for all* $t \in [\varepsilon, \varepsilon + \delta]$, *and*
3. $\int_0^{d(T^n x, T^n y)} \psi(t)dt < \phi_n(\int_0^{d(x,y)} \psi(t)dt)$ *for all* $n \in \mathbb{N}$ *and* $x, y \in X$ *with* $x \neq y$ *where* $\psi \in \Psi$.

2.2 Fixed Point Theorems Under Asymptotic Contractions

In 1962, E. Rakotch obtained the following fixed point theorem.

Theorem 2.2.1. *[132] Let X be a complete metric space and suppose* $T : X \to X$ *satisfies* $d(T(x), T(y)) \leq \alpha(d(x, y))d(x, y)$

for each $x, y \in X$ *where* $\alpha : [0, \infty) \to [0, 1)$ *is monotonically decreasing then* T *has a unique fixed point* x_* *and* $\{T^n x\}$ *converges to* x_* *for each* $x \in X$.

In 1969, Boyd and Wong established a more general result on contraction mapping theorem in metric space which states that

Theorem 2.2.2. *[13] Let (X, d) be a complete metric space. Let* $T : X \to X$ *be a function satisfying* $d(Tx, Ty) \leq \phi(d(x, y))$ *for each* $x, y \in X$ *where* $\phi : [0, \infty) \to [0, \infty)$ *such that* $\phi(t) < t$ *for all* $t > 0$ *and* ϕ *is upper semicontinuous from the right, then* T *has a unique fixed point* x_* *for each* $x \in X$ *and* $\{T^n x\}$ *converges to* x_* *for each* $x \in X$.

In this theorem it is assumed that $\phi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from the right
(i.e $r_j \downarrow r \geq 0 \Rightarrow limsup_{n\to\infty}\phi(r_j) \leq \phi(r)$).

In 1986, M. R. Taskovic established the following results in topological space.

Theorem 2.2.3. *[159] Let* T *be a mapping of topological space X:= (X, d) into itself, where X satisfies the condition of TCS-convergence. Suppose that there exists a sequence of nonnegative real functions* $\{\ \alpha_n(x, y)\}_{n\in\mathbb{N}}$ *such that* $\alpha_n(x, y) \to 0$ *and a positive integer* $m(x, y)$ *such that*

$$d(T^n x, T^n y) \leq \alpha_n(x, y) \quad forall \quad n \geq m(x, y) \tag{2.2}$$

and for all $x, y \in X$ *where* $d : X \times X \to \mathbb{R}^0_+$. *If* $x \mapsto d(x, Tx)$ *is a T-orbitally continuous and* $d(a, b) = 0$ *implies* $a = b$, *then* T *has a unique fixed point* $\xi \in X$ *and* $T^n x \to \xi$ *for each* $x \in X$.

As a localization of condition (2.2) of Theorem (2.2.3), we have the following theorem.

Theorem 2.2.4. *[159] Let T be a mapping of topological space X:= (X, d) into itself, where X satisfies the condition of TCS-convergence. Suppose that there exists a sequence of nonnegative real functions $\{ \alpha_n(x, y)\}_{n\in\mathbb{N}}$ such that $\alpha_n(x, Tx) \to 0$ and a positive integer $m(x)$ such that*

$$d(T^n x, T^{n+1} y) \leq \alpha_n(x, Tx) \quad forall \quad n \geq m(x)$$

and for every $x \in X$ where $d : X \times X \to \mathbb{R}^0_+$. If $x \mapsto d(x, Tx)$ is a T-orbitally lower semicontinuous or T is orbitally continuous and $d(a, b) = 0$ implies $a = b$, then T has at least one fixed point in X.

In 2003, W.A. Kirk obtained a result which is asymptotic version of the Boyd and Wong.The concept of asymptotic contractions is suggested by one of the earliest version of Banach's contraction principle attributed to Cacciopoli [22] whose result asserts that if X is a complete metric space then the Picard iterates of a mapping $T : X \to X$ converges to the unique fixed point of T provided for each $n \geq 1$ there exits a constant c_n such that,

$$d(T^n x, T^n y) \leq c_n d(x, y) \quad x, y \in X \quad with \quad \sum_{n=1}^{\infty} c_n < \infty \quad (2.3)$$

Theorem 2.2.5. *[84] Let (X, d)is a complete metric space and suppose $T : X \to X$ is an asymptotic contraction for which the mappings ϕ_n in (2.1) are continuous. Assume also that some orbit of T is bounded. Then T has a unique fixed point $z \in X$*

and moreover the Picard sequence $\{T^n x\}_{n=1}^{\infty}$ *converges to* z *for some* $x \in X$.

In 2004, Jacek Jachymski and Izabela Jozwik [56] extended and gave a constructive proof of Kirk obtaining a complete characterization of asymptotic contraction on a compact metric space. As a by-product, they have established a separation theorem for upper semicontinuous functions satisfying some limit conditions with suitable example.

Theorem 2.2.6. *[56] Asume that (X, d) is complete metric space and T is a continuous selfmap of X. Then, the following statements are equivalent:*

1. *T is an asymptotic contraction;*
2. *the core* $Y := \cap_{n \in N} T^n(X)$ *is a sigleton;*
3. *T is an asymptotic* ϕ_0 *contraction, where* $\phi_0(t) := 0$ *for all* $t \in \mathbb{R}_+$, *and*
4. *T is a Banach contraction under some metric equivalent to d.*

In 2004, Y-Z Chen proved the theorem of Kirk under weaker assumptions without the use of ultrafilter methods. Kirk's paper assumes the continuity for ϕ and all ϕ_n, but Chen assumes the upper semicontinuity of ϕ and one of the $\phi_n{'s}$ which is weaker condition.

Theorem 2.2.7. *[26] Suppose that (X, d) is a complete metric space and suppose* $T : X \to X$ *such that,*

$$\begin{aligned} d(T^n x, T^n y) &\leq \phi_n(d(x,y)) \quad for \quad all \quad x, y \in X, \\ where \quad \phi_n &: [0,\infty) \to [0,\infty) \end{aligned}$$

and $\phi_n \to \phi$ *uniformly on any bounded interval [0, b]. Suppose that* ϕ *is an upper semicontinuous and* $\phi(t) < t$ *for* $t > 0$*. Furthermore, suppose there exists a positive integer* n_* *such that* ϕ_{n_*} *is upper semicontinuous and* $\phi_{n_*}(0) = 0$*. If there exists* $x_0 \in X$ *which has a bounded orbit* $O_T(x_0) = \{x_0, Tx_0, T^2x_0, ...\}$ *then* T *has a unique fixed point* $x_* \in X$ *such that*

$$lim_{n\to\infty} T^n x = x_* \quad \forall x \in X.$$

In 2004, Philipp Gerhardy [44] using techniques from proof mining [*proof mining* [15] refers to the logical analysis of given mathematical proofs with the help of the tools and insights from the part of mathematical logic known as proof theory, with the aim of obtaining relevant information *hidden* in the proofs.], developed a variant of the notion of asymptotic contraction and established a quantitative version of the corresponding fixed point theorem. Using techniques from proof mining as developed in [[85],[86]], he first derived a suitable generalization of the notion of asymptotic contractivity and subsequently established an elementary proof of Kirk's fixed point theorem, providing an explicit rate of convergence (to the unique fixed point) for sequences $\{T^n x\}$.

In 2006, T. Suzuki [157] introduced the notion of asymptotic contraction of Meir-Keeler type and established a fixed point theorem for such contractions which is generalization of fixed point theorems of Meir-Keleer [97] and Kirk [84]. Suzuki has used the characterization of Meir- Keeler contraction (1.1) proved by Lim [90].

Theorem 2.2.8. *[157] Let (X, d) be a complete metric space. Let T be an asymptotic contraction of Meir-Keeler (ACMK,* *for short) on X. Assume that T^k is continuous for some $k \in \mathbb{N}$. Then, there exists a unique fixed-point $z \in X$. Moreover, $lim_n T^n x = z$ for all $x \in X$*

In 2007, T. Suzuki introduced a more generalized notion of asymptotic contraction of final type (ACF,for short) and established fixed point theorems for such contractions.

Theorem 2.2.9. *[158] Let T be an ACMK on a metric space (X, d) Then T is an ACF.*

Theorem 2.2.10. *[158] Let (X, d) be a complete metric space and let T be an ACF on X. Assume that the following holds if $u \in X$ and $lim_n T^n u = v$, then $\exists \quad \ell \in \mathbb{N}$ such that $T^\ell v = v$. Then there exists a unique fixed point $z \in X$ of T. Moreover, $lim_n T^n x = z$ holds for every $x \in X$.*

Theorem 2.2.11. *[158] Let (X, d) be a complete metric space and let T be an ACF on X. Assume that T^ℓ is continuous for some $\ell \in \mathbb{N}$. Then, there exists a unique fixed point $z \in X$ of T. Moreover, $lim_n T^n x = z$ holds for every $x \in X$.*

In 2007, Marina Arav, Fransisco Eduardo Castillo Santos, Simeon Reich, and Alexander J. Zaslavski provided sufficient condition for the iterates of an asymptotic contraction on a complete metric space X to converge to its unique fixed point uniformly on each bounded subset of X. They improved the theorem of Chen [26] and have established a more general result.

Theorem 2.2.12. *[8] Let X be a metric space. Let $x_* \in X$ be a fixed point of $T : X \to X$. Assume that $d(T^n x, x_*) \leq \phi_n(d(x, x_*)) \quad \forall x \in X$ and all natural numbers n,* *where* $\phi_n : [0, \infty) \to [0, \infty)$ *and* $\phi_n \to \phi$ *uniformly on any bounded interval [0, b]. Suppose that* ϕ *is an upper semicontinuous and* $\phi(t) < t$ *for* $t > 0$*, then* $lim_{n\to\infty} T^n x = x_*$ *uniformly on each bounded subset of X.*

Theorem 2.2.13. *[8] let X be a metric space. Let* $T : X \to X$ *such that*

$$d(T^n x, T^n y) \leq \phi_n(d(x, y))$$

for all $x, y \in X$ *and all the natural numbers n, where* $\phi_n : [0, \infty) \to [0, \infty)$ *and* $\lim_{n\to\infty} \phi_n = \phi$*, uniformly on any bounded interval [0, b]. Suppose that* ϕ *is upper semicontinuous and that* $\phi(t) < t$ *for all* $t > 0$*. Furthermore, suppose that there exists a positive integer* n_* *such that* ϕ_{n_*} *is upper semicontinuous and* $\phi_{n_*}(0) = 0$*. If there exists* $x_0 \in X$ *which has a bounded orbit* $O_T(x_0) = \{x_o, Tx_o, T^2x_o \ldots\}$*, then T has a unique fixed point* $x_* \in X$ *and we have* $\lim_{n\to\infty} T^n x = x_*$ *uniformly on each bounded subset of X*

In 2007, Ivan D. Arandelovic established a fixed point theorem of Kirk's type unifying and generalizing the results of [[26],[56], [84]].

Theorem 2.2.14. *[7] Let (X,d) be a complete metric space,* $T : X \to X$ *continuous function and* (ϕ_i) *sequence of functions such that* $\phi_i : [0, \infty) \to [0, \infty)$ *and for each* $x, y \in X$
$d(T^i(x), T^i(y) < \phi_i(d(x, y))$.
Assume also that there exists upper semicontinuous function $\phi : [0, \infty) \to [0, \infty)$ *such that for any* $r > 0$
$\phi(r) < r, \psi(0) = 0$ *and* $\phi_i \to \psi$ *uniformly on any bounded interval [0, b]. If one of the following conditions is satisfying:*

1. *there exists* $x \in X$ *such that the orbit of* T *at* x *is bounded; or*

2. $\underline{lim}_{t\to\infty}(t - \phi(t)) > 0$, *or;*

3. $\overline{lim}_{t\to\infty} \frac{\phi(t)}{t} < 1$

then T *has a unique fixed point* $y \in X$ *and all sequences of Picard iterates defined by* T *converges to* y*, uniformly on each bounded subset of* X*.*

In 2007, the results established by E. M. Briseid [16] build on the analysis of Kirk's fixed point theorem for asymptotic contractions given by Gerhardy [44]. He had proved fixed point theorems on asymptotic contractions which give an explicit rate of convergence to the fixed point for a sequence. The rate of convergence depends on the space, the mapping and the starting point through a bound on the iteration sequence and some moduli for the mapping appearing as parameters.

In 2007, K.P.R. Sastry, G.V.R. Babu, S. Ismail and M. Balaiah [144] established a fixed point theorem with hypothesis slightly different from that of Chen [[26], theorem 2.2].

In 2011, Behazad Djafari Rauhani and Jennifer Love [136] introduced the weaker condition $liminf_{n\to\infty} d(x, T^n x) = 0$ for some x in X, and proved that this condition implies the existence of a fixed point and the convergence of the Picard iterates to this fixed point.

Theorem 2.2.15. *[136] Let (X, d) be a complete metric space. Let* $T : X \to X$ *such that* $d(T^n x, T^n y) \le \phi_n(d(x, y))$ *for all*

$x, y \in X$ where $\phi_n : [0, \infty) \to [0, \infty)$ and $\phi_n \to \phi$ uniformly on any bounded interval [0, b]. Suppose that ϕ is upper semi-continuous and $\phi(t) < t$ for $t > 0$ and assume that there is a positive integer n^* such that ϕ_{n^*} is upper semicontinuous and $\phi_{n^*}(0) = 0$. If $liminf_{n\to\infty} d(x, T^n x) = 0$, then T has a unique fixed point $x \in X$,and $lim_{n\to\infty} T^n y = x$ for all $y \in X$.

In 2012, E. Canzoneri and P. Vetro introduced the notion of asymptotic contraction of integral Meir-Keeler type on a metric space and proved a theorem which ensures existence and uniqueness of fixed points for such contractions.

Theorem 2.2.16. *[24] Let (X, d) be a complete metric space and T be an ACIMK on X. Assume that T^m is continuous for some $m \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover, $lim_{n\to\infty} T^n x = z$ for all $x \in X$.*

Remarks: On the basis of the above results, we observe that weaker forms of contractive conditions for the existence and uniqueness of fixed point are rapidly being developed. The condition of mappings to be continuous is necessary for the existence of fixed point but for the convergence to such a fixed point, it is not necessary. The rate of convergence depends upon the space, mapping and the starting point through a bound on iteration sequence. The notion of asymptotic contraction has been developed towards Boyd- Wong type and Meir -Keeler type conditions with applications to rate of convergence. Finally, the notion of *Asymptotic Contraction* and its development have become a necessary tool for the existence of fixed point and it extends several existing results in the literature.

Chapter 3

Fixed Point Results in Dislocated Metric Space

In this chapter, we present the introduction of dislocated metric space and the fixed point theorems which has been established in this space with examples.

3.1 Introduction

In 1922, S. Banach proved a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by different authors and many generalizations of this notion have been established. In 1986, S. G. Matthews [96] in his Ph.D. thesis introduced the notion of dislocated metric in the context of metric domains, in which self distance of a point need not be equal to zero. In 2000, P. Hitzler and A. K. Seda generalized the notion of topology by relaxing the requirement that neighborhoods of a point includes the point itself and by allowing neighborhoods of points to be empty which evolved out of applications in the area of logic programming semantics. Corresponding generalized notion of metric is obtained by allowing points to have nonzero distance to themselves. The study of common fixed points of mappings in dislocated metric space satisfying certain contractive conditions has been the center of vigorous research activities. Dislocated metric space plays very important role in topology, logic programming and electronics engineering.

C. T. Aage and J. N. Salunke [1], A. Isufati [53], K. P. R. Rao and P. Rangaswamy [134] established some fixed point theorems for single and pair of mappings in dislocated metric space.

Now, we start with the following definitions, lemmas and theorems.

3.2 Basic Definitions

Definition 3.2.1. *[49] Let X be a non empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the following conditions:*

1. *$d(x, y) = d(y, x)$*
2. *$d(x, y) = d(y, x) = 0$ implies $x = y$.*
3. *$d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.*
   Then d is called dislocated metric(or simply d-metric) on X.

Definition 3.2.2. *[49] A sequence $\{x_n\}$ in a dislocated metric space (X, d) is called a Cauchy sequence if for given $\epsilon > 0$, there corresponds $n_0 \in N$ such that for all $m, n \geq n_0$, we have $d(x_m, x_n) < \epsilon$.*

Definition 3.2.3. *[49] A sequence in dislocated metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$.*

In this case, x is called limit of $\{x_n\}$ (in d)and we write $x_n \to x$.

Definition 3.2.4. *[49] A dislocated metric space (X, d) is called complete if every Cauchy sequence in it is convergent with respect to d.*

Definition 3.2.5. *[49] Let (X, d) be a dislocated metric space. A mapping $T : X \to X$ is called contraction if there exists a number λ with $0 \le \lambda <$ 1 such that $d(Tx, Ty) \le \lambda\ d(x, y)$.*

We state the following lemmas without proofs.

Lemma 3.2.6. *Let (X, d) be a dislocated metric space. If $T : X \to X$ is a contraction function, then $\{T^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.*

Lemma 3.2.7. *[49] Limits in a dislocated metric space are unique.*

3.3 Fixed Point Theorems in Dislocated Metric Space

Theorem 3.3.1. *[49] Let (X, d) be a complete dislocated metric space. Let $T : X \to X$ be a continuous mapping satisfying*

$$d(Tx, Ty) \le \lambda d(x, y), \quad 0 \le \lambda < 1.$$

Then, T has a unique fixed point.

In 2008, C. T. Aage and J. N. Salunke established following fixed point theorems in complete d-metric space.

Theorem 3.3.2. *[1] Let (X, d) be a complete dislocated metric space. Let $T : X \to X$ be continuous mapping satisfying,*

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \\ &+ \delta \frac{d(x, Tx)d(y, Ty)}{d(x, y)} + \mu \frac{d(x, Tx)d(y, Ty)}{d(x, y)} \end{aligned}$$

for all $x, y \in X$ and $\alpha + \beta + \gamma + \delta + 4\mu < 1$. Then, T has a unique fixed point.

Theorem 3.3.3. *[1] Let (X, d) be a complete dislocated metric space. Let S, $T : X \to X$ be continuous mappings satisfying,*

$$d(Sx, Ty) \leq h\, max\{d(x, y), d(x, Sx), d(y, Ty)\}$$

for all $x, y \in X$ and $0 < h < 1$ then S and T have unique common fixed point.

In 2010, K.P.R. Rao and P. Ranga Swamy [134] established the following theorem for four mappings in d-metric space.

Theorem 3.3.4. *[134] Let (X, d) be a complete dislocated metric space. Let A, B, S, T : X→X be continuous mappings satisfying,*

1. *$S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$ and T(X) or S(X) is a closed subset of X, and*
2. *The pairs (S, A) and (T, B) are weakly compatible such that*

$$d(Sx,Ty) \leq h\, max\{\, d(Ax,By), d(Ax,Sx), d(By,Ty), \frac{d(Ax,Ty)+d(By,Sx)}{2}\}$$

for all $x, y \in X$ and $0 < h < 1$ then the mappings A, B, S and T have a common fixed point.

Theorem 3.3.5. *[133] Let (X, d) be a complete dislocated metric space. Let A, B, S, T : X→X be continuous mappings satisfying*

1. *$S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$*
2. *$SA = AS$ and $TB = BT$ and*
3. $$d(Sx,Ty) \leq \phi(max\{\, d(Ax,By), d(Ax,Sx), d(By,Ty), \frac{d(Ax,Sx)d(By,Ty)}{d(Ax,By)}\})$$

for all x, $y \in X$, where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotonically non decreasing and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$, then
(i) A and S (or) B and T have coincidence point or,
(ii) the pairs (A, S) and (B, T) have a common coincidence point.

Daffer and Kaneko [31] proved the following fixed point theorem in complete metric space.

Theorem 3.3.6. *[31] Let (X, d) be a complete metric space. Let S be a surjective self mapping and T an injective self map of X which satisfy the conditions that for a number $\alpha > 1$ we have $d(Sx,Sy) \geq \alpha d(Tx,Ty)$ for each $x,y \in X$.*
Then, S and T have a unique common fixed point.

B.E Rhoades [137] extended the Theorem (3.3.6) into compatible mappings which is as follows.

Theorem 3.3.7. *[137] Let (X, d) be a complete metric space. Let S and T be compatible self mappings of X satisfying the condition* $d(Sx, Sy) \geq \alpha d(Tx, Ty)$ *for all* $x, y \in X$ *and* $T(X) \subseteqq S(X), S$ *continuous. Then, S and T have a unique common fixed point.*

S. Kumar [88] generalized the theorem for weakly compatible mappings in metric space. Motivated with above theorems, we now prove a fixed point theorem for a single pair of weakly compatible mappings in dislocated metric space.

Theorem 3.3.8. *[62] Let (X, d) be a complete dislocated metric space. Let* $S, T : X \to X$ *be two continuous self mappings such that*

1. $T(X) \subseteqq S(X)$, *the pair (S, T) is weakly compatible maps, and*
2. *there exists a number* $\alpha < \frac{1}{2}$, $d(Tx, Ty) \leq \alpha d(Sx, Sy) \quad \forall \; x, y \in X$,

If one of the subspaces T(X) or S(X) is complete then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$. Using condition (1), choose $x_1 \in X$ such that $Sx_1 = Tx_0$. We define sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = Sx_{n+1} = Tx_n$. Let S(X) be complete.

Now, using condition (2), we get

$$\begin{aligned} d(y_n, y_{n+1}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \alpha d(Sx_n, Sx_{n+1}) \\ &= \alpha d(Tx_{n-1}, Tx_n) \\ &= \alpha d(y_{n-1}, y_n) \end{aligned}$$

Hence, we get

$$d(y_n, y_{n+1}) \leq \alpha d(y_{n-1}, y_n) \leq \alpha^2 d(y_{n-2}, y_{n-1}) \cdots \leq \alpha^n d(y_0, y_1)$$
$$since \quad \alpha < \frac{1}{2}. \quad So, \quad as \quad n \to \infty, \alpha^n \to 0$$

This implies that $\{y_n\}$ is a Cauchy sequence. So, it converges to some element $z \in X$. So, the subsequences $\{Sx_{n+1}\} \to z$ and $\{Tx_n\} \to z$.

Since $S(X)$ is complete, so there exists a point $u \in X$ such that $Su = z$. Now, by condition (2), we get
$d(Tu, Tx_n) \leq \alpha d(Su, Sx_n)$.
Now, taking limit as $n \to \infty$, we get $d(Tu, z) \leq 0$. So, this implies that $Tu = z$. Hence, $Su = Tu = z$. Since the pair (S, T) is weakly compatible, so we have $STu = TSu \Rightarrow Sz = Tz$.
Now, we claim that z is the fixed point of T. For this, by condition (2), we get

$$d(Tz, Tx_n) \leq \alpha d(Sz, Sx_n).$$

Taking limit as $n \to \infty$, we get

$$\begin{aligned} d(Tz, z) &\leq \alpha d(Sz, z) \\ &= \alpha d(Tz, z) \end{aligned}$$

a contradiction. Therefore, we have $Tz = z$.
Hence, we get $Sz = Tz = z$. This shows that z is the common fixed point of the maps S and T.

To prove the uniqueness of fixed point, let u and v be two common fixed points of the mappings S and T. Then, using condition (2), we get

$$\begin{aligned} d(u,v) = d(Tu,Tv) &\leq \alpha d(Su,Sv) \\ &= \alpha d(u,v), \end{aligned}$$

which is a contradiction. Hence, we get $u = v$.
This completes the proof of the theorem. □

We have the following example in favor of above theorem.

Example 3.3.9. *Let* $X = [0,1]$ *and let* d *be defined by* $d(x,y) = |x-y|$ *then* d *is a dislocated metric. Let us define the mappings* S *and* T *by*

$$Sx = \frac{x}{2} \quad \text{and} \quad Tx = \frac{x}{8}.$$

Then, we can observe that

$$d(Tx,Ty) \leq \alpha d(Sx,Sy) \quad for \quad \frac{1}{4} < \alpha < \frac{1}{2}.$$

The mappings S *and* T *are weakly compatible at* $x = 0$ *and hence* $x = 0$ *is the unique common fixed point of the maps* S *and* T.

Motivated with the theorem in [161], we prove the following common fixed point theorem for two pairs of weakly compatible mappings in dislocated metric space.

Theorem 3.3.10. *[60] Let (X, d) be a complete dislocated metric space. Let $A, B, S, T : X \to X$ be continuous mappings satisfying*

1. *$T(X) \subset A(X), \quad S(X) \subset B(X), \quad$ and*
2. *The pairs (S, A) and (T,B) are weakly compatible,*
3. *$d(Sx, Ty) \leq \alpha\, d(Ax, Ty) + \beta\, d(By, Sx) + \gamma\, d(Ax, By)$*

for all $x, y \in X$ where $\alpha,\ \beta,\ \gamma\ \geq 0, \quad 0 \leq \alpha + \beta + \gamma < \frac{1}{2}$.

Then, A, B, S, and T have a unique common fixed point.

Proof. Using condition(1), we define sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Bx_{2n+1} = Sx_{2n}, \quad \textit{and}$$
$$y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}, n = 0, 1, 2...$$

If $y_{2n} = y_{2n+1}$ for some n, then $Bx_{2n+1} = Tx_{2n+1}$.Therefore x_{2n+1} is a coincidence point of B and T.

Also, if $y_{2n+1} = y_{2n+2}$ for some n, then $Ax_{2n+2} = Sx_{2n+2}$. Hence x_{2n+2} is a coincidence point of S and A.

We assume that $y_{2n} \neq y_{2n+1}$ for all n. Then, using condition (3) we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \alpha d(Ax_{2n}, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Sx_{2n}) \\
&+ \gamma d(Ax_{2n}, Bx_{2n+1}) \\
&= \alpha d(y_{2n-1}, y_{2n+1}) + \beta d(y_{2n}, y_{2n}) + \gamma d(y_{2n-1}, y_{2n}) \\
&\leq \alpha[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] \\
&+ \beta[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \gamma d(y_{2n-1}, y_{2n}) \\
&= (\alpha + \beta + \gamma) d(y_{2n-1}, y_{2n}) + (\alpha + \beta) d(y_{2n}, y_{2n+1})
\end{aligned}$$

So, we have

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &\leq \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta} d(y_{2n-1}, y_{2n}) \\
&= h\, d(y_{2n-1}, y_{2n})
\end{aligned}$$

$$where \quad h = \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta} < 1.$$

This shows that

$$d(y_n, y_{n+1}) \leq h\, d(y_{n-1}, y_n) \leq ... \leq h^n\, d(y_0, y_1).$$

For each integer $q > 0$, we have

$$\begin{aligned}
d(y_n, y_{n+q}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + ... \\
... &\quad ... + d(y_{n+q-1}, y_{n+q}) \\
&\leq (1 + h + h^2 + ... + h^{q-1}) d(y_n, y_{n+1}) \\
&\leq \frac{h^n}{1 - h} d(y_0, y_1)
\end{aligned}$$

$$Since, \quad 0 < h < 1, \quad so \quad we \quad have \quad h^n \to 0 \quad as \quad n \to \infty.$$

So, we get $d(y_n, y_{n+q}) \to 0$. This implies $\{y_n\}$ is a Cauchy sequence in a complete dislocated metric space. So, there exists a point $z \in X$ such that $\{y_n\} \to z$. Therefore, the subsequences

$$\{Sx_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\}, \quad and \quad \{Ax_{2n+2}\}$$

all converge to z. Since $T(X) \subset A(X)$, there exists a point $u \in X$ such that $z = Au$. So, using condition (3), we get

$$\begin{aligned} d(Su, z) &= d(Su, Tx_{2n+1}) \\ &\leq \alpha d(Au, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Su) + \gamma d(Au, Bx_{2n+1}) \\ &= \alpha d(z, Tx_{2n+1}) + \beta d(Bx_{2n+1}, Su) + \gamma d(z, Bx_{2n+1}) \end{aligned}$$

Now, taking limit as n $\to \infty$, we get

$$d(Su, z) \leq \beta d(z, Su)$$

which is a contradiction, since $2\alpha + \beta + 2\gamma < 1$. So, we have $Su = Au = z$. Again, since $S(X) \subset B(X)$, there exists a point $v \in X$ such that $z = Bv$. We claim that $z = Tv$.
If $z \neq Tv$, then using condition (3), we get

$$\begin{aligned} d(z, Tv) &= d(Su, Tv) \\ &\leq \alpha d(Au, Tv) + \beta d(Bv, Su) + \gamma d(Au, Bv) \\ &= \alpha d(z, Tv) + \beta d(z, z) + \gamma d(z, z) \\ &\leq \alpha d(z, Tv) + 2(\beta + \gamma) d(z, Tv) \\ &= (\alpha + 2\beta + 2\gamma) d(z, Tv) \end{aligned}$$

a contradiction, since $\alpha + 2\beta + 2\gamma < 1$. So, we get $z = Tv$.
Hence, we have $Su = Au = Tv = Bv = z$.
Since the pair (S, A) are weakly compatible so by definition

$SAu = ASu$ implies $Sz = Az$.
Now, we show that z is the fixed point of S .
If $Sz \neq z$, then, using condition (3), we get

$$\begin{aligned} d(Sz, z) &= d(Sz, Tv) \\ &\leq \alpha d(Az, Tv) + \beta d(Bv, Sz) + \gamma d(Az, Bv) \\ &= (\alpha + \beta + \gamma) d(Sz, z) \end{aligned}$$

which is a contradiction. So, we have $Sz = z$.
This implies that $Az = Sz = z$. Again, the pair (T, B) are weakly compatible, so by definition $TBv = BTv$ implies $Tz = Bz$. Now, we show that z is the fixed point of T.
If $Tz \neq z$, then using condition (3), we get

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \\ &\leq \alpha d(Az, Tz) + \beta d(Bz, Sz) + \gamma d(Az, Sz) \\ &\leq (\alpha + \beta + 2\gamma) d(z, Tz) \end{aligned}$$

which is a contradiction. This implies that $z = Tz$.
Hence, we have $Az = Bz = Sz = Tz = z$. This shows that z is the common fixed point of the self mappings A, B, S and T.

To prove the uniqueness of fixed point, let $u \neq v$ be two common fixed points of the mappings A, B, S and T. Then using condition (3), we have

$$\begin{aligned} d(u, v) &= d(Su, Tv) \\ &\leq \alpha d(Au, Tv) + \beta d(Bv, Su) + \gamma d(Au, Bv) \\ &= \alpha d(u, v) + \beta d(v, u) + \gamma d(u, v) \\ &= (\alpha + \beta + \gamma) d(u, v). \end{aligned}$$

a contradiction. This shows that $d(u,v)=0$
Since (X,d) is a dislocated metric space, so we have $u=v$. This establishes the theorem. $\square$

Example 3.3.11. *Let $X=[0,1]$ and let d be defined by $d(x,y)=|x-y|$ then d is a dislocated metric. Let the mappings A,B,S and T be defined by*

$$Sx=0, \quad Ax=\frac{x}{2}, \quad Tx=\frac{x}{5} \quad and \quad Bx=x.$$
$$Then, for \quad some \quad \alpha=\frac{1}{5}, \quad \beta=\frac{1}{6}, \quad \gamma=\frac{1}{8},$$

the mappings $A,B,S,$ and T satisfy all conditions of above theorem . Since, $T(X)\subset A(X)$ and $S(X)\subset B(X)$. The pairs (S,A) and (T,B) are weakly compatible at $x=0$, since $S0=A0$ implies that $SA0=AS0$ and $T0=B0$ implies that $TB0=BT0$. The contractive condition holds for any two points of X, so $x=0$ is the unique common fixed point of mappings A, B, S and T.

We have following corollaries:

Corollary 3.3.12. *Let (X, d) be a complete dislocated metric space. Let $S, T : X \to X$ be continuous mappings satisfying*

$$d(Sx, Ty) \leq \alpha\, d(x, Ty) + \beta\, d(y, Sx) + \gamma\, d(x, y)$$

for all x, $y \in X$, where $\alpha, \beta, \gamma \geq 0, \quad 0 \leq (\alpha + \beta + \gamma) < \frac{1}{2}$. Then S and T have a unique common fixed point.

Proof: Let us take A = B = I an identity mapping in the theorem (3.3.11). The sequences $\{x_n\}$ and $\{y_n\}$ will reduce to $y_{2n+1} = Tx_{2n+1}$ and $y_{2n} = Sx_{2n}$. Then, following the procedure as in the theorem, we can show that $\{y_n\}$ is a Cauchy sequence and hence converges to a point z in X. Consequently, the subsequences converge to z. Since $T(X) \subset X$, so there exists a point $u \in X$ such that $z = u$. Using condition (3) and letting $n \to \infty$ we get $Su = z$, so $Sz = z$. Similarly, we show that $Tz = z$. So, we have $Sz = Tz = z$. Hence, z is the common fixed point of the mappings S and T.

To prove the uniqueness of fixed point, let u and v be two common fixed points of the mappings S and T. Then, we have

$$\begin{aligned} d(u, v) = d(Su, Tv) &\leq \alpha d(u, v) + \beta d(v, u) + \gamma d(u, v) \\ &= (\alpha + \beta + \gamma) d(u, v) \end{aligned}$$

a contradiction. Hence, $d(u, v) = 0$ implies $u = v$.

If we take $S = T$, then the above corollary (3.3.12) reduces to the following result.

Corollary 3.3.13. *Let (X, d) be a complete dislocated metric space. Let $T : X \to X$ be a continuous mapping satisfying,*

$$d(Tx, Ty) \leq \alpha\, d(x, Ty) + \beta\, d(y, Tx) + \gamma\, d(x, y)$$

for all x, $y \in X$,where $\alpha, \beta, \gamma \geq 0$, $\quad 0 \leq \alpha + \beta + \gamma < \frac{1}{2}$.
Then T has a unique fixed point.

Proof:
Let $\{x_n\}$ be a sequence in X such that $Tx_n = x_{n+1}$. Then, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha d(x_{n-1}, x_{n+1}) + \beta d(x_n, x_n) + \gamma d(x_{n-1}, x_n) \\ &\leq \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) + 2\beta d(x_n, x_{n-1}) \\ &+ \gamma d(x_{n-1}, x_n). \end{aligned}$$

This implies

$$(1 - \alpha) d(x_n, x_{n+1}) \leq \frac{\alpha + 2\beta + \gamma}{1 - \alpha} d(x_{n-1}, x_n),$$

where $h = \frac{\alpha+2\beta+\gamma}{1-\alpha} < 1$.
So, we get $d(x_n, x_{n+1}) \leq h\, d(x_{n-1}, x_n)$.
Hence, we have
$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \leq h^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq h^n d(x_0, x_1)$.
Since $0 \leq h < 1$,so for $n \to \infty$, we have $d(x_n, x_{n+1}) \to 0$. Hence, the sequence $\{x_n\}$ is a Cauchy sequence in complete metric space. So, there exists a point $z \in X$ such that $\{x_n\} \to z$. Since T is continuous, we have
$T(z) = Lim_{n\to\infty} T(x_n) = Lim_{n\to\infty} x_{n+1} = z$.

To prove the uniqueness of fixed point, let u and v are two fixed points of T. Then, using contractive condition, we get

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\leq \alpha d(u,Tv)+\beta d(v,Tu)+\gamma d(u,v) \\ &= \alpha d(u,v)+\beta d(v,u)+\gamma d(u,v) \\ &= (\alpha+\beta+\gamma)d(u,v) \end{aligned}$$

which is a contradiction, so $d(u,v)$ implies that $u=v$.

If we take $A=T$ and $B=S$ in Theorem (3.3.11) then it reduces to the following corollary.

Corollary 3.3.14. *Let (X,d) be a complete dislocated metric space. Let $S,T:X\to X$ be continuous mappings satisfying*

$$d(Sx,Ty)\leq \alpha\, d(Tx,Ty)+\beta\, d(Sy,Sx)+\gamma\, d(Tx,Sy)$$

for all $x,\ y\in X$, where $\alpha,\beta,\gamma\geq 0,\quad 0\leq\alpha+\beta+\gamma<\frac{1}{2}$.
Then S and T have a unique common fixed point.

Remarks:

Since, our theorem (3.3.10) gives the common fixed point of four mappings so it generalizes the results of A. Isufati [53] and improves the K. Jha and D. Panthi [60], C. T. Aage and J. N. Salunke [[1], [2]], R. Shrivastava, Z. K. Ansari and M. Sharma [149], K. P. R. Rao and P. Rangaswamy [134], K. Jha, K. P. R. Rao and D. Panthi [63] and similar other results of fixed point in the literature.

Now, we prove a common fixed point theorem four mappings in a complete dislocated metric space.

Theorem 3.3.15. *[120] Let (X, d) be a complete dislocated metric space. Let $A, B, S, T : X \to X$ be continuous mappings satisfying,*

1. *$T(X) \subset A(X), \quad S(X) \subset B(X)$*
2. *The pairs (S, A) and (T, B) are weakly compatible and*
3. *$d(Sx, Ty) \leq \alpha\,[d(Ax, Ty) + d(By, Sx)] + \beta[d(By, Ty) + d(Ax, Sx)] + \gamma\, d(Ax, By)$*

for all $x, y \in X$ where $\alpha, \beta, \gamma \geq 0, \quad 0 \leq \alpha + \beta + \gamma < \frac{1}{4}$.

Then the mappings A, B, S, and T have a unique common fixed point.

Proof:
Using condition (1), we define sequences $\{x_n\}$ and $\{y_n\}$ in X such that,

$$y_{2n} = Bx_{2n+1} = Sx_{2n}, \quad and \quad y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}, n = 0, 1, 2...$$

If $y_{2n} = y_{2n+1}$ for some n, then $Bx_{2n+1} = Tx_{2n+1}$. Therefore, x_{2n+1} is a coincidence point of B and T. Also, if $y_{2n+1} = y_{2n+2}$ for some n, then $Ax_{2n+2} = Sx_{2n+2}$. Hence, x_{2n+2} is a coincidence point of S and A. Assume that $y_{2n} \neq y_{2n+1}$ for all n. Then using condition (3), we have

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) & \\
&= d(Sx_{2n}, Tx_{2n+1}) \\
&\leq \alpha[d(Ax_{2n}, Tx_{2n+1}) + d(Bx_{2n+1}, Sx_{2n})] \\
&+ \beta[d(Bx_{2n+1}, Tx_{2n+1}) + d(Ax_{2n}, Sx_{2n})] \\
&+ \gamma d(Ax_{2n}, Bx_{2n+1}) \\
&\leq \alpha[d(y_{2n-1}, y_{2n+1})] + d(y_{2n}, y_{2n})] \\
&+ \beta[d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})] + \gamma d(y_{2n-1}, y_{2n}) \\
&\leq \alpha[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n}) \\
&+ d(y_{2n}, y_{2n+1})] + \beta[d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})] \\
&+ \gamma d(y_{2n-1}, y_{2n}) \\
&= (2\alpha + \beta + \gamma) d(y_{2n-1}, y_{2n}) + (2\alpha + \beta) d(y_{2n}, y_{2n+1})
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
d(y_{2n}, y_{2n+1}) &\leq \frac{2\alpha + \beta + \gamma}{1 - 2\alpha - \beta} d(y_{2n-1}, y_{2n}) \\
&= h\, d(y_{2n-1}, y_{2n}),
\end{aligned}
$$

where,

$$
h = \frac{2\alpha + \beta + \gamma}{1 - 2\alpha - \beta} < 1.
$$

This shows that

$$
d(y_n, y_{n+1}) \leq h\, d(y_{n-1}, y_n) \leq ... \leq h^n\, d(y_0, y_1).
$$

For every integer $q > 0$, we have

$$
\begin{aligned}
d(y_n, y_{n+q}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + ... \\
&... \; ... + d(y_{n+q-1}, y_{n+q}) \\
&\leq (1 + h + h^2 + ... + h^{q-1}) d(y_n, y_{n+1}) \\
&\leq \frac{h^n}{1 - h} d(y_0, y_1).
\end{aligned}
$$

$$Since, \quad 0 < h < 1, \quad so \quad h^n \to 0 \quad as \quad n \to \infty.$$

So, we get $d(y_n, y_{n+q}) \to 0$. This implies $\{y_n\}$ is a Cauchy sequence in a complete dislocated metric space. So, there exists a point $z \in X$ such that $\{y_n\} \to z$. Therefore, the subsequences

$$\{Sx_{2n}\}, \{Bx_{2n+1}\}, \{Tx_{2n+1}\} \quad and \quad \{Ax_{2n+2}\}$$

all converge to z. Since $T(X) \subset A(X)$, there exists a point $u \in X$ such that $z = Au$. So, using condition (3), we get

$$\begin{aligned} d(Su, z) &= d(Su, Tx_{2n+1}) \\ &\leq \alpha[d(Au, Tx_{2n+1}) + d(Bx_{2n+1}, Su)] \\ &+ \beta[d(Bx_{2n+1}, Tx_{2n+1}) + d(Au, Su)] \\ &+ \gamma d(Au, Bx_{2n+1}) \\ &= \alpha[d(z, Tx_{2n+1}) + d(Bx_{2n+1}, Su)] + \beta[d(z, Su)] \\ &+ \gamma d(z, Bx_{2n+1}). \end{aligned}$$

Now, taking limit as n $\to \infty$, we get

$$d(Su, z) < \beta d(z, Su)$$

which is a contradiction. So, we have $Su = Au = z$.
Again, since $S(X) \subset B(X)$, there exists a point $v \in X$ such that $z = Bv$. We claim that $z = Tv$. If $z \neq Tv$, then using condition (3), we get

$$\begin{aligned} d(z, Tv) &= d(Su, Tv) \\ &\leq \alpha[d(Au, Tv) + d(Bv, Su)] + \beta[d(Bv, Tv) + d(Au, Su)] \\ &+ \gamma d(Au, Bv) \\ &= \alpha[d(z, Tv) + d(z, z)] + \beta[d(z, Tv) + d(z, z)] + \gamma d(z, z) \\ &= (3\alpha + 3\beta + 2\gamma) d(z, Tv), \end{aligned}$$

a contradiction. So, we get $z = Tv$. Hence, we have $Su = Au = Tv = Bv = z$. Since the pair (S, A) are weakly compatible so by definition $SAu = ASu$ implies $Sz = Az$. Now, we show that z is the fixed point of S.
If $Sz \neq z$, then using condition (3), we get

$$\begin{aligned} d(Sz, z) &= d(Sz, Tv) \\ &\leq \alpha[d(Az, Tv) + d(Bv, Sz)] + \beta[d(Bv, Tv) \\ &+ d(Az, Sz)] + \gamma d(Az, Bv) \\ &= \alpha[d(Sz, z) + d(z, Sz)] + \beta[d(z, z) + d(Sz, Sz)] \\ &+ \gamma d(Sz, z) \\ &\leq (2\alpha + 4\beta + \gamma)d(Sz, z) \end{aligned}$$

which is a contradiction. So, we have $Sz = z$.
This implies that $Az = Sz = z$.
Again, the pair (T, B) are weakly compatible, so by definition $TBv = BTv$ implies $Tz = Bz$.
Now, we show that z is the fixed point of T. If $Tz \neq z$, then using condition (3) we get

$$\begin{aligned} d(z, Tz) &= d(Sz, Tz) \\ &\leq \alpha[d(Az, Tz) + d(Bz, Sz)] + \beta[d(Bz, Tz) + d(Az, Sz)] \\ &+ \gamma d(Az, Sz) \\ &= \alpha[d(z, Tz) + d(Tz, z)] + \beta[d(Tz, Tz) + d(z, z)] \\ &+ \gamma d(z, Tz) \\ &\leq (2\alpha + 4\beta + 2\gamma)d(z, Tz) \end{aligned}$$

which is a contradiction. This implies that $z = Tz$.
Hence, we have $Az = Bz = Sz = Tz = z$.
This shows that z is the common fixed point of the self mappings A, B, S and T.

To prove the uniqueness of fixed point, let $u \neq v$ be two common fixed points of the mappings A, B, S and T. Then using condition (3), we have

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\leq \alpha[d(Au,Tv)+d(Bv,Su)]+\beta[d(Bv,Tv)+d(Au,Su)] \\ &+ \gamma d(Au,Bv) \\ &= \alpha[d(u,v)+d(v,u)]+\beta[d(v,v)+d(u,u)]+\gamma d(u,v) \\ &= (2\alpha+4\beta+\gamma)d(u,v), \end{aligned}$$

which is a contradiction. This shows that $d(u,v) = 0$. Since (X,d) is a dislocated metric space, so we have $u = v$. This establishes the theorem.

Now we have the following corollaries

Corollary 3.3.16. *Let (X,d) be a complete dislocated metric space. Let $S,T : X \to X$ be continuous mappings satisfying*

$$\begin{aligned} d(Sx,Ty) &\leq \alpha\,[d(x,Ty)+d(y,Sx)]+\beta\,[d(y,Ty)+d(x,Sx)] \\ &+ \gamma\,d(x,y) \end{aligned}$$

$\forall x,y \in X$, where $\alpha,\beta,\gamma \geq 0$, $\quad 0 \leq \alpha+\beta+\gamma < \frac{1}{4}$. Then the mappings S and T have a unique common fixed point.

Proof:

Let A = B = I an identity mapping in the theorem. The sequences $\{x_n\}$ and $\{y_n\}$ will reduce to, $y_{2n+1} = Tx_{2n+1}$ and $y_{2n} = Sx_{2n}$. Then, following the procedure as in theorem, we can show that $\{y_n\}$ is a Cauchy sequence and hence converges to a point z in X. Consequently,the subsequences converge to

z. Since $T(X) \subset X$, so there exists a point $u \in X$ such that $z = u$. Using condition (3) and letting $n \to \infty$ we get $Su = z$, which implies $Sz = z$. Similarly, we show that $Tz = z$. So, $Sz = Tz = z$. Hence, z is the common fixed point of the mappings S and T.

To prove the uniqueness of fixed point, let u and v be two common fixed points of the mappings S and T. Then, we have

$$\begin{aligned} d(u,v) &\leq \alpha[d(u,v) + d(v,u)] + \beta[d(v,v) + d(u,u)] + \gamma d(u,v) \\ &\leq 2\alpha d(u,v) + 4\beta d(u,v) + \gamma d(u,v) \\ &= (2\alpha + 4\beta + \gamma) d(u,v) \end{aligned}$$

which is a contradiction. So, $d(u,v) = 0$ implies $u = v$.

If we take $S = T$ then the above corollary (3.3.16) is reduced to the following result.

Corollary 3.3.17. *Let (X,d) be a complete dislocated metric space. Let $T : X \to X$ be a continuous mapping satisfying*

$$\begin{aligned} d(Tx,Ty) &\leq \alpha\,[d(x,Ty) + d(y,Tx)] + \beta\,[d(y,Ty) + d(x,Tx)] \\ &+ \gamma d(x,y) \end{aligned}$$

$\forall x, y \in X$, *where* $\alpha, \beta, \gamma \geq 0, \quad 0 \leq \alpha + \beta + \gamma < \frac{1}{4}$.
Then the mapping T has a unique fixed point.

Proof:

Let us define sequence $\{x_n\}$ such that $T(x_n) = x_{n+1}$.

Then using contractive condition, we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq (\alpha + \beta + \gamma) d(x_{n-1}, x_n) \\ &+ (3\alpha + \beta) d(x_n, x_{n+1}) \end{aligned}$$

which gives $d(x_n, x_{n+1}) \leq \frac{\alpha+\beta+\gamma}{(1-3\alpha-\beta)} d(x_{n-1}, x_n)$,
where $h = \frac{\alpha+\beta+\gamma}{(1-3\alpha-\beta)} < 1$.
So, we have $d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n)$
Hence, we get
$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \leq h^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq h^n d(x_0, x_1)$.
Since, $0 \leq h < 1$ so for $n \to \infty$ we have $d(x_n, x_{n+1}) \to 0$.
Therefore, $\{x_n\}$ is a Cauchy sequence in complete metric space.
So there exists a point $z \in X$ such that $\{x_n\} \to z$. Since T is continuous, we have
$T(z) = Lim_{n\to\infty} T(x_n) = Lim_{n\to\infty} x_{n+1} = z$.

To prove the uniqueness of fixed point, let u and v be two common fixed points of the mappings S and T. Using contractive condition, we have

$$d(u, v) = d(Tu, Tv) \leq (2\alpha + 4\beta + \gamma) d(u, v)$$

a contradiction. Hence $d(u, v) = 0$ implies $u = v$.

If we take $A = T$ and $B = S$ in above Theorem (3.3.15), then it reduces to the following result.

Corollary 3.3.18. *Let (X, d) be a complete dislocated metric space. Let $S, T : X \to X$ be continuous mappings satisfying*

$$\begin{aligned} d(Sx, Ty) &\leq \alpha [d(Tx, Ty) + d(Sy, Sx)] + \beta [d(Sy, Ty) + d(Tx, Sx)] \\ &+ \gamma \, d(Tx, Sy) \end{aligned}$$

$\forall x, y \in X$, *where* $\alpha, \beta, \gamma \geq 0, \quad 0 \leq \alpha + \beta + \gamma < \frac{1}{4}$.
Then the mappings S and T have a unique common fixed point.

We prove the following common fixed point theorem for two pairs of weakly compatible mappings in complete dislocated metric space.

Theorem 3.3.19. *[63] Let (X, d) be a complete dislocated metric space. Let A, B, S, T:X→ X be continuous mappings satisfying,*

1. *A(X)⊆ T(X), B(X)⊆ S(X) and B(X) or A(X) is closed subset of X.*
2. *the pairs (A, S) and (B, T) are weakly compatible and*

$$d(Ax, By) \leq \phi\Big(max(d(Sx, Ty), d(Ax, Sx), d(By, Ty), \frac{d(Sx, By) + d(Ax, Ty)}{2})\Big)$$

for all $x, y \in X$ *where* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *is monotonically nondecreasing and,*

$$\sum_{n=1}^{\infty} \phi^n(t) < \infty$$

for all $t > 0$*, then the mappings A, B, S, and T have a unique common fixed point.*

Proof:

Since, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotonically nondecreasing and,

$$\sum_{n=1}^{\infty} \phi^n(t) < \infty$$

so $\phi(t) < t$ and consequently $\phi^n(t) \to 0$.
Let x_0 be an arbitrary point in X. Then, using condition (1), we define sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$y_{2n} = \mathrm{A}x_{2n} = \mathrm{T}x_{2n+1}$, and
$y_{2n+1} = \mathrm{B}x_{2n+1} = Sx_{2n+2}$, $n = 0, 1, 2, ...$

If $y_{2n} = y_{2n+1}$ for some n, then $Tx_{2n+1} = Bx_{2n+1}$. Therefore x_{2n+1} is a coincidence point of B and T. Also, if $y_{2n+1} = y_{2n+2}$ for some n, then $Sx_{2n+2} = Ax_{2n+2}$. Hence, x_{2n+2} is a coincidence point of S and A. Assume that $y_{2n} \neq y_{2n+1}$ for all n. Now, we consider,

$$
\begin{aligned}
&d(y_{2n}, y_{2n+1})\\
&= d(Ax_{2n}, Bx_{2n+1})\\
&\le \phi\Big(max\big(d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1}),\\
&\quad \frac{d(Sx_{2n}, Bx_{2n+1}) + d(Ax_{2n}, Tx_{2n+1})}{2}\big)\Big)\\
&= \phi\Big(max\big(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}),\\
&\quad \frac{d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})}{2}\big)\Big)\\
&= \phi\Big(max\big(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\big)\Big)
\end{aligned}
$$

If, max $(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})) = d(y_{2n}, y_{2n+1})$
then, $d(y_{2n}, y_{2n+1}) \le \phi\Big(d\big(y_{2n}, y_{2n+1}\big)\Big)$
which is a contradiction. So, we have
max $(d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})) = d(y_{2n-1}, y_{2n})$

Hence, we get

$$d(y_{2n}, y_{2n+1}) \le \phi\big(d(y_{2n-1}, y_{2n})\big) \tag{3.1}$$

Again, using contractive condition (2), we get

$$
\begin{aligned}
&d(y_{2n-1}, y_{2n}) \\
&\quad = d(Ax_{2n-1}, Bx_{2n}) \\
&\quad \leq \phi\Big(max(d(Sx_{2n-1}, Tx_{2n}), d(Ax_{2n-1}, Sx_{2n-1}), d(Bx_{2n}, Tx_{2n}), \\
&\qquad \frac{d(Sx_{2n-1}, Bx_{2n}) + d(Ax_{2n-1}, Tx_{2n})}{2})\Big) \\
&\quad = \phi\Big(max(d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n-2}), d(y_{2n}, y_{2n-1}), \\
&\qquad \frac{d(y_{2n-2}, y_{2n}) + d(y_{2n-1}, y_{2n-1})}{2}\Big)
\end{aligned}
$$

Hence, we get

$$
d(y_{2n-1}, y_{2n}) \leq \phi\big(d(y_{2n-2}, y_{2n-1})\big) \tag{3.2}
$$

Therefore, using relations (3.1) and (3.2), we have

$$
\begin{aligned}
d(y_n, y_{n+1}) &\leq \phi\big(d(y_{n-1}, y_n)\big) \\
&\leq \phi^2\big(d(y_{n-2}, y_{n-1})\big) \\
&\vdots \\
&\leq \phi^n\big(d(y_0, y_1)\big)
\end{aligned}
$$

Now, for $n, m \in \mathbb{N}$ with $n < m$, from the triangle inequality, we have

$$
\begin{aligned}
d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + ... + d(y_{m-1}, y_m) \\
&\leq \phi^n\big(d(y_0, y_1)\big) + \phi^{n+1}\big(d(y_0, y_1)\big) + ... + \phi^{m-1}\big(d(y_0, y_1)\big) \\
&= \sum_{i=n}^{m-1} \phi^i\big(d(y_0, y_1)\big)
\end{aligned}
$$

converge to 0 as n , m tend to ∞ as $\phi(t) < t$ for $t > 0$. So, $\{y_n\}$ is a Cauchy sequence in the complete dislocated metric space

X. Hence, there exists a point z in X such that $\{y_n\} \to z$.
So, the subsequences
$\{Ax_{2n}\} \to z$, $\{Bx_{2n+1}\} \to z$, $\{Tx_{2n+1}\} \to z$ and $\{Sx_{2n+2}\} \to z$.
Now, assume that $A(X)$ is a closed subset of X, and we have $A(X) \subseteq T(X)$, then there exist $v \in X$ such that $Tv = z$.
If $Bv \neq z$ then,using contractive condition (2), we get

$$\begin{aligned} & d(Ax_{2n}, Bv) \\ & \quad \leq \phi\Big(max\big(d(Sx_{2n}, Tv), d(Ax_{2n}, Sx_{2n}), d(Bv, Tv), \\ & \qquad \frac{d(Sx_{2n}, Bv) + d(Ax_{2n}, Tv)}{2})\Big) \\ & \quad = \phi\Big(max\big(d(y_{2n-1}, Tv), d(y_{2n}, y_{2n-1}), d(Bv, Tv), \\ & \qquad \frac{d(y_{2n-1}, Bv) + d(y_{2n}, Tv)}{2})\Big) \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(z, Bv) & \leq \phi\Big(max\big((d(z, Tv), d(z, z), d(Bv, Tv), \\ & \qquad \frac{d(z, Bv) + d(z, Tv)}{2})\Big) \\ & < d(z, Bv) \end{aligned}$$

therefore, we have $Bv = z = Tv$.
Since B and T are weakly compatible so by definition we have $BTv = TBv$ implies $Bz = Tz$
Also if, $z \neq Bz$ then using contractive condition (2), we have

$$
\begin{aligned}
&d(Ax_{2n}, Bz) \\
&\quad \le \phi\Big(max\big(d(Sx_{2n}, Tz), d(Ax_{2n}, Sx_{2n}), d(Bz, Tz), \\
&\quad \frac{d(Sx_{2n}, Bz) + d(Ax_{2n}, Tz)}{2})\Big) \\
&\quad = \phi\Big(max\big(d(y_{2n-1}, Tz), d(y_{2n}, y_{2n-1}), d(Bz, Tz), \\
&\quad \frac{d(y_{2n-1}, Bz) + d(y_{2n}, Tz)}{2})\Big).
\end{aligned}
$$

Letting $n \to \infty$, we have

$$
\begin{aligned}
d(z, Bz) &\le \phi\Big(max\big(d(z, Tz), d(z, z), d(Bz, Tz), \\
&\quad \frac{d(z, Bz) + d(z, Tz)}{2})\Big) \\
&< d(z, Bz).
\end{aligned}
$$

So, we get Bz = z. Since $B(X) \subseteq S(X)$ there exists $w \in X$ such that Sw = z.
If $Aw \neq z$, then using contractive condition (2), we have

$$
\begin{aligned}
d(Aw, Bz) &\le \phi\Big(max\big(d(Sw, Tz), d(Aw, Sw), d(Bz, Tz) \\
&\quad \frac{d(Sw, Bz) + d(Aw, Tz)}{2})\Big).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
d(Aw, z) &= \phi\Big(max\big(d(z, Tz), d(Aw, z), d(z, Tz), \\
&\quad \frac{d(z, z) + d(Aw, z)}{2})\Big) \\
&< d(Aw, z)
\end{aligned}
$$

This implies that $Aw = z$. So, we have $Aw = z = Sw$.
Since the pair (A, S) is weakly compatible so, $ASw = SAw$ implies $Az = Sz$.
If $Az \neq z$, then we have

$$\begin{aligned} d(Az, z) &= d(Az, Bz) \\ &\leq \phi\Big(max\big(d(Sz, Tz), d(Az, Sz), d(Bz, Tz), \\ &\quad \frac{d(Sz, Bz) + d(Az, Tz)}{2}\big)\Big) \\ &< d(Az, z) \end{aligned}$$

which is a contradiction. Thus, we have $Az = z$. Therefore we have $Az = Sz = Bz = Tz = z$. This shows that z is the common fixed point of the mappings A, B, S and T.

To prove the uniqueness of fixed point, let z and u be the common fixed point of the mappings A, B, S, T. Then, using contractive condition (2), we have

$$\begin{aligned} d(z, u) &= d(Az, Bu) \\ &\leq \phi\Big(max\big(d(Sz, Tu), d(Az, Sz), d(Bu, Tu), \\ &\quad \frac{d(Sz, Bu) + d(Az, Tu)}{2}\big)\Big) \\ &= \phi\Big(max\big(d(z, u), d(z, z), d(u, u), \\ &\quad \frac{d(z, u) + d(z, u)}{2}\big)\Big) \\ &= \phi\Big(max\big(d(z, u), d(z, z), d(u, u)\big)\Big). \end{aligned}$$

Now, replacing u by z, we get
$d(z, z) \leq \phi\Big(\max\big(d(z, z), d(z, z), d(z, z)\big)\Big)$.
since $\phi(t) < t$, so we have a contradiction.

Hence, $d(z,z) = 0$. Similarly, we can get $d(u,u) = 0$.
In the same way, we can show that
$d(z,u) \leq \phi\left(d(z,u)\right)$ implies $d(z,u) = 0$.

This shows that z is the unique common fixed point of the mappings A, B, S and T.
This completes the proof of the theorem.

Remarks:
Since the condition on ϕ and on mappings in theorem (3.3.19) are improved, so our result generalizes the result of of K. P. R. Rao and P. Rangaswamy [134] and improves the results of C. T. Aage and J. N. Salunke [2], A. Isufati [53] and other similar results of fixed points in dislocated metric space.

Chapter 4

Fixed Point Results in Dislocated Quasi Metric Space

In this chapter, we introduce dislocated quasi metric space and prove fixed point theorems which generalize, extend and unify some well-known similar results in literature.

4.1 Introduction

In 2006, F. M. Zeyada, G. H. Hassan and M. A. Ahmed [162] introduced various definitions and generalized the result of P. Hitzler and A. K. Seda [49] in dislocated quasi- metric space. In 2008, C. T. Aage and J. N. Salunke [1] proved some results in dislocated and dislocated quasi-metric spaces. In 2010, A. Isufati [53] has also proved some results in these spaces.

4.2 Basic Definitions

We start with the following definitions.

Definition 4.2.1. *[162] Let X be a nonempty set and let* $d : X \times X \to [0, \infty)$ *be a function satisfying following conditions:*
(i) d(x, y) = d(y, x) = 0, implies x = y, and
(ii) $d(x, y) \leq d(x, z) + d(z, y)$*, for all x, y, z* $\in X$*.*
Then, d is called a dislocated quasi-metric (or simply dq-metric) on X.

In this metric, the symmetric property of dislocated metric space has been relaxed.

Definition 4.2.2. *[162] A sequence* $\{x_n\}$ *in dislocated quasi-metric space(dq-metric space) (X , d) is called Cauchy sequence if for given* $\epsilon > 0$*, there corresponds* $n_0 \in N$*, such that for all m, n* $\geqslant n_0$*, we have* $d(x_m, x_n) < \epsilon$*; or* $d(x_n, x_m) < \epsilon$*.*

Definition 4.2.3. *[162] A sequence* $\{x_n\}$ *in a dislocated quasi metric space* (X, d) *is said to be dislocated quasi convergent (for short dq-convergent) to x if* $lim_{n\to\infty}$ $d(x_n, x) = lim_{n\to\infty}$ $d(x, x_n) = 0$.
In this case, x is called a dq-limit of $\{x_n\}$ *and we write* $x_n \to x$*.*

Definition 4.2.4. *[162] A dislocated quasi metric space (X, d) is called complete if every Cauchy sequence in it is a dq- convergent.*

Definition 4.2.5. *[162] Let* (X, d_1) *and* (Y, d_2) *be dislocated quasi metric spaces and let* $T : X \to Y$ *be a function. Then T is continuous if for each sequence* $\{x_n\}$ *which is* d_1q*-convergent to* x_0 *in X, the sequence* $\{Tx_n\}$ *is* d_2q*-convergent to* Tx_0*.*

We state the following lemmas without proof.

Lemma 4.2.6. *[162] Every subsequence of dq-convergent sequence to a point x_0 is dq-convergent to x_0.*

Lemma 4.2.7. *[162] Let (X, d) be a dislocated quasi metric space. If $T:X\rightarrow X$ is a contraction function, then $\{T^n(x_0)\}$ is a Cauchy sequence for each $x_0 \in X$.*

Lemma 4.2.8. *[162] dq-limits in a dislocated quasi metric space are unique.*

Definition 4.2.9. *[162] Let (X, d) be a dislocated quasi metric space. A map $T : X \rightarrow X$ is called contraction if there exists $0 \leq \lambda < 1$ such that $d(Tx, Ty) \leq \lambda\ d(x, y)$.*

4.3 Fixed Point Theorems in Dislocated Quasi Metric Space

In 1975, B. K. Dass and S. Gupta [32] generalized Banach Contraction Mapping Theorem through rational expressions in metric space. In 1977, D. S. Jaggi [57] established the Theorem in complete metric space.

In 2006, F. M. Zeyada, G. H. Hassan and M. A. Ahmed [162] established the following fixed point theorem in dislocated quasi metric space.

Theorem 4.3.1. *[162] Let (X, d) be a complete dislocated metric space. Let $T : X \rightarrow X$ be a continuous mapping satisfying*

$$d(Tx, Ty) \leq \lambda d(x, y), \quad 0 \leq \lambda < 1.$$

Then, T has a unique fixed point.

In 2008, C. T. Aage and J. N. Salunke established the following theorem in dislocated quasi metric space.

Theorem 4.3.2. *[1] Let (X, d) be a complete dislocated quasi metric space. Let $T : X \to X$ be continuous mapping satisfying the condition*

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$,, $0 \leq \alpha < \frac{1}{2}$, then T has a unique fixed point.

Theorem 4.3.3. *[1] Let (X, d) be a complete dislocated quasi metric space.*
Let $T : X \to X$ be continuous mapping satisfying the condition

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta d(x, Tx) + \gamma d(y, Ty) \\ &+ \delta[d(x, Ty) + d(y, Tx)] \end{aligned}$$

for all $x, y \in X$, $\alpha, \beta, \gamma, \delta \geq 0$ which may depend on both x and y such that sup$\{\alpha+\beta+\gamma+2\delta\} < 1$. Then, T has a unique fixed point.

In 2010, A. Isufati established the following theorem in dislocated quasi metric space.

Theorem 4.3.4. *[53] Let (X, d) be a complete dislocated quasi metric space.Let $T : X \to X$ be continuous mapping satisfying the condition*

$$d(Tx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Tx) + \gamma d(x, y)$$

for all $x, y \in X$, $\alpha, \beta, \gamma \geq 0$ which may depend on both x and y such that sup$\{2\alpha + 2\beta + \gamma\} < 1$. Then, T has a unique fixed point.

R. Shrivastava, Z. K. Ansari and M. Sharma proved the following fixed point theorems in 2012.

Theorem 4.3.5. *[149] Let T be a continuous self map defined on a complete dislocated quasi metric space (X, d). Further, let T satisfies the contractive condition* (1.2) *for all* $x, y \in X, x \neq y$ *for some* $\alpha, \beta \in [0, 1)$ *with* $\alpha + \beta < 1$*, then T has a unique fixed point.*

Theorem 4.3.6. *[149] Let (X, d) be a complete dislocated quasi metric space.Let* $T : X \to X$ *be continuous mapping satisfying the condition,*

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta \frac{d(x, Tx).d(y, Ty)}{d(x, y)} \\ &+ \gamma[d(x, Tx) + d(y, Ty)] + \delta[d(x, Ty) + d(y, Tx)] \end{aligned}$$

for all $x, y \in X, \quad \alpha, \beta, \gamma, \delta > 0$*, with* $0 \leq \alpha + \beta + 2\gamma + 2\delta < 1$*, then T has a unique fixed point.*

K. Zoto, E. Hoxha and A. Isufati established the following fixed point theorem in 2012.

Theorem 4.3.7. *[163] Let (X, d) be a complete dislocated quasi metric space. Let* $T : X \to X$ *be continuous mapping satisfying the condition,*

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta \frac{d(x, Tx).d(y, Ty)}{d(x, y)} \\ &+ \gamma[d(x, Tx) + d(y, Ty)] + \delta[d(x, Ty) + d(y, Tx)] \\ &+ \eta[d(x, Tx) + d(x, y)] \end{aligned}$$

for all $x, y \in X, \quad \alpha, \beta, \gamma, \delta, \eta \geq 0,$
with $0 \leq \alpha + \beta + 2\gamma + 2\delta + 2\eta < 1,$ *then* T *has a unique fixed point.*

Now, motivated with above theorems we establish the following fixed point theorem which unifies and generalizes the above mentioned theorems in dislocated quasi metric spaces.

Theorem 4.3.8. *[121] Let (X, d) be a complete dislocated quasi metric space. Let* $T : X \to X$ *be continuous mapping satisfying the condition*

$$\begin{aligned} d(Tx, Ty) &\leq \alpha d(x, y) + \beta \frac{d(x, Tx).d(y, Ty)}{d(x, y)} \\ &+ \gamma[d(x, Tx) + d(y, Ty)] \\ &+ \delta[d(x, Ty) + d(y, Tx)] + \eta[d(x, Tx) + d(x, y)] \\ &+ \kappa[d(y, Ty) + d(x, y)] \end{aligned} \tag{4.1}$$

for all $x, y \in X, \quad \alpha, \beta, \gamma, \delta, \eta, \kappa \geq 0,$ *with*
$0 \leq \alpha + \beta + 2\gamma + 4\delta + 2\eta + 2\kappa < 1,$ *then* T *has a unique fixed point.*

Proof:
Let us define a sequence $\{x_n\}$ as follows:
$T(x_n) = x_{n+1}$, for n = 0, 1, 2, ...
Also, let $x = x_{n-1}, and \quad y = x_n$, Then, by condition (4.1) and

using triangle inequality, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, Tx_{n-1}).d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \\
&+ \gamma[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\
&+ \delta[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) + \eta[d(x_{n-1}, Tx_{n-1}) \\
&+ d(x_{n-1}, x_n)] + \kappa[d(x_n, Tx_n) + d(x_{n-1}, x_n)] \\
&= \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \\
&+ \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&+ \delta[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + \eta[d(x_{n-1}, x_n) \\
&+ d(x_{n-1}, x_n)] + \kappa[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\
&\leq (\alpha + \gamma + 2\delta + 2\eta + \kappa) d(x_{n-1}, x_n) \\
&+ (\beta + \gamma + 2\delta + \kappa) d(x_n, x_{n+1}).
\end{aligned}$$

Hence, we have

$$d(x_n, x_{n+1}) \leq \frac{\alpha + \gamma + 2\delta + 2\eta + \kappa}{1 - (\beta + \gamma + 2\delta + \kappa)} d(x_{n-1}, x_n).$$

Thus, we have

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n),$$

where

$$\lambda = \frac{\alpha + \gamma + 2\delta + 2\eta + \kappa}{1 - (\beta + \gamma + 2\delta + \kappa)}, \qquad 0 \leq \lambda < 1.$$

Similarly, we get $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$.
Hence, we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1),$$

so, for $n \to \infty$, we get $d(x_n, x_{n+1}) \to 0$. Similarly, we get $d(x_{n+1}, x_n) \to 0$.

Hence, $\{x_n\}$ is a Cauchy sequence in complete dislocated quasi metric space (X, d). So, there exists a point $u \in X$ such that $\{x_n\} \to u$. Since T is continuous, so we have

$$T(u) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

To prove the uniqueness of fixed point, let u and v are two fixed points of T so that, by definition, $Tu = u$ and $Tv = v$. Let u be fixed. Then, the condition (4.1) gives

$$\begin{aligned} d(u,u) &= d(Tu,Tu) \\ &\leq \alpha d(u,u) + \beta d(u,u) + 2\gamma d(u,u) + 2\delta d(u,u) + 2\eta d(u,u) \\ &+ 2\kappa d(u,u) \\ &= (\alpha + \beta + 2\gamma + 2\delta + 2\eta + 2\kappa) d(u,u) \end{aligned}$$

which implies that $d(u,u) = 0$,
since, $0 \leq \alpha + \beta + 2\gamma + 4\delta + 2\eta + 2\kappa < 1$.
Thus, we have $d(u,u) = 0$.

Similarly, we can get $d(v,v) = 0$, for v fixed. Again, from (4.1), we have

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &\leq \alpha d(u,v) + \beta \frac{d(u,u).d(v,v)}{d(u,v)} + \gamma[d(u,u) + d(v,v)] \\ &+ \delta[d(u,v) + d(v,u)] + \eta[d(u,u) + d(u,v)] \\ &+ \kappa[d(v,v) + d(u,v)] \\ &= (\alpha + \delta + \eta + \kappa) d(u,v) + \delta d(v,u). \end{aligned}$$

Similarly, we get

$$d(v,u) \leq (\alpha + \delta + \eta + \kappa) d(v,u) + \delta d(u,v).$$

Hence, we obtain

$$|d(u,v) - d(v,u)| \leq (\alpha + \eta + \kappa)|d(u,v) - d(v,u)|,$$

which is a contradiction. So, we have $d(u,v) = d(v,u)$.
Again, by (4.1), $d(u,v) \leq (\alpha + 2\delta + \eta + \kappa)d(u,v)$.
which implies that $d(u,v) = 0$. Hence, we have $d(u,v) = d(v,u) = 0$.
Therefore, we have $u = v$.
This completes the proof of theorem.

Remarks: In our theorem (4.3.8),

1. If we put $\kappa = 0$, we get the Theorem 3.1 of K. Zoto, E. Hoxha and A. Isufati [163].
2. If we put $\eta = \kappa = 0$, we obtain the Theorem 3.5 of R. Shrivastav, Z. K. Ansari and M. Sharma [149].
3. If we put $\gamma = \delta = \eta = \kappa = 0$, we obtain Theorem 3.3 of [149].
4. If we put $\beta = \eta = \kappa = 0$, we obtain the Theorem 3.5 of C. T. Aage and J. N. Salunke [1].
5. If we put $\beta = \gamma = \eta = \kappa = 0$, we get Theorem 3.2 of A. Isufati [53].
6. If we put $\beta = \gamma = \delta = \eta = \kappa = 0$, we get the Theorem 2.1 of F. M. Zeyada, G. H. Hassan and M. A. Ahmed [162].

Thus, our result extends and unifies the results of [1],[53],[149], [162],[163] and other similar results.

Again, we have obtained the following fixed point theorem in dislocated quasi metric space.

Theorem 4.3.9. *Let (X, d) be a complete dislocated quasi metric space. Let $T : X \to X$ be continuous mapping satisfying the condition*

$$\begin{aligned} d(Tx,Ty) &\leq \alpha d(x,y) + \beta \frac{d(x,Tx).d(y,Ty)}{d(x,y)} + \gamma[d(x,Tx) + d(y,Ty)] \\ &+ \delta[d(x,Ty) + d(y,Tx)] + \eta[d(x,Tx) + d(x,y)] \\ &+ \kappa[d(y,Ty) + d(x,y)] + \mu \frac{d(x,Ty).d(x,Tx)}{d(x,y)} \end{aligned} \tag{4.2}$$

for all $x, y \in X, \quad \alpha, \beta, \gamma, \delta, \eta, \kappa, \mu \geq 0$,
with $0 \leq \alpha+\beta+2\gamma+4\delta+2\eta+2\kappa+2\mu < 1$, *then* T *has a unique fixed point.*

Proof:
Let us define a sequence $\{x_n\}$ as follows:
$T(x_n) = x_{n+1}$, for n = 0, 1, 2, ...
Also, let $x = x_{n-1}, and \quad y = x_n$, Then, by condition (4.2) and

using triangle inequality, we have

$$
\begin{aligned}
d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\
&\leq \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, Tx_{n-1}).d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \\
&+ \gamma[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\
&+ \delta[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1}) + \eta[d(x_{n-1}, Tx_{n-1}) \\
&+ d(x_{n-1}, x_n)] + \kappa[d(x_n, Tx_n) + d(x_{n-1}, x_n)] \\
&+ \mu \frac{d(x_{n-1}, Tx_n).d(x_{n-1}, Tx_{n-1})}{d(x_{n-1}, x_n)} \\
&= \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, x_n).d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \\
&+ \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
&+ \delta[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + \eta[d(x_{n-1}, x_n) \\
&+ d(x_{n-1}, x_n)] + \kappa[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] \\
&+ \mu \frac{d(x_{n-1}, x_{n+1}).d(x_{n-1}, x_n)}{d(x_{n-1}, x_n)} \\
&\leq (\alpha + \gamma + 2\delta + 2\eta + \kappa + \mu) d(x_{n-1}, x_n) \\
&+ (\beta + \gamma + 2\delta + \kappa + \mu) d(x_n, x_{n+1})
\end{aligned}
$$

hence,

$$d(x_n, x_{n+1}) \leq \frac{\alpha + \gamma + 2\delta + 2\eta + \kappa + \mu}{1 - (\beta + \gamma + 2\delta + \kappa + \mu)} d(x_{n-1}, x_n), \qquad 0 \leq \lambda < 1.$$

Thus, we have

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n).$$

where

$$\lambda = \frac{\alpha + \gamma + 2\delta + 2\eta + \kappa + \mu}{1 - (\beta + \gamma + 2\delta + \kappa + \mu)}.$$

Similarly, we get $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$.
Hence, we have

$$d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1).$$

Now, for any m, n with $m > n$ and using triangle inequality, we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \lambda^n d(x_0, x_1) + \lambda^{n+1} d(x_0, x_1) + \cdots + \lambda^{m-1} d(x_0, x_1) \\ &\leq (\lambda^n + \lambda^{n+1} + \lambda^{n+2} \ldots) d(x_0, x_1) \\ &= \frac{\lambda^n}{1-\lambda} d(x_0, x_1). \end{aligned}$$

For any ε, we choose $N \geq 0$ such that $\frac{\lambda^N}{1-\lambda} d(x_0, x_1) < \varepsilon$.
Then, for any $m > n \geq N$, we have

$$d(x_n, x_m) \leq \frac{\lambda^N}{1-\lambda} d(x_0, x_1) \leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1).$$

Similarly, we can show that $d(x_m, x_n) < \varepsilon$.

Hence, the sequence $\{x_n\}$ is a Cauchy sequence in complete dislocated quasi metric space (X, d). So, there exists a point $u \in X$ such that $\{x_n\} \to u$. Since T is continuous, so we have

$$T(u) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = u.$$

To prove the uniqueness of fixed point, if possible, let u and v are two fixed points of T so that, by definition, $Tu = u$ and $Tv = v$. Let u be fixed. Then, the condition (4.2) gives

$$\begin{aligned} d(u, u) &= d(Tu, Tu) \\ &\leq \alpha d(u, u) + \beta d(u, u) + 2\gamma d(u, u) + 2\delta d(u, u) \\ &+ 2\eta d(u, u) + 2\kappa d(u, u) + \mu d(u, u) \\ &= (\alpha + \beta + 2\gamma + 2\delta + 2\eta + 2\kappa + \mu) d(u, u) \end{aligned}$$

which implies that $d(u,u) = 0$,
since $0 \leq \alpha + \beta + 2\gamma + 4\delta + 2\eta + 2\kappa + 2\mu < 1$.
Thus, we have $d(u,u) = 0$
Similarly, we can get $d(v,v) = 0$, for v fixed. Again, from (4.2), we have

$$\begin{aligned}
d(u,v) &= d(Tu,Tv) \\
&\leq \alpha d(u,v) + \beta \frac{d(u,u).d(v,v)}{d(u,v)} + \gamma[d(u,u) + d(v,v)] \\
&+ \delta[d(u,v) + d(v,u)] + \eta[d(u,u) + d(u,v)] \\
&+ \kappa[d(v,v) + d(u,v)] + \mu d(u,u) \\
&= (\alpha + \delta + \eta + \kappa)d(u,v) + \delta d(v,u).
\end{aligned}$$

Similarly, we get

$$d(v,u) \leq (\alpha + \delta + \eta + \kappa)d(v,u) + \delta d(u,v).$$

Hence, we obtain

$$|d(u,v) - d(v,u)| \leq (\alpha + \eta + \kappa)|d(u,v) - d(v,u)|,$$

which is a contradiction. So, we have $d(u,v) = d(v,u)$.
Again, by (4.2) with substitutions, we obtain
$d(u,v) \leq (\alpha + 2\delta + \eta + \kappa)d(u,v)$.
which implies that $d(u,v) = 0$. Hence, we have $d(u,v) = d(v,u) = 0$. Therefore, we have $u = v$.
This completes the proof of theorem.

Future Research Scope

Some of the future aspects of fixed point results in dislocated and dislocated quasi metric spaces are as follows:

1. To ensure the fixed point results for rational type contractive conditions in dislocated and dislocated quasi-metric spaces.
2. Dislocated and dislocated quasi metric spaces are open wide areas of research activities for the establishment of fixed point results under weaker contractive definitions like compatible, semicompatible, weakly compatible, occasionally weakly compatible and non compatible mapppings etc.
3. There is a wide scope to study common fixed point theorems for pairs of mappings and even for sequence of mappings in these spaces.

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